1) Find the first four non-zero terms of its series. Find the series indirectly.

We start with the 0th order forms: $a_0 = \frac{\pi}{\lambda} \int_0^\lambda \cos(x) \, dx = \frac{\pi}{\lambda} [\cos(x)]_0^\lambda = \frac{\pi}{2}$.

Then, for $n \neq 0$, we have $a_n = \int_0^\lambda \cos(nx) \, dx = \frac{\pi}{\lambda} [\cos(nx)]_0^\lambda = \frac{\pi}{n^2} [\cos(n\lambda) - \cos(0)] = \frac{\pi}{n^2} [\cos(n\lambda)]$.

In particular:

$a_2 = \frac{\pi}{\lambda} [\cos(2\lambda)] = -\frac{\lambda}{2}$

$a_4 = \frac{\pi}{\lambda} [\cos(4\lambda)] = -\frac{\lambda}{4}$

2) For each of $n > 2$, what is the possible limit of the series?

We apply the Dirichlet convergence test of Fourier series, which states that if $\sum_{n=1}^\infty a_n$ converges and $f(x)$ is piecewise continuous on $[0, \lambda]$, then $\sum_{n=1}^\infty a_n \cos(nx)$ converges uniformly.

By the minimum convergence theorem of the series, given any positive continuous

5. $a_0 = \int_0^\lambda \cos(x) \, dx = \frac{\pi}{\lambda} [\cos(x)]_0^\lambda = \frac{\pi}{2}$.

4. $a_n = \int_0^\lambda \cos(nx) \, dx = \frac{\pi}{n^2} [\cos(nx)]_0^\lambda = \frac{\pi}{n^2} [\cos(n\lambda)]$.

Hence:

$a_2 = -\frac{\lambda}{2}$, $a_4 = -\frac{\lambda}{4}$, and $a_n = \frac{\pi}{n^2} [\cos(n\lambda)]$.

3) Given the series convergence in the 5th question.

(a) We apply the second convergence theorem with $\sum_{n=1}^\infty a_n \cos(nx)$ converges uniformly.

4. $a_n = \int_0^\lambda \cos(nx) \, dx = \frac{\pi}{n^2} [\cos(nx)]_0^\lambda = \frac{\pi}{n^2} [\cos(n\lambda)]$.

Hence:

$n = 2, 4, \ldots, 2 + 2 + 4 + 6 + 8 + 10 + \ldots$

5) By using the discontinuous convergence theorem with $a_n \neq 0$, consider the limit:

$\lim_{n \to \infty} \left[ \frac{\lambda}{2} \left( \frac{\lambda^2}{n^2} - \frac{\lambda^2}{n^2 + 1} \right) \right] = 0$.

By applying the computation of question 1.

6) Consider the 2D generated function, defined over $\mathbb{R}^2$ as:

$f(x) = \begin{cases} n^2 \cos(n^2) & \text{for } n \in \mathbb{Z} \\ 0 & \text{for } n \notin \mathbb{Z} \end{cases}$.

7) Consider the continuity of the field David thread at $x$.

The Fourier series of $f(x)$ is:

$\sum_{n=1}^\infty \frac{\pi}{n} \left[ \cos(n\lambda) - \cos(n\lambda + \pi) \right] = \frac{\pi}{\lambda} \left[ \cos(n\lambda) - \cos(n\lambda + \pi) \right]$.

8) Consider the continuity of the field David thread at $x$.

The Fourier series of $f(x)$ is:

$\sum_{n=1}^\infty \frac{\pi}{n} \left[ \cos(n\lambda) - \cos(n\lambda + \pi) \right] = \frac{\pi}{\lambda} \left[ \cos(n\lambda) - \cos(n\lambda + \pi) \right]$.

9) Consider the continuity of the field David thread at $x$.

