Working Backwards and Delta-Epsilon Proofs

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1 Introduction

Often, when solving an existence problem, we have to work backwards from the properties we want our object to have. Writing up these problems can be particularly tricky: when we first solve the problem we work backwards, but when we write up the solution we have to work in the other direction.

Here's an example of what I mean:

Example 1.1. Suppose $f(x) = x^2 - 12x + 5$ and $g(x) = x^3 - 5x^2 - 3$. Then, there exists some $x_0 \in \mathbb{R}$ such that $f(x_0) = g(x_0)$.

We could use Intermediate Value Theorem to solve this problem, but what if instead, we wanted to compute an exact value for x_0? Pull out a peice of scratch paper, and write at the top of it what we want to be true:

$$f(x) = g(x).$$

Since we know what f and g are, this becomes

$$x^2 - 12x + 5 = x^3 - 5x - 3.$$

Using the algebraic techniques we are all familiar with, this becomes

$$\begin{aligned} 0 &= x^3 - 6x^2 + 12x - 8 \\ &= (x-2)^3 \end{aligned}$$

We know that $x = 2$ solves the above equation, so we think that the x_0 we want is $x_0 = 2$.

Okay, so we found our x_0. Are we done?

Well, yes and no.

We've figured out what the solution is going to be, but everything we've written down at this point is scratch work. If we want to write down a formal proof, we should start with this solution (namely $x_0 = 2$) and then prove that it satisfies the property we want.

Proof. Suppose $x_0 = 2$. Certainly, $2 \in \mathbb{R}$. Furthermore,

$$\begin{aligned} f(x_0) &= f(2) \\ &= (2)^2 - 12(2) + 5 \\ &= 4 - 24 + 5 \\ &= -15 \end{aligned}$$

and

$$\begin{aligned} g(x_0) &= g(2) \\ &= (2)^3 - 5(2)^2 - 3 \\ &= 8 - 20 - 3 \\ &= -15 \end{aligned}$$

Hence, $f(x_0) = g(x_0)$, and our proof is complete. □

This situation doesn't just apply to existence proofs. There are many other problems whose scratch work is backwards from the proof writeup.

Example 1.2. For all $x \in \mathbb{R}$ with $0 < x < 4$,

$$\frac{4}{x(4-x)} \geq 1$$

Again, let's pull out a piece of scratch paper, write down what we want to be true, and start doing some algebra:

$$\frac{4}{x(4-x)} \geq 1$$

Let's multiply both sides by $x(4-x)$ to clear the denominator (note that this won't change the direction of the inequality, since $0 < x < 4$, so both x and $4 - x$ are positive numbers). Then, simplify.

$$4 \geq x(4-x) = 4x - x^2$$

Moving everything to the left-hand side, we obtain

$$x^2 - 4x + 4 \geq 0$$

However, we know that $x^2 - 4x + 4 = (x-2)^2$. Squares of real numbers are always nonnegative, so our above inequality must be true.

Cool, we started with what we wanted and ended up with a true statement. So, we're done... right?

Well, again, yes and no. We figured out what the logic should be since all of our algebraic steps can be reversed, but just as before, our formal proof needs to start with something we know to be true, and end with the conclusion we want.

Proof. Observe that for all real numbers x,

$$0 \leq (x-2)^2 = x^2 - 4x + 4$$

Subtracting 4 from both sides and multiplying by -1, we obtain

$$4 \geq 4x - x^2 = x(4-x)$$

For $0 < x < 4$, both x and $4-x$ are positive numbers, so we can divide both sides of the inequality by $x(4-x)$ without changing the direction of the inequality. So, we obtain

$$\frac{4}{x(4-x)} \geq 1$$

which is what we wanted to show, so we are done. □

This backwards-scratch-work-forwards-proof business can be unsettling for students seeing proof for the first time. In your previous courses, like Algebra or Calculus, the computation *was* the solution you turned in. There was no second step.

Proof writing is a whole new animal. This is what research Mathematicians are doing every day. Often, we have to wander down a lot of rabbit holes before we find something that works, and attempting to work backwards is one of many options in our toolkit. So, scratch work is incredibly important at this stage in your mathematical work.

