Workshop 6

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1 Goal

The goal of this workshop is to practice proving limits of functions. We will also review sequences by exploring their properties and constructing counterexamples to false statements.

2 Limits of Functions

1. Let $f : \mathbb{R} - \{0\} \to \mathbb{R}$ be defined by $f(x) = x\sin(\frac{1}{x})$. Prove that $\lim\limits_{x\to 0} f(x) = 0$.

2. Suppose $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

   Prove that $\lim\limits_{x\to 0} g(x) = 0$.

3. Let $D \subseteq \mathbb{R}$ and let x_0 be an accumulation point of D. Let $f : D \to \mathbb{R}$. Prove that if

$$\lim_{x\to x_0} f(x) = L,$$

   then for every sequence $\{a_n\}$ in D which converges to x_0,

$$\lim_{n\to\infty} f(a_n) = L.$$

4. Suppose $D \subseteq \mathbb{R}$ and let x_0 be an accumulation point of D. Consider two functins $g, f : D \to \mathbb{R}$ such that

$$\lim_{x\to x_0} g(x) = 0$$

   and f is bounded, in the sense that $\exists M > 0$ such that $\left|f(x)\right| \leq M$ for all $x \in D$. Prove that

$$\lim_{x\to x_0} g(x)f(x) = 0.$$

3 Sequence Review

Determine whether each of the following statements about sequences of real numbers is true or false. Then, give a proof or counterexample.

1. If $\lim_{n\to\infty}(a_n - b_n) = 0$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.
2. If $a_n \to a$, then $|a_n| \to |a|$.
3. If $a_n \to a$ and $(a_n - b_n) \to 0$, then $b_n \to a$.
4. If $a_n \to 0$, $a_n > 0$ for all n, and $|b_n - b| < a_n$ for all n, then $b_n \to b$.
5. If $a_n \to a$, where $a > 0$ and $a_n > 0$ for all n, then $\sqrt{a_n} \to \sqrt{a}$.
6. If $\{a_n\}$ converges to a, then every subsequence also converges to a.
7. If $\{a_n\}$ is a sequence such that every proper subsequence converges, then $\{a_n\}$ also converges.
8. If $\{a_n\}$ is a monotone sequence with a convergent subsequence, then $\{a_n\}$ converges.
9. Every convergent sequence is Cauchy.
10. Every bounded sequence is convergent.
11. Every convergent sequence is bounded.
12. If $\{a_n\}$ is a Cauchy sequence, then so is $\{(-1)^n a_n\}$.
13. If $\{a_n\}$ is a bounded sequence and $\{b_n\}$ is a convergent sequence, then $\{a_nb_n\}$ converges.