Workshop Review 1

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1 Major Theorems

For each of the theorems described below, you should know the precise statement and how to apply it. You should also know the precise definitions of the terms in the below theorems. For those theorems marked with *, you should also know at least the key points of the proofs, because those techniques are useful in many other contexts.

- Every convergent sequence is bounded.*
- Every convergent sequence is Cauchy.*
- Balzano-Weierstass
- Every bounded sequence has at least one limit point.
- Every real number is an accumulation point of $\mathbb{Q}$ (ie the rational numbers is dense in $\mathbb{R}$)
- Every Cauchy sequence of real numbers converges to some real number.
- The limit of the sum of two convergent sequences is the sum of the limits.*
- The limit of the product of two convergent sequences is the product of the limits.*
- A sequence converges if and only if each of its subsequences converges.
- Bounded monotone sequences converge.
- f converges to L at x_0 if and only if for every sequence $\{x_n\}$ converging to x_0, $f(x_n)$ also converges to L.
- If f has a limit at x_0, then it is bounded near x_0.*
- The limit of a sum of functions is the sum of the limits, provided they exist.*
- The limit of a product of functions is the product of the limits, provided they exist.*

2 Starter Problems

The problems below are basic questions, which help you assess whether you understand the basic definitions and theorems.

1. Prove that $1 + \frac{1}{n}$ converges to 1.
2. Prove or give a counterexample: Every bounded sequence converges.
3. Prove that $\frac{\sin(n)}{n}$ converges to 0.
4. Let p be any postive number. Prove that $\frac{1}{n^p}$ converges to 0.
5. Let $0 < c < 1$. Prove that $\sqrt[n]{c}$ converges.
6. Prove that $\lim\limits_{x \to 0}(x^2 + 4) = 4$.
7. Prove that $\lim\limits_{x \to 4} \sqrt{x} = 2$.
8. Prove or give a counterexample: If $\lim_{x \to x_0} f(x) = L$ and x_0 is in the domain of f, then $f(x_0) = L$.

3 Study Problems

The problems below are a bit harder than the ones in the section above. These are closer in difficulty to the homework problems and the problems you are likely to see on the exam.

1. Prove that if $c > 1$, then $\sqrt[n]{c}$ converges.
2. Show that a_n converges to A if and only if $a_n - A$ converges to 0.
3. Suppose a_n converges to A. Define a new sequence $\{b_n\}_{n=1}^{\infty}$ by
$$b_n = \frac{a_n + a_{n+1}}{2}.$$
Prove that b_n also converges to A.
4. Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences. Prove directly that $\{a_n + b_n\}_{n=1}^{\infty}$ is Cauchy.
5. Prove that the sequence $\{a_n\}$ defined by
$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$
converges.
6. Prove that the sequence $\{s_n\}$ defined by
$$s_n = \frac{1}{n^2+1} + \frac{1}{n^2+2} \cdots + \frac{1}{n^2+n}$$
converges to 0.

7. Let $c \in \mathbb{R}$ with $|c| < 1$.
   (a) Prove that the sequence c^n converges to 0.
   (b) Let a_1 be any real number, and then define the sequence $\{a_n\}$ recursively by

   $$a_{n+1} = ca_n.$$

   Prove that a_n converges to 0.
   (c) Suppose $\{b_n\}$ is a bounded sequence, and define a sequence $\{s_n\}$ by

   $$s_n = c^n b_n.$$

   Does s_n converge? If so, to what limit?

8. Suppose $|a_n|$ converges to 0. Must a_n also converge to 0? Prove or give a counterexample.

9. Let $c > 0$, and prove that $\sqrt[n]{c}$ converges to 1.

10. Prove that $\sqrt[n]{n}$ converges to 1.

11. Suppose $\{a_n\}$ is a strictly increasing sequence which converges to some $A \in \mathbb{R}$. Suppose $\{b_n\}$ is a sequence such that for all $n \in \mathbb{N}$,

    $$a_n < b_n < a_{n+1}.$$

    Prove that b_n also converges to A.

12. Does every sequence have at most countably many subsequences? Does there exist a subsequence with uncountable many subsequences?

13. Suppose $\{a_n\}$ is a bounded sequence, and let M be such that $|a_n| \leq M$ for all n. Suppose a_n converges to a. Must $|a| \leq M$? Prove or give a counterexample.

14. Suppose $f : D \to \mathbb{R}$ with x_0 an accumulation point of D. Assume L_1 and L_2 are limits of f at x_0. Prove that $L_1 = L_2$.

15. Let $D \subset \mathbb{R}$ and x_0 be an accumulation point of D. Suppose f, g, and h are all real-valued functions with domain D. Suppose further that

    $$f(x) \leq g(x) \leq h(x)$$

    for all $x \in D$, and that

    $$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L.$$

    Prove that

    $$\lim_{x \to x_0} g(x) = L.$$

16. Let $D_1, D_2 \subset \mathbb{R}$, and suppose $x_1 \in D_1$ is an accumulation point of D_1. Let $f : D_1 \to D_2$ be such that

$$\lim_{x \to x_1} f(x) = p \in D_2.$$

Suppose that p is an accumulation point of D_2 and that $g : D_2 \to \mathbb{R}$ is such that

$$\lim_{x \to p} g(x) = q.$$

Is it true that

$$\lim_{x \to x_1} g(f(x)) = q?$$

Prove or give a counterexample.