Workshop 2

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1 Goal

The goal of this workshop is to first review some sticky points about proof-writing and then to play with some properties of real numbers. When discussing real numbers, we will practice working with supremums and infemums.

2 Proof-Writing Review

2.1 Quantifiers

1. What's the difference between the following two statements?
   - For every person in this class, there is a teacher who taught him/her Calculus.
   - There is a teacher who, for every person in this class, this teacher taught him/her Calculus.
2. Let X denote the set of all people. Let $P(x, y)$ stand for "x is y's mother", where $x, y \in X$. Translate the following statements into English, and explain their difference
   (a) $(\forall x \in X)(\exists y \in X)P(x, y)$
   (b) $(\exists x \in X)(\forall y \in X)P(x, y)$
3. Let $\mathcal{F}$ denote a set of functions, $f \in \mathcal{F}$, and let $S \subset \mathbb{R}$ be some set on which all the functions in $\mathcal{F}$ are defined.
   (a) We say f is <u>continuous at $x \in S$</u> if and only if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall y \in \mathbb{R}, |x - y| < \delta$ implies $\left|f(x) - f(y)\right| < \epsilon$.
   What does it mean to say that f is *not* continuous at x?
   (b) We say f is <u>continuous on S</u> if and only if $\forall x \in S$ and $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall y \in S$, if $|x - y| < \delta$ then $\left|f(x) - f(y)\right| < \epsilon$.
   What does it mean to say that f is *not* continuous on S?
   (c) We say f is <u>uniformly continuous on S</u> if and only if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in S$, if $|x - y| < \delta$ then $\left|f(x) - f(y)\right| < \epsilon$.
   What does it mean to say that f is *not* uniformly continuous on S?
   (d) We say $\mathcal{F}$ is <u>equicontinuous</u> if and only if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall f \in \mathcal{F}$ and $\forall x, y \in S$, if $|x - y| < \delta$ then $\left|f(x) - f(y)\right| < \epsilon$.
   What does it mean to say that $\mathcal{F}$ is *not* equicontinuous on S?

4. Consider the definitions above.

   (a) What is the difference between continuity and uniform continuity?

   (b) What is the difference between the statements "All functions in $\mathcal{F}$ are uniformly continuous," and "$\mathcal{F}$ is equicontinuous"?

2.2 Find-The-Error

Find the error in each of the following "proofs". Note that in some cases, the proposition is false, while in other cases, the proposition is true. You are *not* being asked whether or not the proposition is true. You are being asked to find the logical flaw in the "proof" presented.

1.

   Proposition 2.1. *Let $a, b, c \in \mathbb{Z}$. Then $ac|bc$ if and only if $a|b$.*

   Proof. If $ac|bc$, then

   $$bc = ack \tag{1}$$

   for some $k \in \mathbb{Z}$. Then, dividing both sides of equation (1) by c, we obtain

   $$b = ak.$$

   Since $k \in \mathbb{Z}$, we see that $a|b$, as desired. □

   Note that the proof of Proposition 2.1 has two errors. One is more obvious than the other.

2.

   Proposition 2.2. *Every integer is rational.*

   Proof. Suppose not. Then, every integer is irrational. But then $1 = \frac{1}{1}$, which is an integer, is rational. This is a contradiction. Hence, every integer must be rational. □

3.

   Proposition 2.3. $\exists x \in \mathbb{Q}$ *such that* $\forall k \in \mathbb{Z}, |x - k| > \frac{1}{4}$.

   Proof. Let $x = \frac{1}{2} \in \mathbb{Q}$ and $k = 2 \in \mathbb{Z}$. Then

   $$|x - k| = \left|\frac{1}{2} - 2\right| = \frac{3}{2} > \frac{1}{4}$$

   □

4.

   Proposition 2.4. *Take $n \in \mathbb{N}$. Then $n^3 + 1$ is composite.*

   Proof. Suppose $n^3 + 1$ is prime. For notational simplicity, call it p. Then, since p is prime, it has no divisors. But p is an integer, and 1 divides every integer, so p has a divisor. Therefore, we have shown that p has a divisor and that p has no divisors, which is a contradiction. Therefore, $n^3 + 1$ must have been composite. □

3 Real Numbers

1. Prove that for all $x \in \mathbb{R}$ with $0 < x < 4$,

$$\frac{4}{x(4-x)} \geq 1$$

2. Let $S = \{\frac{n-1}{n+1} : n \in \mathbb{N}\}$.

   (a) Prove that 1 is an upper bound for S.

   (b) Prove that each of the following is not the supremum of S.

      i. 10

      ii. 0

      iii. $\frac{1}{2}$

3. Give an example of each of the following, or state that it cannot exist.

   (a) A set B such that $\inf B \geq \sup B$.

   (b) A finite set that contains its infemum but not its supremum.

   (c) A bounded subset of $\mathbb{Q}$ that contains it supremum but not its infemum

4. Given two sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to show that if A and B are nonempty and bounded above, then $\sup(A + B) = \sup(A) + \sup(B)$.

   (a) Let $s = \sup(A)$ and $t = \sup(B)$. Prove that $s + t$ is an upper bound for $A + B$.

   (b) Let u be any upper bound for $A + B$, and let a be some element of A. Prove that $t \leq u - a$.

   (c) Finally, show that $\sup(A + B) = s + t$.