Workshop 5

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1 Goal

The goal of this workshop is to explore the relationship between accumulation points of a set and limits/limit points of a sequence, and to introduce the notions of $\liminf$ and $\limsup$.

2 Accumulation Points of a Set

Recall that for a set $S \subseteq \mathbb{R}$, an **accumulation point** for the set is a point $x \in \mathbb{R}$ such that every neighborhood of x contains infinitely many elements of S. It is worth noting that this definition is equivalent to saying that for all $\varepsilon > 0$, the interval $(x-\varepsilon, x+\varepsilon)$ contains infinitely many elements of S.

1. How many accumulation points does a finite set have?
2. Find an uncountable set whose set of accumulation points is all of $\mathbb{R}$.
3. Find a countable set whose set of accumulation points is all of $\mathbb{R}$.
4. Construct a set with exactly two accumulation points.
5. Let $S \subset \mathbb{R}$ be nonempty. Suppose $x \in \mathbb{R}$ such that $\forall \varepsilon > 0$, $S \cap (x-\varepsilon, x+\varepsilon) - \{x\} \neq \emptyset$.
   (a) Prove that x is an accumulation point of S.
   (b) Suppose we changed our assumption to say that $\forall \varepsilon > 0$, $S \cap (x-\varepsilon, x+\varepsilon) \neq \emptyset$. Give an example to show that x may not be an accumulation point of S in this case.

3 Limit Points: Accumulation Points of a Sequence

For a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$, a **limit point** (sometimes called an accumulation point) of the sequence is a real number x such that there exists a subsequence $\{a_{n_k}\}$ that converges to x.

1. How many limit points does the sequence $\{a_n\}_{n=1}^{\infty}$ have in each of the below cases?
   (a) $a_n = 1$
   (b) $a_n = (-1)^n$
   (c) a_n is the unique integer between 1 and 3 that is congruent to $n \mod 3$.
   (d) $a_n = 100$ for $n = 1, \dots 100$, and $a_n = \frac{1}{n}$ for $n > 100$.

2. Construct a sequence with one limit point.

3. Construct a sequence with no limit points.

4. Construct a sequence with infinitely many limit points.

5. Prove that a sequence of real numbers converges to x if and only if every subsequence converges to x.

6. Construct a sequence that has exactly one limit point, but which does not converge.

7. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers and suppose $x \in \mathbb{R}$ is such that $\forall \varepsilon > 0$, $\forall k \in \mathbb{N}$, $\{a_n : n \geq k\} \cap (x - \varepsilon, x + \varepsilon) \neq \varnothing$. Show that x is a limit point of the sequence $\{a_n\}_{n=1}^{\infty}$. Compare this to problem 5 in the first section.

4 $\liminf$ and $\limsup$

Definition 4.1. The $\limsup$ of a sequence $\{a_n\}$, denoted $\limsup_{n\to\infty} a_n$ or sometimes $\overline{\lim_{n\to\infty}} a_n$, is the supremum of the set of limit points of the the sequence. The $\liminf$ of a sequence $\{a_n\}$, denoted $\liminf_{n\to\infty} a_n$ or sometimes $\underline{\lim_{n\to\infty}} a_n$, is the infemum of the set of limit points of the the sequence.

5. For this problem, we will investigate the sequence $\sin(\frac{\pi}{4}), \sin(\frac{\pi}{2}), \sin(\frac{3\pi}{4}), \ldots \sin(\frac{n\pi}{4}), \ldots$.
   (a) Write out the set of limit points for the sequence.
   (b) What is $\limsup_{n\to\infty} \sin(\frac{n\pi}{4})$?
   (c) What is $\liminf_{n\to\infty} \sin(\frac{n\pi}{4})$?

6. For this problem, we will investigate the sequence $1, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \ldots, (-1)^n(1 - \frac{1}{n}), \ldots$
   (a) Write out the set of limit points for the sequence.
   (b) What is $\limsup_{n\to\infty}(-1)^n(1 - \frac{1}{n})$?
   (c) What is $\liminf_{n\to\infty}(-1)^n(1 - \frac{1}{n})$?

7. Suppose $\{a_n\}$ is a sequence of real numbers satisfying $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. How many limit points does the sequence $\{a_n\}$ have? Prove your answer.

8. Let $\{a_n\}$ be a bounded sequence of real numbers. Prove that the sequence converges if and only if $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. You may use the fact that a bounded sequence converges if and only if it has exactly one limit point.

5 Challenge Questions

For this section, let $\{a_n\}$ denote a bounded sequence of real numbers. Construct a new sequence $\{b_k\}$ defined by

$$b_k = \sup_{n\geq k} a_n$$

1. For each of the below sequences, find b_1, b_2, b_3, and a general term b_k of the sequence defined above.

   (a) $a_n = (-1)^n(1 + \frac{1}{n})$
   (b) $a_n = \sin(\frac{2^n \pi}{8})$
   (c) $a_n = (-1)^n$
   (d) $a_n = \frac{(5-n)(n-12)}{n^3}$ Hint: Look at the graph of $f(x) = \frac{(5-x)(x-12)}{x^3}$

2. Prove that for any bounded sequence $\{a_n\}$ of real numbers, the sequence $\{b_k\}$ converges.

3. Prove that $\limsup_{n\to\infty} a_n = \lim_{k\to\infty} b_k$.

   Hint: You may find it useful to construct a subsequence $\{a_{n_j}\}$ which converges to the same limit as $\{b_k\}$.