Elementary partial differential equations: partial correction of Homework 4

Exercise 2

We consider the solution \( u(t, x) \) to the wave equation:

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.
\]

(1) Let us define the function \( v(t, x) \) as: \( v(t, x) = u(t, x - y) \), and let us compute its partial derivatives, owing to the chain rule:

\[
\frac{\partial v}{\partial t}(t, x) = \frac{\partial u}{\partial t}(t, x - y), \quad \frac{\partial^2 v}{\partial t^2}(t, x) = \frac{\partial^2 u}{\partial t^2}(t, x - y),
\]
\[
\frac{\partial v}{\partial x}(t, x) = \frac{\partial u}{\partial x}(t, x - y), \quad \frac{\partial^2 v}{\partial x^2}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x - y).
\]

Thus:

\[
\left( \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} \right)(t, x) = \left( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \right)(t, x - y) = 0,
\]

and \( v \) is also a solution to the wave equation (with different initial data, of course!).

(2) Let us define the function \( v(t, x) \) as: \( v(t, x) = \frac{\partial u}{\partial x}(t, x) \), we have:

\[
\frac{\partial^2 v}{\partial t^2}(t, x) = \frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial t^2}(t, x) \right), \quad \frac{\partial^2 v}{\partial x^2}(t, x) = \frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial x^2}(t, x) \right),
\]

Thus:

\[
\left( \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} \right)(t, x) = \frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \right)(t, x) = 0,
\]

and \( v \) is also a solution to the wave equation.

(3) Let us define the function \( v(t, x) \) as: \( v(t, x) = u(at, ax) \), and let us compute its partial derivatives, owing to the chain rule:

\[
\frac{\partial v}{\partial t}(t, x) = a \frac{\partial u}{\partial t}(at, ax), \quad \frac{\partial^2 v}{\partial t^2}(t, x) = a^2 \frac{\partial^2 u}{\partial t^2}(at, ax),
\]
\[
\frac{\partial v}{\partial x}(t, x) = a \frac{\partial u}{\partial x}(at, ax), \quad \frac{\partial^2 v}{\partial x^2}(t, x) = a^2 \frac{\partial^2 u}{\partial x^2}(at, ax).
\]

Thus:

\[
\left( \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} \right)(t, x) = a^2 \left( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \right)(at, ax) = 0,
\]

and \( v \) is also a solution to the wave equation.

Exercise 4

Let us consider the PDE:

(1) \[
3 \frac{\partial^2 u}{\partial t^2} + 10 \frac{\partial^2 u}{\partial x \partial t} + 3 \frac{\partial^2 u}{\partial x^2} = \sin(x + t),
\]

(1) This is a simple computation which resembles very much the factorization of second-order polynomial equations; for any function \( u(t, x) \), one has:

\[
3 \frac{\partial^2 u}{\partial t^2} + 10 \frac{\partial^2 u}{\partial x \partial t} + 3 \frac{\partial^2 u}{\partial x^2} = 3 \left( \frac{\partial^2 u}{\partial t^2} + \frac{10}{3} \frac{\partial^2 u}{\partial x \partial t} + \frac{3}{1} \frac{\partial^2 u}{\partial x^2} \right) = 3 \left( \frac{\partial^2 u}{\partial t^2} + \frac{10}{3} \frac{\partial^2 u}{\partial x \partial t} + \frac{25}{9} \frac{\partial^2 u}{\partial x^2} - \frac{16}{9} \frac{\partial u}{\partial x} \right),
\]
where the second line consists in making appear the end of the perfect square term corresponding to the first two terms. We can now write:

\[
3 \frac{\partial^2 u}{\partial t^2} + 10 \frac{\partial^2 u}{\partial x \partial t} + 3 \frac{\partial^2 u}{\partial x^2} = 3 \left( \left( \frac{\partial}{\partial t} + \frac{5}{3} \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + \frac{5}{3} \frac{\partial}{\partial x} \right) u - \frac{16}{9} \frac{\partial^2 u}{\partial x^2} \right),
\]

which leads immediately to the desired equation:

\[
\left( \frac{\partial}{\partial t} + \frac{5}{3} \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + \frac{5}{3} \frac{\partial}{\partial x} \right) u - \frac{16}{9} \frac{\partial^2 u}{\partial x^2} = \frac{1}{3} \sin(x + t).
\]

