Exercise 1
For each of the following statements, state whether it is true or false; if it is true, prove it (or invoke a theorem of the lectures), and if it is false, provide a counterexample.

1. Each sequence \( \{a_n\} \) of real numbers has a convergent subsequence.
2. If a sequence \( \{a_n\} \) of real numbers is increasing and not bounded from above, then it goes to \( +\infty \).
3. A sequence \( \{a_n\} \) of real numbers is convergent if and only if it is bounded.
4. A sequence \( \{a_n\} \) of real numbers is convergent if and only if it is Cauchy.

Exercise 2
Let \( \{a_n\} \) be a sequence of integers; show that, if \( \{a_n\} \) converges, then it is stationary after a certain rank.

Exercise 3
Show that, if \( A \) is uncountable, and \( B \) is countable, then \( A \cap B \) is countable.

Exercise 4
Let \( \{a_n\} \) be a monotone sequence of real numbers. Show that if \( \{a_n\} \) has a convergent subsequence, then it is convergent. Is it true in the case that \( \{a_n\} \) is not monotone?

Exercise 5
Let \( A \subset \mathbb{R} \) be a non empty set which is bounded from above.

1. Does \( A \) have a maximum? A supremum?
2. Show that, if \( x \in A \) is such that \( x < \sup(A) \), then \( \sup(A \setminus \{x\}) = \sup(A) \).
3. Show that, if \( x \in A \) is such that \( \sup(A \setminus \{x\}) < \sup(A) \), then \( x = \sup(A) \).

Exercise 6
(1) Recall the definition of a Cauchy sequence.
(2) Show the identity:
\[ \forall x, y \in \mathbb{R}, \quad \sin(x) - \sin(y) = 2 \sin \left( \frac{x - y}{2} \right) \cos \left( \frac{x + y}{2} \right). \]

[Hint: introduce \( p = \frac{x+y}{2}, \quad q = \frac{x-y}{2} \), and remark that \( x = p + q, \quad y = p - q \).]

(3) Deduce from the answer to (2) that:
\[ \forall x, y \in \mathbb{R}, \quad |\sin(x) - \sin(y)| \leq |x - y|. \]

(4) Let \( b \in (0,1) \) and \( c \in \mathbb{R} \). Let \( \{a_n\}_{n \in \mathbb{N}} \) be the sequence of real numbers defined by:
\[ u_0 \in \mathbb{R}, \quad \forall n \in \mathbb{N}, \quad a_{n+1} = b \sin(a_n) + c. \]

Show that, for all \( n \in \mathbb{N} \),
\[ |a_{n+1} - a_n| \leq b^n |a_1 - a_0|. \]

(5) Show that \( \{a_n\} \) is a Cauchy sequence.

[Hint: For \( n < m \), decompose \( |a_m - a_n| \leq |a_m - a_{m+1}| + \ldots + |a_{n+1} - a_n| \), and use the previous question with the inequality \( 1 + b + b^2 + \ldots + b^n \leq \frac{1}{1-b} \).

(6) Infer that \( \{a_n\} \) converges.