Exercise 1:
In this exercise, we are interested in the set of algebraic numbers. An algebraic number is a real number \( x \in \mathbb{R} \) which arises as a root of a polynomial with integer coefficients i.e. such that there exist \( k \in \mathbb{N}^* \) and \( a_0, \ldots, a_k \in \mathbb{Z} \) with:
\[
a_k x^k + a_{k-1} x^{k-1} + \ldots + a_1 x + a_0 = 0.
\]
For instance, \( \sqrt{2} \) is an algebraic number, since it is a root of the polynomial \( X^2 - 2 \). On the other hand, one can show (but it is difficult) that \( \pi, e, \) and \( \log(2) \) are not algebraic (they are called transcendental numbers).

1. Show the following variant of a proposition of the lectures: let \( S \) be a countable set, and let \( \{A_s\}_{s \in S} \) be a family of countable sets indexed by \( S \). Then the union set \( \bigcup_{s \in S} A_s \) is countable.
2. Show that the set \( P_k := \{ a_k X^k + a_{k-1} X^{k-1} + \ldots + a_1 X + a_0, a_0, \ldots, a_k \in \mathbb{Z} \} \) of polynomials of order \( k \) with integer coefficients is countable.
3. Show that the set of algebraic numbers is countable.
   [Hint: remark that, for a given polynomial with integer coefficients \( P \), the set \( \{ x \in \mathbb{R}, P(x) = 0 \} \) is countable, then use Questions (1) and (2).]

Exercise 2:
Let \( A \) and \( B \) be two bounded subsets of \( \mathbb{R} \).
1. Let \( -A \) be the subset of \( \mathbb{R} \) defined by:
   \[
   -A = \{-a, a \in A\}.
   \]
   Show that \( \sup (-A) = -\inf A \).
2. Let \( A + B \) be the subset of \( \mathbb{R} \) defined by:
   \[
   A + B = \{a + b, a \in A, b \in B\}.
   \]
   Show that \( \sup (A + B) = \sup A + \sup B \).
Let now \( X \) be a non-empty subset of \( \mathbb{R} \), and let \( f, g : X \to \mathbb{R} \) be two functions with bounded range.
3. Show that:
   \[
   \sup \{ f(x) + g(x), x \in X \} \leq \sup \{ f(x), x \in X \} + \sup \{ g(x), x \in X \}.
   \]
4. Construct an example where strict inequality holds.