Workshop 7

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1 Goal

The goal of this workshop is to recall limits of functions from Calculus and to explore properties of limits that Calculus isn't enough to address. In this workshop, we're going to restrict our attention to real-valued functions on $\mathbb{R}$. This will simplify our definitions and allow us to focus on the key ideas.

2 Calculus Review

Definition 2.1. Let $f : \mathbb{R} \to \mathbb{R}$, and let $x_0 \in \mathbb{R}$. We say that f has a limit L at x_0, denoted

$$\lim_{x \to x_0} f(x) = L$$

if and only if for all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $0 < |x - x_0| < \delta$, $\left|f(X) - L\right| < \varepsilon$.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3x & \text{if } x \neq 0 \\ 4 & \text{if } x = 0 \end{cases}.$$

   (a) Graph this function.

   (b) Based on this picture and from what you remember from Calculus, what would you say is $\lim_{x \to 0} f(x)$?

   (c) Use the formal definition of the limit above to prove your claim.

2. Prove that

$$\lim_{x \to 2}(3x + 1) = 7$$

   using the limit definition above.

3. Prove that

$$\lim_{x \to 2} x^2 = 4$$

   using the limit definition above.

3 More Complicated Functions

1. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

   Prove that

$$\lim_{x \to 0} f(x) = 0$$

   using the limit definition above.

2. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

   (a) Graph f.

   (b) Does f have a limit at 0? Prove your claim using the formal definition of a limit above.

3. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}.$$

   Prove that

$$\lim_{x \to 0} f(x) = 0$$

   using the limit definition above.

4. **Challenge Problem:** Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ (in lowest terms)} \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}.$$

   Prove that for all $x_0 \in \mathbb{R} - \mathbb{Q}$,

$$\lim_{x \to x_0} f(x) = 0.$$

   What about for $x_0 \in \mathbb{Q}$?

   Hint: You may use the fact that every open interval of $\mathbb{R}$ contains at least one rational point and at least one irrational point.