It may seem odd to start our proofs with something like "Let $x_0 = 2$." Someone reading your proof for the first time might be confused. Where the heck did 2 come from? This is one reason that textbooks include the scratch work when presenting these kinds of problems. Pedagogically, it is important to include intuition and motivation for students to understand where these numbers came from. However, logically, it is unnecessary to include it.

The next few sections will cover a few senarios where we nearly always have to work backwards in our scratch work.

2 Surjective Functions

Definition 2.1. Let A and B be sets. Then a function $f : A \to B$ is **surjective** if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$.

So, if we want to prove that a function is surjective, we can start with the equation $f(a) = b$ and try to solve for a in terms of b. Then, that a should be what we want.

But the logic there is backwards from what we're supposed to show: we have to start with an arbitrary $b \in B$, find an $a \in A$, and then prove that the a we came up with satisfies $f(a) = b$.

Example 2.2. Define the function $f : \mathbb{R} \to [0, \infty)$ by $f(x) = x^2$. Then, f is surjective.

We start with some $y \in [0, \infty)$ with the goal of finding some $x \in \mathbb{R}$ such that $f(x) = y$. So, we pull out a piece of scratch paper to figure out what kind of x we might want:

If $f(x) = y$, then $x^2 = y$. Taking square roots of both sides, we see that a good option could be to pick $x = \sqrt{y}$. We do have to be careful when taking square roots, because we want the x we pick to be a real number. Since $y \geq 0$, its square root will indeed be real, so we're good here.

Okay, we found our x, and we even double checked that it's a real number. So, we're good, right? Not quite (sensing a pattern here?). We have to write up our proof in the other direction.

Proof. Let $y \in [0, \infty)$, and let $x = \sqrt{y}$. Since $y \geq 0$, we know that its square root is real, so $x \in \mathbb{R}$. Furthermore,

$$f(x) = f(\sqrt{y}) = (\sqrt{y})^2 = |y| \,.$$

However, since $y \geq 0$, $|y| = y$, so $f(x) = y$, thus demonstrating that f is surjective. $\square$

3 Delta-Epsilon Proofs

If you took Calculus, there's a good chance you've seen the formal definition of a limit. If you've only seen it in Calculus courses, there's also a pretty good chance that it didn't make a whole lot of sense. This is likely the first mathematically precise definition you see in your coursework, and it takes some time to get used to. However, one of the other reasons Calculus students typically struggle with the formal definition of a limit is the fact that for nearly every proof using it, your scratch work is backwards from the writeup - something you likely wouldn't have seen before!

For simplicity, we are only considering real-valued functions on $\mathbb{R}$, but these definitions can be extended to functions whose domains are subsets of $\mathbb{R}$, or to other systems such as $\mathbb{R}^n$, $\mathbb{C}$, or general metric spaces.

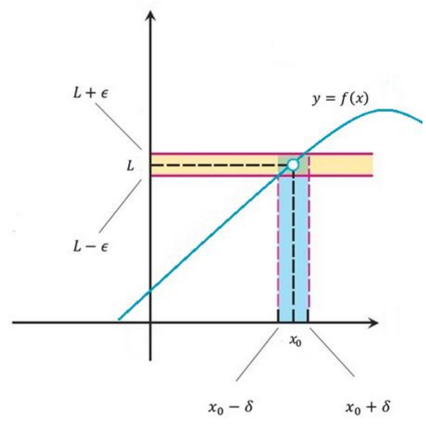
Definition 3.1. Let f be a real-valued function on $\mathbb{R}$, and let $a \in \mathbb{R}$. Then, we say that

$$\lim_{x \to a} f(x) = L$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|x - a| < \delta \Rightarrow \left| f(x) - L \right| < \varepsilon.$$

Let's unpack that definition a little bit before we jump into some examples. You can think of this as a sort of game. In this game, you are given some positive number ε, and based on that ε, the function, and a, you have to find a δ. What kind of δ do you want to pick? Well, you want to make sure that whenever you plug x-values into f that are at most δ away from a, the outputs are at most ε away from L.



Okay, so you know what you want δ to do, but how do you go about actually picking it? Well, let's do some scratch work. Start with $\left|f(x) - L\right| < \varepsilon$, then work backwards to get $|x - a|$, since that's what we know is related to δ.