(2) The purpose of this question is to find a new set of variables \((\xi, \eta)\) such that, for any twice differentiable function \(u\), one has:

\[
(\xi) \frac{\partial u}{\partial t} = \frac{5}{3} \frac{\partial u}{\partial x}, \quad \text{and} \quad (\eta) \frac{\partial u}{\partial \eta} = \frac{4}{3} \frac{\partial u}{\partial x}.
\]

But, we know that, because of chain rule, for any such change of variables, one has:

\[
(\xi) \frac{\partial u}{\partial \xi} = (\xi) \frac{\partial u}{\partial t} \frac{\partial t}{\partial \xi} + (\xi) \frac{\partial u}{\partial \xi} \frac{\partial x}{\partial \xi}, \quad \text{and} \quad (\eta) \frac{\partial u}{\partial \eta} = (\eta) \frac{\partial u}{\partial t} \frac{\partial t}{\partial \eta} + (\eta) \frac{\partial u}{\partial \eta} \frac{\partial x}{\partial \eta}.
\]

Consequently, so that (2) holds, by identification with the general expression (3), it is enough that the new set of variables \((\xi, \eta)\) satisfies:

\[
\frac{\partial t}{\partial \xi} = \frac{1}{3}, \quad \frac{\partial x}{\partial \xi} = \frac{5}{3},
\]

and

\[
\frac{\partial t}{\partial \eta} = 0, \quad \frac{\partial x}{\partial \eta} = \frac{4}{3}.
\]

These formulae bring only constants into play, which suggests a linear change of variables. Let us search for \((\xi, \eta)\) under the form:

\[
t = a \xi + b \eta, \quad x = c \xi + d \eta,
\]

for some constants \(a, b, c, d\) to be found. In view of the above equations, one simply has:

\[
a = \frac{\partial t}{\partial \xi} = \frac{1}{3}, \quad b = \frac{\partial t}{\partial \eta} = 0,
\]

and

\[
c = \frac{\partial x}{\partial \xi} = \frac{5}{3}, \quad d = \frac{\partial x}{\partial \eta} = \frac{4}{3}.
\]

Eventually, the new set of variables is defined by:

\[
t = \xi, \quad \text{and} \quad x = \frac{5}{3} \xi + \frac{4}{3} \eta,
\]

whence, inverting this system to get \((\xi, \eta)\) in terms of \((t, x)\):

\[
\xi = t, \quad \text{and} \quad \eta = \frac{5}{4} t + \frac{3}{4} x.
\]

Now, with these new variables, (2) holds by construction, and (1) rewrites:

\[
(4) \quad \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} = \frac{1}{3} \sin(t + x) = \frac{1}{3} \sin \left( \frac{8}{3} \xi + \frac{4}{3} \eta \right). \]

(3) Denote \(v(\xi, \eta) = A \sin \left( \frac{8}{3} \xi + \frac{4}{3} \eta \right)\), for some constant \(A\) to be found. Then,

\[
\frac{\partial^2 v}{\partial \xi^2}(\xi, \eta) = -\frac{64}{9} A \sin \left( \frac{8}{3} \xi + \frac{4}{3} \eta \right), \quad \text{and} \quad \frac{\partial^2 v}{\partial \eta^2}(\xi, \eta) = -\frac{16}{9} A \sin \left( \frac{8}{3} \xi + \frac{4}{3} \eta \right).
\]

Consequently:

\[
\frac{\partial^2 v}{\partial \xi^2}(\xi, \eta) - \frac{\partial^2 v}{\partial \eta^2}(\xi, \eta) = -\frac{48}{9} A \sin \left( \frac{8}{3} \xi + \frac{4}{3} \eta \right).
\]

