Workshop 11

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1 Goal

The goal of this workshop is to practice proving statements about continuity.

2 Continuity

1. Define $f : \mathbb{R} \to \mathbb{R}$ as below:

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   Prove that f is continuous at 0.

2. Let $f(x) = e^x$ for all $x \in \mathbb{R}$. Prove that f is continuous. You may use the fact that $\lim_{x \to 0} e^x = 1$.

3. Prove that

$$\lim_{x \to 0} |x|^x = 1.$$

   You may use L'Hôpital's rule.

4. Suppose the functions $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and that $f(r) = g(r)$ for every $r \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for every $x \in \mathbb{R}$. Hint: the rationals are dense in $\mathbb{R}$.

5. Suppose the function $g : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{h \to 0} (g(x + h) - g(x - h)) = 0$$

   for every $x \in \mathbb{R}$. Does that imply that g is continuous? Prove or give a counterexample.

6. Define the function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \ . \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

   Prove that f is continuous at every irrational point and discontinuous at every rational point.