Exercise 1:
Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Prove that there exist two real numbers $m, p \in \mathbb{R}$ such that:
\[ \forall x \in \mathbb{R}, \ |f(x)| \leq m|x|+p. \]

Exercise 2:
Let $f : (0, 1) \to \mathbb{R}$ be an increasing function, which is bounded from above. Show that the limit $\lim_{x \to 1} f(x)$ exists.

Exercise 3:
Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Show that there exists $c \in \left[0, \frac{1}{2}\right]$ such that:
\[ f(c) = f \left( c + \frac{1}{2} \right). \]

Exercise 4:
(1) Give an example of a continuous and bounded function $f : (0, 1) \to \mathbb{R}$ which has neither a maximum, nor a minimum on $(0, 1)$.
(2) Give an example of a bounded function $f : [0, 1] \to \mathbb{R}$ which has neither a maximum, nor a minimum on $[0, 1]$.

Exercise 5:
Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function, whose derivative is continuous, such that $f(0) = 0$, and, for all $x \in [0, 1]$, $f'(x) > 0$. Show that there exists a real number $m > 0$ such that:
\[ \forall x \in [0, 1], \ f(x) \geq mx. \]

Exercise 6: (Around the constant of Euler-Mascheroni)
(1) By using the mean-value theorem, show that, for any natural number $n \in \mathbb{N}^*$, one has:
\[ \frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}. \]
Let $\{x_n\}_{n \in \mathbb{N}^*}$ be the sequence defined by:
\[ x_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - \log(n). \]
(2) Show that the sequence $\{x_n\}_{n \in \mathbb{N}^*}$ is strictly decreasing.
(3) Show that, for any $n \in \mathbb{N}^*$, one has:
\[ 0 \leq x_n \leq 1. \]
(4) Conclude that $\{x_n\}_{n \in \mathbb{N}^*}$ has a limit $\gamma \in (0, 1)$. This limit is called the Euler-Mascheroni constant.

Exercise 7:
Let $a < b$ be two real numbers, and let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $f(a) = f(b)$ and $f'(a) = f'(b) = 0$. By applying Rolle’s theorem to the auxiliary function $h(x) = e^{-x}(f(x) + f'(x))$, show that there exists a number $c \in (a, b)$ such that:
\[ f''(c) = f(c). \]

Exercise 8:
Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions. We assume that:
\[ \forall x \in \mathbb{Q}, \ f(x) < g(x). \]
(1) Show that, for all \( x \in \mathbb{R} \), \( f(x) \leq g(x) \).

[Hint: Start by observing that, for any real number \( x \), there exists a sequence \( \{r_n\} \) of elements of \( \mathbb{Q} \) such that \( r_n \to x \).]

(2) Does it necessarily hold that, for all \( x \in \mathbb{R} \), one has: \( f(x) < g(x) \)? If your answer is yes, prove it; else, provide a counterexample.

**Exercise 9:**

Let \( f : \mathbb{R} \to \mathbb{R} \) be the function defined by:
\[
\forall x \in \mathbb{R}, \quad f(x) = \frac{1}{3}(4-x^2).
\]

Let also \( \{u_n\}_{n \in \mathbb{N}} \) be the sequence defined recursively by:
- \( u_0 = \frac{1}{2} \);
- \( \forall n \in \mathbb{N}, \quad u_{n+1} = f(u_n) \).

(1) Calculate \( u_1, u_2 \).
(2) Show by induction that, for any \( n \in \mathbb{N} \), \( u_n \in \left[0, \frac{4}{3}\right] \).
(3) Calculate the derivative of \( f \).
(4) Show that, for any \( x \in \left[0, \frac{4}{3}\right] \), one has:
\[
|f(x) - 1| \leq \frac{8}{9}|x - 1|.
\]

[Hint: apply the mean-value theorem to \( f \).]
(5) Infer from your answer to the previous question that, for any \( n \in \mathbb{N} \):
\[
|u_{n+1} - 1| \leq \frac{8}{9}|u_n - 1|.
\]
(6) Show that, for any \( n \in \mathbb{N} \), \( |u_n - 1| \leq \left(\frac{8}{9}\right)^n |u_0 - 1| \).
(7) Conclude that \( \{u_n\}_{n \in \mathbb{N}} \) converges to 1.

**Exercise 10:**

For any natural number \( n \geq 2 \), let \( f_n : [1, +\infty) \to \mathbb{R} \) be the function defined by:
\[
f_n(x) = x^n - x - 1.
\]

(1) Show that, for a given \( n \geq 2 \), the function \( f_n \) is strictly increasing on \([1, +\infty)\).
(2) Show that, for a given \( n \geq 2 \), there exists a unique real \( x_n \in [1, +\infty) \) such that \( f_n(x_n) = 0 \).
(3) Show that, for any \( n \geq 2 \), one has: \( f_{n+1}(x_n) > 0 \).
(4) Infer that the sequence \( \{x_n\} \) is decreasing.
(5) Show that the sequence \( \{x_n\} \) has a limit \( \ell \).
(6) Show that \( \ell = 1 \).

[Hint: Argue by contradiction; if \( \ell \neq 1 \), show that there is a fixed number \( \alpha > 0 \) and a rank \( N \in \mathbb{N} \) in the sequence such that, for \( n \geq N \), \( x_n > 1 + \alpha \), and infer a contradiction from this last fact.]

**Exercise 11:**

Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function such that \( f(0) = 0 \) and \( f(1) > 0 \).

(1) Let \( C \subset [0, 1] \) be defined by:
\[
C = \{x \in [0, 1], \; f(x) = 0\}.
\]

Show that \( C \) is compact.
(2) infer that there exists a number \( x_0 \in [0, 1) \) such that:
\[
f(x_0) = 0 \quad \text{and} \quad \forall x > x_0, \; f(x) > 0.
\]

**Exercise 12:**

Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function which satisfies the following property:
\[
\forall x, y \in \mathbb{R}, \quad f(x + y) = f(x) + f(y).
\]

Denote as \( m = f(1) \).
(1) Show that, for any rational number \( x \in \mathbb{Q} \), one has \( f(x) = mx \).

[Hint: Start by proving this property for \( x \in \mathbb{N} \), then for \( x \in \mathbb{Z} \).]

(2) infer that, for any real number \( x \in \mathbb{R} \), one has \( f(x) = mx \).

**Exercise 13:**

Let \( a < b \) be two real numbers, and \( f, g : [a, b] \to \mathbb{R} \) be two continuous functions such that:

\[ f(a) \leq g(a) \quad \text{and} \quad f(b) \geq g(b). \]

Show that the equation \( f(x) = g(x) \) has a solution in \([a, b]\).

**Exercise 14**

A function \( f : \mathbb{R} \to \mathbb{R} \) is said to be even if, for any \( x \in \mathbb{R} \), \( f(x) = f(-x) \), and is said to be odd if \( f(x) = -f(-x) \).

(1) Give examples of non constant even and odd functions, and draw their graphs.

(2) Show that the derivative \( f' \) of a differentiable, odd function, is even.

(3) Does the converse necessarily hold (i.e. if \( f' \) is even, is \( f \) necessarily odd)? If your answer is yes, prove it; else, provide a counterexample.

(4) Show that the derivative \( f' \) of a differentiable, odd function, is even.