# Workshop 10

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## 1 Goal

The goal of this workshop is to understand how limits behave with usual algebraic operators.

## 2 Limit Theorems

Throughout this workshop, let $D \subseteq \mathbb{R}$, let $f, g : D \to \mathbb{R}$, and let $x_0$ be an accumulation point of $D$. You may *not* use theorem 2.1 from the textbook in your proofs.

1. Prove that if $f$ and $g$ have limits at $x_0$, then

$$\lim_{x \to x_0} (f + g)(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x).$$

2. Is it true that if $\lim_{x \to x_0}(f + g)(x)$ exists, then both $f$ and $g$ have limits at $x_0$? Prove or give a counterexample.

3. Prove that if $f$ and $g$ have limits at $x_0$, then

$$\lim_{x \to x_0} (fg)(x) = \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} g(x)\right).$$

You may use theorem 2.3 from the textbook in this problem.

4. Is it true that if $\lim_{x \to x_0}(fg)(x)$ exists, then both $f$ and $g$ have limits at $x_0$? Prove or give a counterexample.

5. Suppose that $f$ and $g$ both have limits at $x_0$ and that $f(x) \leq g(x)$ for all $x \in D$. Prove that

$$\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} g(x)$$

6. Suppose $\lim_{x \to x_0} g(x) = L \neq 0$ and that $g(x) \neq 0$ for all $x \in D$. Prove that there is some $M > 0$ and some $r > 0$ such that $\left|g(x)\right| \geq M$ for all $x \in D \cap (x_0 - r, x_0 + r)$.

## 3 Challenge Problems

1. Suppose that $g(x) \neq 0$ for all $x \in D$. Prove that if $f$ and $g$ have limits at $x_0$, then

$$\lim_{x \to x_0} \left( \frac{f}{g} \right)(x) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}.$$

2. Suppose that $g(x) \neq 0$ for all $x \in D$. Is it true that if $\lim_{x \to x_0} (\frac{f}{g})(x)$ exists, then both $f$ and $g$ have limits at $x_0$? Prove or give a counterexample.

3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If $f$ has a limit at 0, prove that $f$ has a limit at every point and either $\lim_{x \to x_0} f(x) = 1$ or $f(x) = 0$ for all $x \in \mathbb{R}$.