Therefore, a particular solution to (4) is \(v(t, x) = -\frac{3}{48} \sin \left( \frac{8}{3} \xi + \frac{4}{3} \eta \right)\).
The PDE (1) is linear. Thus, its general solution is given by the sum of the general solution to the associated homogeneous equation

\[ \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} = 0 \]

and of a particular solution to (1). By the results of the lectures over the wave equation, we know that the general solution to (5) is of the form:

\[ f(\xi - \eta) + g(\xi + \eta), \]

where \( f \) and \( g \) are two arbitrary twice differentiable functions. Hence, the general solution of (1) is of the form:

\[ u(\xi, \eta) = f(\xi - \eta) + g(\xi + \eta) - \frac{3}{48} \sin \left( \frac{8}{3} \xi + \frac{4}{3} \eta \right), \]

or, in term of \( t, x \):

\[ u(t, x) = f \left( \frac{1}{4} t + \frac{3}{4} x \right) + g \left( \frac{9}{4} t + \frac{3}{4} x \right) - \frac{3}{48} \sin (t + x), \]

where \( f \) and \( g \) are two arbitrary twice differentiable functions.

**Exercise 5**

Our purpose is to solve the PDE:

\[ \frac{\partial^2 u}{\partial t^2} - 2 \frac{\partial^2 u}{\partial x \partial t} - 3 \frac{\partial^2 u}{\partial x^2} = 0. \]

As suggested, let us remark that the differential operator at play here admits the following factorization, for any twice differentiable function \( u \):

\[ \frac{\partial^2 u}{\partial t^2} - 2 \frac{\partial^2 u}{\partial x \partial t} - 3 \frac{\partial^2 u}{\partial x^2} = \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - 3 \frac{\partial}{\partial x} \right) u. \]

We now proceed exactly as during the lecture over the wave equation, and remark that, introducing

\[ v = \frac{\partial u}{\partial t} - 3 \frac{\partial u}{\partial x}, \]

one has:

\[ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0. \]

Thus, solving the second-order equation (6) boils down to solving two first-order linear equations:

\[ \begin{cases} \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0 \\ \frac{\partial v}{\partial x} - 3 \frac{\partial u}{\partial x} = v \end{cases} \]

But we know that the general solution to (8) is of the form:

\[ v(t, x) = f(x - t), \]

where \( f \) is an arbitrary function (we have been solving this transport equation at least a dozen times during the lectures, but if you are not confident with it, use the method of characteristics !). Now, (7) becomes:

\[ \frac{\partial u}{\partial t} - 3 \frac{\partial u}{\partial x} = f(x - t). \]

As we have seen before, this is an inhomogeneous first-order linear PDE. Its solution is thus the sum of a particular solution, and of the general solution of the the associated homogeneous equation \( \frac{\partial u}{\partial t} - 3 \frac{\partial u}{\partial x} = 0. \)

The general solution to this homogeneous equation is \( u_{hom}(t, x) = g(x - 3t) \), where \( g \) is an arbitrary function (same as above, use the method of characteristics !). We search now for a particular solution to (9) under the form \( u_{part}(t, x) = h(x - t) \), where \( h \) is a function to be found. But, under this form:

\[ \frac{\partial u_{part}}{\partial t} - 3 \frac{\partial u_{part}}{\partial x} = -h'(x - t) - 3h'(x - t) = -4h'(x - t). \]
We may therefore take: \( u_{part}(t, x) = -\frac{1}{4}F(x - t) \), where \( F \) is a primitive of \( f \) over \( \mathbb{R} \). But, since \( f \) is arbitrary, so is \( F \).

All things considered, the solution \( u(t, x) \) to (6) is of the form:

\[
(10) \quad u(t, x) = f(x - t) + g(x + 3t),
\]

where \( f, g \) are two arbitrary twice differentiable functions.

If we eventually assume that \( u(0, x) = x^2 \) and \( \frac{\partial u}{\partial t}(0, x) = e^x \), we have, using (10) in combination with chain rule:

\[
\forall x \in \mathbb{R}, \ f(x) + g(x) = x^2, \quad \text{and} \quad -f(x) + 3g(x) = e^x.
\]

Hence,

\[
f(x) = \frac{1}{4} (3x^2 - e^x), \quad \text{and} \quad g(x) = \frac{1}{4} (x^2 + e^x),
\]

and:

\[
u(t, x) = \frac{1}{4} \left( 3(x - t)^2 - e^{x-t} \right) + \frac{1}{4} \left( (x + 3t)^2 + e^{x+3t} \right).
\]