Exercise 7:
The purpose of this exercise is to provide an alternative proof of the Heine theorem. Let $K \subset \mathbb{R}$ be a compact set, and let $f : K \to \mathbb{R}$ be a continuous function. The proof goes by a contradiction argument.

(1) Negate the definition of uniform continuity for $f$.

(2) Show that, if $f$ is not uniformly continuous on $K$, then there exist $\varepsilon > 0$, as well as two sequences \( \{x_n\}_{n \in \mathbb{N}^*} \) and \( \{y_n\}_{n \in \mathbb{N}^*} \) of elements of $K$ which satisfy the following properties:

\[
\forall n \in \mathbb{N}^*, \ |x_n - y_n| < \frac{1}{n}, \text{ and } |f(x_n) - f(y_n)| > \varepsilon.
\]

(3) Show that there exist two subsequences \( \{x_{n_k}\} \) and \( \{y_{n_k}\} \) of \( \{x_n\}_{n \in \mathbb{N}^*} \) and \( \{y_n\}_{n \in \mathbb{N}^*} \) respectively, which converge to a common limit $\alpha \in K$.

(4) End the proof by obtaining a contradiction between this fact and the properties of Question (2).

Solution:

(1) Saying that $f$ is uniformly continuous over $K$ reads, in terms of quantifiers:

\[
\forall \varepsilon > 0, \exists \delta > 0, \ \forall x, y \in K, \ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.
\]

Hence, saying that $f$ is not uniformly continuous over $K$ can be written as:

\[
\exists \varepsilon > 0, \ \forall \delta > 0, \ \exists x, y \in K, \text{ such that } |x - y| < \delta \text{ and } |f(x) - f(y)| > \varepsilon,
\]

that is: ‘there is (at least) one value of $\varepsilon$ for which there exists a pair $(x, y)$ of elements of $K$ that are arbitrarily close from one another, but whose images $f(x), f(y)$ are distant from at least $\varepsilon$.’

(2) We just explained why there exists $\varepsilon > 0$ such that:

\[
\forall \delta > 0, \ \exists x, y \in K, \text{ such that } |x - y| < \delta \text{ and } |f(x) - f(y)| > \varepsilon.
\]

In particular, taking $\delta = \frac{1}{n}$, for $n = 1, \ldots$, we obtain that, for any $n \in \mathbb{N}^*$, there exist $x_n, y_n \in K$ such that:

\[
|x_n - y_n| < \frac{1}{n}, \text{ and } |f(x_n) - f(y_n)| > \varepsilon.
\]

(3) $\{x_n\}$ is a sequence of elements of the compact set $K$, so it has a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}^*}$ which converges to an element $\alpha \in K$. Then, the corresponding subsequence $\{y_{n_k}\}_{k \in \mathbb{N}^*}$ of $\{y_n\}$ also converges to $\alpha$, since we have:

\[
\forall k \in \mathbb{N}^*, \ |y_{n_k} - \alpha| = |y_{n_k} - x_{n_k} + x_{n_k} - \alpha| \\
\leq |y_{n_k} - x_{n_k}| + |x_{n_k} - \alpha| \\
\leq \frac{1}{n_k} + |x_{n_k} - \alpha|
\]

and the first term at the right-hand side obviously converges to 0, while the second one also does, by definition of $\alpha$ and the subsequence $\{x_{n_k}\}$. This proves the desired result.

(4) With the help of the previous questions, we are in position to obtain a contradiction with the continuity of $f$. Indeed, we know that, since $f$ is continuous over $K$, for any sequence $\{z_n\}$ of elements of $K$ which converges to $\alpha$, the sequence $\{f(z_n)\}$ converges to $f(\alpha)$. But, using the two sequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of the previous question, which both converge to $\alpha$ leads to a contradiction, for we know, by construction of these sequences that:

\[
\forall k \in \mathbb{N}^*, |f(x_{n_k}) - f(y_{n_k})| > \varepsilon.
\]

Passing to the limit in this inequality, we obtain:

\[
0 = |f(\alpha) - f(\alpha)| \geq \varepsilon,
\]

which is impossible, since $\varepsilon > 0$. Contradiction, and $f$ is uniformly continuous.