Workshop 9

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1 Goal

The goal of this workshop is to construct the Cantor set and to prove some of its important properties.

2 Construction

We construct the standard Cantor set as follows.

Define $A_0 = [0, 1]$, the closed interval from 0 to 1. Construct A_1 by removing the open middle third from A_0. Namely,

$$A_1 = [0, 1] - \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Construct A_2 be removing the open middle thirds from each of the disjoint closed intervals making up A_1. Namely,

$$A_2 = \left(\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]\right) - \left(\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)\right) = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

We continue this process to construct a countably infinite number of sets $A_0, A_1, A_2, A_3, \ldots$. You should convince yourself of the following properties of these A_k:

- Each A_k is a finite union of disjoint closed intervals.
- For $k \geq 1$, $A_k \subset A_{k-1}$.

Exercise 2.1. *Construct A_3, A_4, and A_5 explicitly. Write your answer as a union of disjoint closed intervals.*

We are now ready to define the cantor set.

Definition 2.1. The **Cantor set** is the set C defined by

$$C = \bigcap_{k=0}^{\infty} A_k$$

Exercise 2.2. *Prove the following properties of the Cantor set.*

1. *C is a closed set.*

2. $0 \in C$ *and* $1 \in C$.

3. $C \neq [0,1]$

Exercise 2.3. *Consider the set* T

$$T = \left\{ \sum_{k=1}^{\infty} \frac{\delta_k}{3^k} : \delta_k \in \{0.2\} \right\}.$$

1. *Prove that* $T \subseteq [0,1]$.

2. *Prove that* $T = C$.

3 Properties: Size

Exercise 3.1. *Prove that* C *is uncountable. Hint: You may find Exercise 2.3 and Cantor's diagonalization argument useful.*

In the above exercise, you've shown that the Cantor set is as large as the whole real line., when we're measuring size by cardinality. However, we could "measure" our sets in a different (but still reasonable) way, to see that the Cantor set is actually quite small.

In graduate real analysis, you would learn about **measure theory**. We won't go into the details here, but we will use some important properties of the standard measure on $\mathbb{R}$, **Lebesgue measure**.

Proposition 3.1. *Let* μ *be Lebesgue measure. Then* μ *satisfies the following properties:*

1. *Let* $a < b$. *Then* $\mu(a,b) = b - a$.

2. *Let* $A, B \subseteq \mathbb{R}$ *be disjoint. Then* $\mu(A \cup B) = \mu(A) + \mu(B)$.

3. *If* $A \subseteq B$, *then* $\mu(A) \leq \mu(B)$.

4. *If* $B \subseteq A$, *then* $\mu(A - B) = \mu(A) - \mu(B)$.

5. *Let* $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ *be a sequence of subsets of* $\mathbb{R}$. *Suppose further that* $\mu(A_1) < \infty$. *Then*

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k)$$

I should mention that it is possible to construct particularly bizarre subsets of $\mathbb{R}$ where the above properties don't work. However, we won't come across those sets in this workshop. Furthermore, the condition that needs to be checked for the above proposition is highly technical, and I'm not going to get into that here.

Exercise 3.2. *Use the above proposition to prove the following facts:*

1. *For every* $a \in \mathbb{R}$, $\mu(\{a\}) = 0$.

2. *Let* $a < b$. *Then* $\mu([a,b]) = b - a$.

Exercise 3.3. *We will now use Lebesgue measure to explore how "big" the Cantor set is.*

1. *Compute* $\mu(A_1)$, $\mu(A_2)$, *and* $\mu(A_3)$.
2. *Make a conjecture about a formula for* $\mu(A_k)$. *Then, prove it.*
3. *Compute* $\mu(C)$, *where* C *is the Cantor set.*

4 Properties: Density

One of the bizarre properties of the Cantor set is that its points are simultaneously close together and spread out.

Definition 4.1. A set $A \subseteq \mathbb{R}$ is called **perfect** if A is closed and every $a \in A$ is an accumulation point of A.

Roughly, a set is perfect if all of its points are close together. For example, any closed interval is perfect.

Exercise 4.1. *Prove that the Cantor set is perfect.*

Definition 4.2. A closed set in $\mathbb{R}$ is **nowhere dense** if it doesn't contain any nonempty, open intervals.

Roughly, a set is nowhere dense if its points are spread out. For example, $\mathbb{Z}$ is nowhere dense.

Exercise 4.2. *Prove that the Cantor set is nowhere dense.*