Workshop 10

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1 Goal

The goal of this workshop is to see and prove some important properties of the standard topology of $\mathbb{R}$.

2 Open and Closed Sets

Definition 2.1. A set $G \subseteq \mathbb{R}$ is called **open** if for all $x \in G$, there is some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq G$.

Definition 2.2. A set $F \subseteq \mathbb{R}$ is called **closed** if $F^c = \mathbb{R} - F$ is open.

Exercise 2.1.
1. *Prove that* $(0, 1)$ *is open.*
2. *Prove that* $[0, 1]$ *is closed.*
3. *Prove that* $[0, 1)$ *is neither open nor closed.*

Exercise 2.2. *Prove that* $F \subseteq \mathbb{R}$ *is closed if and only if it contains all of its accumulation points.*

In particular, this tells us that we could have defined a closed set as a set containing all of its accumulation points, and nothing would have changed.

3 Closure, Interior, and Boundaries

Definition 3.1. Let $S \subseteq \mathbb{R}$.

1. The **closure** of S, denoted $\overline{S}$, is the union of S and the set of accumulation points of S.
2. The **interior** of S, denoted S^o, is defined as follows:

   For $x \in \mathbb{R}$ and $r > 0$, let $B(x, r)$ denote the interval of radius r centered at x, ie $B(x, r) = (x - r, x + r)$. Then

$$S^o = \{x \in \mathbb{R} : \exists r > 0 \text{ such that } B(x, r) \subseteq S\}$$

Exercise 3.1. *Prove that for all* $S \subset \mathbb{R}$, $\overline{S}$ *is closed and* S^o *is open.*

Exercise 3.2. *Prove that for any* $S \subseteq \mathbb{R}$, $\overline{S}$ *is the smallest closed set containing* S*. By "smallest", I mean the following: for any closed set* F *containing* S, $\overline{S} \subseteq F$.

Formulate and prove an analagous statement for S^o.

Exercise 3.3. *Let $S \subseteq \mathbb{R}$.*

1. *Prove that S is open if and only if $S = S^o$.*
2. *Prove that S is closed if and only if $S = \overline{S}$.*

Definition 3.2. For $S \subseteq \mathbb{R}$, its **boundary**, denoted ∂S is defined as $\overline{S} \cap \overline{S^c}$.

Exercise 3.4. *Let $S \subseteq \mathbb{R}$.*

1. *Prove that ∂S is closed.*
2. *Prove that $\overline{S} = \partial S \cup S$.*
3. *Prove that $S^o = S - \partial S$.*

4 Definition of a Topology

We've been throwing around the word *topology* a lot these last few weeks, but we've only been taking about it in the context of $\mathbb{R}$. Topology has its own definition independent of the real numbers, and is a subject of interest in its own right.

Definition 4.1. Let X be some set, and let T be some collection of subsets of X. Then we say T is a **topology** on X if all of the following conditions hold:

1. $\varnothing, X \in T$
2. The union of artibrarily many elements of T is also an element of T (sometimes phrased as "T is closed under arbitrary unions")
3. The intersection of finitely many elements of T is also an element of T (sometimes phrased as "T is closed under finite intersections")

We call the pair (X, T) a topological space. Sometimes, just say X is a topological space if there is no chance for confusion about what T is.

Definition 4.2. If T is a topology on X, we call the sets in T **open sets**.

Exercise 4.1. *Prove that the definition of open set in $\mathbb{R}$ that was given in section 1 defines a topology on $\mathbb{R}$.*

The topology on $\mathbb{R}$ defined above is the **standard topology** on $\mathbb{R}$. We could, however, have defined other topologies with different, sometimes interesting, properties.

Exercise 4.2. *For each of the T_k below, prove that it defines a topology on $\mathbb{R}$.*

1. *$T_1 = \{\varnothing, \mathbb{R}\}$. This is sometimes called the **trivial topology**.*
2. *$T_2 = \mathcal{P}(\mathbb{R})$, the power set of $\mathbb{R}$. This is sometimes called the **discrete topology**.*
3. *$T_3 = \{U \subseteq \mathbb{R} : U^c$ is finite or is all of $\mathbb{R}\}$. This is sometimes called the **finite complement topology**.*

For more examples, I suggest you take a look at Munkres' topology book, sections 12 and 13.

The notion of a topology also makes sense in cases where X is not $\mathbb{R}$.

Exercise 4.3. *Let* $X = \{a, b, c\}$.

1. *Let* $T_1 = \{X, \varnothing, \{a, b\}, \{b, c\}, \{b\}, \{c\}\}$. *Prove that this defines a topology on* X.
2. *Let* $T_2 = \{X, \varnothing, \{a\}, \{b\}\}$. *Prove that this does not define a topology on* X.

Definition 4.3. Let T be a topology on X. We say that a subset of X is **closed** when its complement is in T.

Exercise 4.4. *Let* T *be a topology on* X.

1. *Prove that* $\varnothing$ *and* X *are closed.*
2. *Prove that an arbitrary intersection of closed sets is closed.*
3. *Prove that a finite union of closed sets is closed.*
4. *Construct an example using the standard topology of* $\mathbb{R}$ *to show that an infinite union of closed sets may not be closed.*

5 Continuous Functions

The definition of continuous functions that we've been working with so far depends on limits. However, we can define what it means for a function to be continuous on a more general topological space where limits may not make sense.

Definition 5.1. Let (X, T_X) and (Y, T_Y) be topological spaces. Let $f : X \to Y$. We say that f is **continuous** on X if for every $\mathcal{O} \in T_Y$, we have that $f^{-1}(\mathcal{O}) \in T_X$, ie the inverse image of any open set is open.

Exercise 5.1. *Prove that every constant function is continuous function. Namely, let* $f : X \to Y$ *and suppose there exists* $y_0 \in Y$ *such that* $f(x) = y_0$ *for all* $x \in X$. *Prove that* f *is continuous. Hint: Consider two cases, depending on whether* $y_0 \in \mathcal{O}$ *or not.*

Exercise 5.2. *Prove that our definition of a continuous function on* $\mathbb{R}$ *using limits is equivalent to this definition.*

Exercise 5.3. *Suppose* $f : X \to Y$ *is continuous.*

1. *Prove that for every* $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.
2. *Prove that for every* $B \subseteq Y$ *which is closed,* $f^{-1}(B)$ *is closed in* X.

Exercise 5.4. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and suppose* x_0 *is an accumulation point of* $A \subseteq \mathbb{R}$. *Must* $f(x_0)$ *be an accumulation point of* $f(A)$*? Prove or give a counterexample.*

Exercise 5.5. *Let* X, Y, *and* Z *be topological spaces. Suppose* $f : X \to Y$ *and* $g : Y \to Z$ *are continuous. Prove that* $g \circ f : X \to Z$ *is continuous.*

Recall that one equivalent definition of a set being compact that we've looked at only relies on the definitions of open and closed sets. So, we can define what we mean for a subset of a general topological space to be compact.

Definition 5.2. Let $K \subseteq X$. We say K is **compact** if for any collection $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ of open sets in X such that

$$K \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha,$$

there is a finite sub-collection $\wr_{\alpha_1}, \ldots, \mathcal{O}_{\alpha_m}$ such that

$$X \subseteq \bigcup_{j=1}^{m} \mathcal{O}_{\alpha_j}.$$

Exercise 5.6. *Suppose that $f : X \to Y$ is continuous. Prove that the image of a compact set is compact. Namely, let $K \subseteq X$ be compact and prove that $f(K)$ is compact in Y.*