Exercise 0:

(1) Let $D$ be a subset of $\mathbb{R}$, and $x_0 \in D$. Let $f : D \to \mathbb{R}$ be continuous at $x_0$, and such that $f(x_0) \neq 0$. Show that there exists a neighborhood $Q$ of $x_0$ such that:

$$\forall x \in D \cap Q, \ f(x) \neq 0.$$ 

(2) Define precisely the notion of a uniformly continuous function $f$ on a subset $D \subset \mathbb{R}$.

(3) Let $D \subset \mathbb{R}$, and $x_0 \in D$ be an accumulation point of $D$. Prove that, if $f, g : D \to \mathbb{R}$ are two functions which are differentiable at $x_0$, then so is their product $fg$, and provide its derivative.

(4) Let $D \subset \mathbb{R}$, $x_0 \in D$, and consider a function $f : D \to \mathbb{R}$. Express, in terms of quantifiers, what it means for $f$ not to be continuous at $x_0$.

(5) Recall the definition of a compact set; then, state the Heine-Borel theorem.

(6) Let $D \subset \mathbb{R}$, and $f : D \to \mathbb{R}$ be a function; prove that, if $f$ is uniformly continuous on $D$, then it is continuous at any point $x_0 \in D$.

(7) State Rolle’s theorem.

(8) State the intermediate-value theorem.

(9) State the mean-value theorem.

(10) State the theorem of sequential characterization of the limit of a function at a point.

Exercise 1:

(1) State the theorem around sequences of elements lying in a compact set $K \subset \mathbb{R}$.

(2) Show that the interval $A = [0, 2)$ is not compact by finding a sequence $\{x_n\}$ of elements of $A$ which does not satisfy the criterion stated in (1).

(3) State the definition of a compact subset of $\mathbb{R}$.

(4) Show that this interval is not compact by using only the definition of compactness.

(5) Show that this interval is not compact by using the Heine-Borel theorem.

Exercise 2:

$f, g$ continuous on $[0, 1]$, such that:

$$0 < f(x) < g(x).$$

Show that $h = f/g$ is well-defined and continuous. Show that its image is in $(0, 1)$. Show that there exist $m, M$ st

$$mg \leq f \leq Mg.$$

Exercise 3:

Let $a < b$ be two real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on $(a, b)$. Assume in addition that there exists a real number $M > 0$ such that, for all $x \in (a, b)$:

$$|f'(x)| \leq M.$$

Show that, for all $x, y \in [a, b]$, one has:

$$|f(x) - f(y)| \leq M|x - y|.$$

Exercise 4:

(1) Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function defined by:

$$\forall x \neq 0, \ f(x) = \frac{\sin(x)}{x}.$$

Show that $f$ has a limit at 0 and calculate this limit.
(2) Let \( g : \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be the function defined by:
\[
\forall x \neq 0, \quad f(x) = \frac{\cos(x) - 1}{x}.
\]
Show that \( f \) has a limit at 0 and calculate this limit.

**Exercise 5:**
Let \( f : [0, 2] \to \mathbb{R} \) be the function defined by:
\[
f(x) = |x - 1|.
\]
Show that \( f(0) = f(2) \), but that there does not exists any \( c \in [0, 2] \) such that \( f'(c) = 0 \). Why does it not come in contradiction with Rolle’s theorem?

**Exercise 6:**
By using the mean-value theorem, show that:
1. For any real numbers \( 0 < a < b \), one has the inequality:
\[
\frac{1}{b} < \frac{\log(b) - \log(a)}{b - a} < \frac{1}{a}.
\]
2. For all \( x > 0 \), one has:
\[
\frac{1}{x + 1} < \log(1 + x) - \log(x) < \frac{1}{x}.
\]

**Exercise 7:**
The purpose of this exercise is to provide an alternative proof of the Heine theorem. Let \( K \subset \mathbb{R} \) be a compact set, and let \( f : K \to \mathbb{R} \) be a continuous function. The proof goes by a contradiction argument.

1. Negate the definition of uniform continuity for \( f \).
2. Show that, if \( f \) is not uniformly continuous on \( K \), then there exist \( \varepsilon > 0 \), as well as two sequences \( \{x_n\} \) and \( \{y_n\} \) of elements of \( K \) which satisfy the following properties:
\[
\forall n \in \mathbb{N}^*, \ |x_n - y_n| < \frac{1}{n}, \text{ and } |f(x_n) - f(y_n)| > \varepsilon.
\]
3. Show that there exist two subsequences \( \{x_{n_k}\} \) and \( \{y_{n_k}\} \) of \( \{x_n\} \) and \( \{y_n\} \) respectively, which converge to a common limit \( \alpha \in K \).
4. End the proof by obtaining a contradiction between this fact and the properties of Question (2).

**Exercise 8:**
1. Is the function \( f(x) = \frac{1}{x^2} \) uniformly continuous on \((0, +\infty)\)? In any case, prove your answer.
2. Is it true that, if \( f : [0, 1] \to \mathbb{R} \) is a continuous function such that, for all \( x \in [0, 1] \), \( f(x) \neq 0 \), then \( g(x) = \frac{1}{f(x)} \) is a uniformly continuous function? If your answer is yes, prove it; else, provide a counterexample.

**Exercise 9:**
Let \( I \) be an interval of \( \mathbb{R} \), \( f, g : I \to \mathbb{R} \) be two continuous functions such that:
\[
\forall x \in I, \ |f(x)| = |g(x)| \neq 0.
\]
Show that, either
\[
\forall x \in I, \ f(x) = g(x),
\]
or
\[
\forall x \in I, \ f(x) = -g(x).
\]

**Exercise 10:**
Let \( a < b \) be two real numbers, and \( f : [a, b] \to \mathbb{R} \) be a function such that \( f \) is differentiable on \((a, b)\), and:
\[
\forall x \in [a, b], \ f(x) > 0.
\]
Show that there exists \( c \in (a, b) \) such that:

\[
\frac{f(a)}{f(b)} = e^{(b-a)\frac{f'(c)}{f(c)}}.
\]