Exercise 1

Let $a < b$ be two real numbers, and $f : (a, b) \to \mathbb{R}$ be an increasing function, that is:
\[ \forall x, y \in \mathbb{R}, \ x \leq y \Rightarrow f(x) \leq f(y). \]
We also assume that $f$ is bounded over $(a, b)$.

1. Give an example of such a bounded and increasing function defined over some interval $(a, b) \subset \mathbb{R}$ of your choice.
2. Recall the proof (seen during the lectures) of the following fact:
   ‘If $\{a_n\}$ is an increasing sequence of real numbers which is bounded from above, then it is convergent.’
3. Why do the following supremum and infimum exist:
   \[ \inf (\{f(x), x \in (a, b)\}), \sup (\{f(x), x \in (a, b)\})? \]
   in the following, they are denoted as $m$ and $M$ respectively.
4. Drawing inspiration from the answer to (2), show that $f$ has limit $M$ at $b$ and limit $m$ at $a$.

Exercise 2

1. Show that, for any real numbers $x, y$, one has:
   \[ \min(x, y) = \frac{x + y - |x - y|}{2}, \text{ and } \max(x, y) = \frac{x + y + |x - y|}{2}. \]
2. Let $D \subset \mathbb{R}$, and $f : D \to \mathbb{R}$. Define the maximum function $\max(f, g)$ as:
   \[ \forall x \in D, (\max(f, g))(x) = \max(f(x), g(x)), \]
   and similarly for the minimum function $\min(f, g)$. Infer from (1) that, if $f$ and $g$ are two continuous functions on $D$, then so are $\min(f, g)$, $\max(f, g)$. 