Correction of Exercise 3

Solve the partial differential equation

\[ -(1 + x^2) \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \]

by using the method of characteristics, where the unknown function \( u \) is a function of two variables \( u \equiv u(x, y) \).

1st step: derivation of the characteristic curves: Let \( s \mapsto (x(s), y(s)) \) be any curve in the space of variables. We are interested in the value \( z(s) := u(x(s), y(s)) \) taken by the solution \( u \) to (1) along this curve. In particular, we would like to find particular curves along which this variation has a simple expression.

To this end, note that the derivative of the value \( z(s) \) along any such curve has the following expression:

\[ z'(s) = x'(s) \frac{\partial u}{\partial x}(x(s), y(s)) + y'(s) \frac{\partial u}{\partial y}(x(s), y(s)) \]

which follows from a use of chain rule.

Now, inspired by the equation (1), a natural choice seems to consider the curves \( s \mapsto (x(s), y(s)) \) which satisfy the following relations:

\[ \begin{cases} x'(s) = -(1 + x^2(s)) \\ y'(s) = 1 \end{cases} \]

which are the characteristic equations for (1). They consist of a system of two ODE, which we now have to solve. The solution reads:

\[ \begin{cases} \arctan(x(s)) = -s + c_1 \\ y(s) = s + c_2 \end{cases} \]

where \( c_1, c_2 \) are arbitrary constants. Now, up to changing the parameter \( s \) into \( t := s + c_2 \), one can rewrite the characteristic equations as:

\[ \begin{cases} \arctan(x(t)) = -t + c_2 + c_1 \\ y(t) = t \end{cases} \]

and, now denoting again as \( s \) the parameter of curves (so as to keep notations simple), and \( c := c_1 + c_2 \) the arbitrary constant, we have found the expression for the characteristic curves:

\[ \begin{cases} \arctan(x(s)) = -s + c \\ y(s) = s \end{cases} \]

Note that this is a system of non intersecting curves in the \( (x, y) \) plane, parametrized by a single constant \( c \). Now, a point in the \( (x, y) \) plane can be equivalently described by means of \( (x, y) \) coordinates,

- or the parameter \( c \) of the characteristic curves in which it lies, and the parameter \( s \) of the point on this particular curve.

Of course, both representations are related by means of the characteristic equations (4). Note also that each characteristic curve (indexed by \( c \)) can be represented as a usual \( x \mapsto y \) function: \( y = -\arctan(x) + c \).

2nd step: computation of the value \( z(s) \) of the solution \( u \) along the characteristic curves: This step is generally easier than the first one and requires the solution of another ODE. Consider any fixed characteristic curve \( s \mapsto (x(s), y(s)) \) given by (4) (indexed by the parameter \( c \)). Because of the relation (3) for the characteristic curves, and of the general relation (2), we have:

\[ z'(s) = 0, \]
an admittedly very simple ODE! Thus, \( z(s) = u(x(s), y(s)) \) is constant along the considered curve, and its value of course depends on the particular curve we are solving on. Namely, there exists an arbitrary function \( f(c) \) such that:

\[
  u(x(s), y(s)) = f(c).
\]

The above relation should be understood as follows: for any point \((x, y)\) in the space of variables, the value \( u(x(s), y(s)) \) equals \( f(c) \), where \( c \) is the parameter of the curve \((x, y)\) is lying on.

3rd step: expression of \( u(x, y) \) in terms of \( x, y \): We have \( u(x, y) = f(c) \), where \( c \) is the parameter of the curve \((x, y)\) lies on, and \( f \) is an arbitrary function. But we know from (4) that a point \((x, y)\) lies on the characteristic curve with parameter \( c = y + \arctan(x) \). Thus,

\[
  u(x, y) = f(y + \arctan(x)),
\]

where \( f \) is an arbitrary function, which can only be determined by the specification of boundary conditions, in addition to the considered PDE (1).

Is this over ? Certainly not ! A very easy thing is left to do, and is crucial in practice: you should check that the solution found above is indeed solution to (1). To this end, take the expression (5) you just proved, compute the partial derivatives of \( u \) using chain-rule, and make sure it does satisfy (1).