Talking about this in the abstract can be difficult to understand, so let's do an example.

Example 3.2. Prove that

$$\lim_{x \to 1} (2x + 3) = 5.$$

Okay, so we know we want $\left|f(x) - 5\right|$ to be less than ε whenever $|x - a|$ is less than the δ that we're picking. I should emphasize at this point that *we are not picking* ε. You only get to pick the δ.

Well, let's start with writing down the equation we want to be true:

$$\left|(2x + 3) - 5\right| < \varepsilon$$

We can simplify this a little to get

$$|2x - 2| < \varepsilon$$

If we factor a 2 out of the left hand side, we get

$$2|x - 1| < \varepsilon$$

If you recall, we are going to be working only with the x-values that satisfy $|x - 1| < \delta$. So, it looks like if we make $\delta = \frac{\varepsilon}{2}$, the above equation will be true whenever $|x - 1| < \delta$. So, we picked our δ. So we're done? Nah, you know better than that by now. We still have to write our formal proof. Our proof needs to state what we claim δ to be, and then prove that as long as $|x - 1|$ is less than that δ, $\left|f(x) - 5\right|$ will be less than ε.

Proof. Let $f(x) = 2x + 3$. Let $\varepsilon > 0$. Then, let $\delta = \frac{\varepsilon}{2}$. Then whenever $|x - 1| < \delta$, we have

$$\begin{aligned}
\big|f(x) - L\big| &= \big|(2x+3) - 5\big| \\
&= |2x - 2| \\
&= 2|x - 1| \\
&< 2\delta \\
&= 2\frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Hence, whenever $|x-1| < \frac{\varepsilon}{2}$, $\big|f(x) - 5\big| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows that

$$\lim_{x\to 1}(2x+3) = 5.$$

□

In the above example, we were able to solve for δ explicitly, but sometimes, we have to be a bit more clever than that. Let's do a more complicated example.

Example 3.3. Prove that

$$\lim_{x\to -2}(x^2 - 6) = -2.$$

Let's follow the same pattern: start with what we want to be true.

$$\Big|(x^2 - 6) - (-2)\Big| < \varepsilon$$

We can simplify a little and factor to get

$$\big|(x-2)(x+2)\big| < \varepsilon$$

Okay, there's our $\big|x - (-2)\big| = |x+2|$ that we want. Last time, we just divided by the rest of the stuff, and got our δ. Unfortunately, this time, we can't do that because the other stuff involves an x. This is where we need to get a little creative.

What would this look like if we picked a δ that was at most 1? Then, we would know that since $|x+2| < \delta \leq 1$,

$$-1 < x + 2 < 1$$

Subtracting 4 from all values, we get

$$-5 < x - 2 < -3 < 5$$

Hence, we see that this gives $|x-2| < 5$. So, plugging into the earlier equation, we get

$$5|x+2| < \varepsilon.$$

So, we should make $\delta = \frac{\varepsilon}{5}$. Right? Well sort of. We only got the 5 by assuming that $\delta \leq 1$, so we need to take that into account in our definition. We deal with that by making $\delta = \min(1, \frac{\varepsilon}{5})$. This gives us that $\delta \leq 1$ *and* $\delta \leq \frac{\varepsilon}{5}$, regardless of what ε is.

Okay, you know the drill by now: we're done with scratch work, so let's start our proof:

Proof. Let $\varepsilon > 0$. Then, let $\delta = \min(1, \frac{\varepsilon}{5})$.

Suppose $|x+2| < \delta$. Since $\delta \leq 1$, this gives

$$-1 \leq -\delta < x + 2 < \delta \leq 1.$$

Subtracting 4, we see that

$$-5 < x - 2 < -3,$$

and hence $|x-2| < 5$. Therefore, we obtain

$$\begin{aligned}
\left|(x^2 - 6) - (-2)\right| &= \left|x^2 - 4\right| \\
&= \left|(x-2)(x+2)\right| \\
&= |x-2||x+2| \\
&< 5|x+2| \\
&< 5\delta \\
&\leq 5\frac{\varepsilon}{5} \\
&= \varepsilon
\end{aligned}$$

Hence,

$$\lim_{x \to -2} (x^2 - 6) = -2.$$

$\square$