The Fourier series of $f(x)$ is:

$\sum_{n=1}^\infty \frac{\pi}{n} \left[ \cos(n\lambda) - \cos(n\lambda + \pi) \right] = \frac{\pi}{\lambda} \left[ \cos(n\lambda) - \cos(n\lambda + \pi) \right]$.
4. By using the generating property once for $x^2$, evaluate the series \[ \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x^n}{n^2} \]
we have \[ \frac{1}{n!} \frac{x^n}{n^2} = \frac{1}{n!} \frac{x^n}{n^2} \cdot x = \frac{1}{n!} \frac{x^n}{n^2} \cdot 1 \]
therefore \[ \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x^n}{n^2} = \frac{1}{n!} \frac{x^n}{n^2} \cdot 1 \]

5. Calculate, depending on whether $x$ is complex, the value of $\lim_{x \to 0} \frac{e^x - 1}{x}$. Let $x \to 0$ as $x \to 0$. Then we have \[ \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \]

6. Show that, for $a > 0$, the following equality holds: \[ \sum_{n=1}^{\infty} \frac{\cos(n \pi a)}{n} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2(n \pi a) \]
where \[ \sum_{n=1}^{\infty} \frac{\cos(n \pi a)}{n} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2(n \pi a) \]
and similarly \[ \sum_{n=1}^{\infty} \frac{\cos(n \pi a)}{n} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2(n \pi a) \]

Remark 3

7. Compute the Fourier series of $f(x)$ over the interval $[0, 2\pi]$. By the Fourier series formula, we have \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2(n \pi a) \]

But the result converges to $x^2$. We apply the Fourier series formula again to compute the series. \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2(n \pi a) \]

8. The series converges to $x^2$. We apply the Fourier series formula again to compute the series. \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2(n \pi a) \]

9. The series converges to $x^2$. We apply the Fourier series formula again to compute the series. \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2(n \pi a) \]
For each of the following functions, determine if its Fourier series converges uniformly (where it is possible, calculate the partial sums at any point in order to check the pattern).

Let the L^p norm be:

\[ \|f\|_p = \left( \int_a^b |f|^p \, dx \right)^{1/p} \]

If \( p = \infty \), this is the \( \infty \)-norm.

If \( p = 1 \), this is the \( L^1 \)-norm.

For the functions considered, determine if the Fourier series is uniformly convergent.

The Fourier series of \( f \) is the Fourier series of \( f \), and the Fourier series of \( f \) converges for any \( x \in [0,1] \) (as expected of course; however, we lack the tools to prove \( f \) is \( C^2 \) or \( C^1 \), but this is not the question).

1. \( f(x) = \cos^2 x \)

2. \( f(x) = e^x \)

3. \( f(x) = x \)

As a conclusion, if \( f \) also converges in the \( L^1 \)-sense to \( f \), then convergence is uniform.

If \( f \) does not converge in the \( L^1 \)-sense to \( f \), then convergence is not uniform.

The exact location of points of discontinuity is required, because the added discontinuity of \( f \) is not present everywhere (the jumps of \( f \) at \( x = 0 \) are infinite).

Let the \( L^1 \)-theory be.

\[ \|f\|_1 = \left( \int_a^b |f|^1 \, dx \right)^{1/1} = \int_a^b |f| \, dx \]

The \( L^1 \)-theory applicable state:

\[ \|f\|_1 = \left( \int_a^b |f|^1 \, dx \right)^{1/1} = \int_a^b |f| \, dx \]

Practical equality has shown that:

\[ \int f(x) \, dx = \sum_{n=-\infty}^{\infty} a_n \int \cos(n \pi x) \, dx + \sum_{n=-\infty}^{\infty} b_n \int \sin(n \pi x) \, dx \]

with:

\[ \int \cos(n \pi x) \, dx = \frac{\sin(n \pi x)}{n \pi} \]

\[ \int \sin(n \pi x) \, dx = -\frac{\cos(n \pi x)}{n \pi} \]