Workshop 1: In Class

Chloe Wawrzyniak

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1 Goal

The goal of this workshop is to review aspects of cardinality that students typically struggle with, but which are crucial for this class. By the end of this workshop, you should understand what it means for a set to be countable, countably infinite, uncountable, finite, or bounded. You will have examples of each type of set and you should be able to describe the difference between these terms.

2 Definitions

Match the following terms with their definitions.

Countable	(a) There exists a bijection between the set and $\{1, \ldots, N\}$, for some $N \in \mathbb{N}$.
Finite	(b) There is a bijection between the set and $\mathbb{N}$.
Countably Infinite	(c) There is a surjection from the set to $\mathbb{N}$, but no injection.
Uncountable	(d) There is an injection from the set to $\mathbb{N}$.

There are other equivalent ways of defining the above terms. List some other definitions that you know.

The above definitions apply to any set. The term *bounded*, however, requires some extra knowledge. We need to know something about the topology of the set, or the context that we're discussing it in. For this class, we'll primarily be looking at subsets of $\mathbb{R}$ or $\mathbb{R}^n$. So, at least for now, we will only worry about the definition in that context. If you take further math courses, however, then you will see more general definitions.

Definition 2.1. A set $S \subseteq \mathbb{R}$ is **bounded** if there exists some $R > 0$ such that $S \subseteq [-R, R]$.

Mark each of the following sets as countable, countably infinite, uncountable, finite, or bounded. More than one term may apply for each set.

Set	Countable	Countably Infinite	Uncountable	Finite	Bounded
$\mathbb{Q}$					
$\{x \in \mathbb{R} : \lvert x\rvert < 2\}$					
$\{n \in \mathbb{Z} : \lvert n\rvert < 2\}$					
$\{0\}$					
$\{x \in \mathbb{R} : x^2 - 1 = 0\}$					
$\{\frac{1}{n} : n \in \mathbb{N}\}$					

Label each of the following statements as true or false. If a statement is false, give a counterexample. You do *not* need to prove the true statements.

1. A bounded set is always finite.

2. A finite set is always bounded.

3. A countably infinite set is never bounded.

4. A countably infinite set is always countable.

5. An uncountable set is never bounded.

3 Practice

Now that we've discussed the definitions, let's put them into practice.

1. In this problem, we will prove that $|O| = |E|$, where O is the set of all odd integers and E is the set of all even integers. Let $f : O \to \mathbb{Z}$ be the function defined by

$$f(n) = n + 1$$

   (a) Prove that f is injective.

   (b) Prove that the image of f is equal to E.

   This proves that f is a bijection from O to E, and therefore $|O| = |E|$

2. Prove that $|\mathbb{N}| = \left|\mathbb{N} \cup \{0\}\right|$

3. Prove that $|\mathbb{N}| = |\mathbb{Z}|$.

4. Suppose $A_1, A_2, \ldots, A_n, \ldots$ is a countable number of countable sets. Let

$$A = \bigcup_{n=1}^{\infty} A_n.$$

   Prove that A is countable. You may use the fact that a subset of a countable set is countable and that there are infinitely many prime numbers.

5. Prove that $\left|\{\frac{1}{n} : n \in \mathbb{N}\}\right| = \left|\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}\right|$.

6. Prove that $\left|(0, 1)\right| = \left|[0, 1)\right|$.

7. Prove that $\left|(0, 1)\right| = \left|[0, 1]\right|$.