Warning. The fact that several questions may be posed inside a single item does not mean that you may choose which one of them you answer to. All of them must be answered.

Exercise 1

This exercise is partly reprinted from [Strauss], §1.6, Exercises 1 – 2.

1. What are the types of the following two second-order PDE?
   
   (a) \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - 3 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial y^2} = 0. \]
   
   (b) \[ 9 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + 3 \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0. \]

2. Find the regions of the \((x, y)\) plane where the PDE:
   
   \[ (1 + x) \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0 \]

   of a function \(u = u(x, y)\) is elliptic, parabolic, and hyperbolic. Sketch those regions.

Exercise 2

This exercise is reprinted from [Strauss], §1.6, Exercise 6.

Consider the partial differential equation:

\[ 3 \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0, \]

where the unknown \(u = u(x, y)\) is a function of two real variables.

1. What is the order of this PDE? What is its type?

2. Find the general solution of this PDE. (Hint: make the change of unknown function \(v := \frac{\partial u}{\partial y}\), and solve first the corresponding PDE of the unknown function \(v\).)

3. If this equation is supplemented with the boundary conditions:

   \[ \forall x \in \mathbb{R}, \ u(x, 0) = e^{-3x}, \text{ and } \frac{\partial u}{\partial y}(x, 0) = 0; \]

   does it admit a solution? Is this solution unique?

Exercise 3

This exercise is partly reprinted from [Strauss], §1.5, Exercise 5.

Consider the following PDE:

\[ \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = 0, \]

of a function \(u = u(x, y)\) of two variables, together with the boundary condition \(u(x, 0) = \varphi(x)\), where \(\varphi(x)\) is a given function.

1. What is the general solution to this equation (i.e. without considering boundary conditions)?

2. Show that, if \(\varphi(x) \equiv x\), the considered system has no solution.

3. Show that, if \(\varphi(x) \equiv 1\), the considered system has an infinity of solutions.
Exercise 4

This exercise is partly reprinted from [Strauss], §1.5, Exercise 4.

Consider the Laplace problem with homogeneous Neumann boundary conditions, posed in a three-dimensional domain $\Omega$:

$$
\begin{cases}
\Delta u = f & \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega,
\end{cases}
$$

where $f \equiv f(x, y, z)$ is a given source term, and $n$ is the unit normal vector to $\partial\Omega$, pointing outward $\Omega$.

1. What can be added to any solution $u$ of the problem (1), so as to end up with another solution of this problem? Conclude that there is no unicity of solutions to (1).

2. Assume that there exists a solution $u$ to (1). By using the Green formula, show that necessarily, one then has:

$$
\int_{\Omega} f(x, y, z) \, dx \, dy \, dz = 0.
$$

Hence, does a solution to (1) always exist?

3. Provide a physical interpretation to your answers to questions (1) and (2).

Exercise 5

This exercise is partly reprinted from [Haberman], §1.4, Exercise 3.

Consider a three dimensional rod oriented along the $x$-axis. The rod is very thin, and lies in the region $(0 < x < 2)$, so that we assume its temperature $u$ only depends on the time $t$ and on $x$. It is composed of two parts (see Figure 1):

- The first one, lying between $0 < x < 1$, is filled with a homogeneous material with properties:
  $$c_1 \rho_1 = 1, \quad \kappa_1 = 1,$$
  and is submitted to a thermal source of intensity $Q = 1$. We denote by $u(t, x), t > 0, 0 < x < 1$ the temperature in this part.

- The second one, lying between $1 < x < 2$ is filled with a homogeneous material with properties:
  $$c_2 \rho_2 = 2, \quad \kappa_2 = 2,$$
  and is submitted to no thermal source. We denote by $v(t, x), t > 0, 1 < x < 2$ the temperature in this part.

![Figure 1. Setting for Exercise 5.](image)

The system is endowed with homogeneous Dirichlet boundary conditions:

$$\forall t > 0, \; u(t, 0) = 0, \; v(t, 2) = 0,$$
and the two materials are assumed to be in perfect thermal contact, i.e.:

\[(2) \quad \forall t > 0, \ u(t, 1) = v(t, 1), \ \kappa_1 \frac{\partial u}{\partial x}(t, 1) = \kappa_2 \frac{\partial v}{\partial x}(t, 1)\]

(1) Provide a physical interpretation of the conditions (2). In the mechanical literature, such conditions are called \textit{transmission boundary conditions}.

(2) Write the two PDEs satisfied by \(u\) and \(v\).

(3) We now assume that a steady state \((u(x), v(x))\) exists for this system. Find the expression of \(u(x)\), \(0 < x < 1\) and \(v(x)\), \(1 < x < 2\).