**Correction of Exercise 5**

Solve the partial differential equation

\[
  \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = e^{x+2y}
\]

of an unknown function of two variables \( u \equiv u(x, y) \), where \( c \in \mathbb{R} \) is a fixed parameter, and with the additional ‘boundary condition’ \( u(x, 0) = f(x) \), where \( f \) is a given function.

Let us proceed along the lines of the previous corrected exercise (a little bit faster however !).

1st step: derivation of the characteristic curves: Given the form of the considered PDE, the characteristic curves \( s \mapsto (x(s), y(s)) \) of (6) satisfy the following equations:

\[
  \begin{cases}
    x'(s) = 1 \\
    y'(s) = 1/2
  \end{cases}
\]
whence:
\[
\begin{cases}
  x(s) = s + c_1 \\
y(s) = s + c_2
\end{cases}
\]
where \(c_1, c_2\) are two arbitrary constants. Up to changing \(s\) into \(s + c_1\), one can rewrite the characteristic equations as:
\[
\begin{cases}
x(s) = s \\
y(s) = s + c
\end{cases}
\]
where \(c\) is an arbitrary constant. Note that the characteristic curves are actually parallel straight lines, since combining the two above equations yields: \(y = x + c\). This ends this first step.

**2nd step: computation of the value** \(z(s) := u(x(s), y(s))\) **of the solution** \(u\) **along the characteristic curves:**

To this end, considered a given characteristic curve, indexed by the parameter \(d\). We know, from the very definition of the characteristic curves (7) that:
\[
z'(s) = x'(s) \frac{\partial u}{\partial x}(x(s), y(s)) + y'(s) \frac{\partial u}{\partial y}(x(s), y(s))
\]
thus, we get:
\[
z'(s) + z(s) = e^{x(s) + 2y(s)} = e^{3s + 2c}.
\]
We now have to solve this ODE.

- The solution to the corresponding homogeneous equation \(z' + z = 0\) is \(z_{\text{hom}}(s) = Ae^{-s}\), where \(A\) is an arbitrary constant.
- To find the general solution to the ODE, we rely on the method of variation of constants (or another one if this one is not your favorite!): we search the solutions to (9) under the form: \(z_{\text{gen}}(s) = A(s)e^{-s}\), where \(A(s)\) is a function yet to be determined. For such a function, we have:
  \[
z_{\text{gen}}'(s) + z_{\text{gen}}(s) = A'(s)e^{-s} - A(s)e^{-s} + A(s)e^{-s} = A'(s)e^{-s}.
\]
  Hence, if \(z_{\text{gen}}\) is to solve (9), one should have: \(A'(s)e^{-s} = e^{3s + 2c}\), that is, \(A'(s) = e^{4s + 2c}\), whence:
  \[
  A(s) = \frac{1}{4}e^{4s + 2c} + B,
\]
  where \(B\) is an arbitrary constant. Eventually, the general solution to (9) reads:
\[
z_{\text{gen}}(s) = A(s)e^{-s} = \frac{1}{4}e^{3s + 2c} + Be^{-s}.
\]
Let us now go back to our PDE. The value function \(z(s) = u(x(s), y(s))\) is thus of the form (10), but the constant \(B\) appearing there depends on the particular characteristic curve we are considering, and is consequently a function of the parameter \(c\).

All in all, there exists an arbitrary function, say \(g\), such that the solution \(u(x, y)\) to (6) reads:
\[
u(x, y) = \frac{1}{4}e^{3s + 2c} + g(c)e^{-s}.
\]

**3rd step: expression of** \(u(x, y)\) **in terms of** \(x, y\): This is easy since \(s = x\) and \(c = y - x\) (see (8)). Consequently:
\[
u(x, y) = \frac{1}{4}e^{x + 2y} + g(y - x)e^{-y},
\]
where \(g\) is an arbitrary function. At this point, you should check your answer by differentiating the obtained expression!

Eventually, if \(u(x, 0) = f(x)\), where \(f\) is a given (known) function, then, we must have, as for \(g\):
\[
u(x, 0) = \frac{1}{4}e^{x} + g(-x) = f(x).
\]
From this relation, one easily gets:

\[ g(x) = f(-x) - \frac{1}{4}e^{-x}, \]

and:

\[ u(x, y) = \frac{1}{4}e^{x+2y} - \frac{1}{4}e^{x-2y} + f(x - y)e^{-y}. \]