Exercise 1
Let $D$ be a subset of $\mathbb{R}$, and $f : D \to \mathbb{R}$ be a continuous function. For each of the following assertions, say if it is true or false. If it is true, prove it, else, supply a counterexample.

(1) If $D$ is an interval of $\mathbb{R}$, then $f(D)$ is also an interval.
(2) If $D$ is an open subset of $\mathbb{R}$, then so is $f(D)$.
(3) If $D$ is a closed subset of $\mathbb{R}$, then so is $f(D)$.
(4) If $D$ is a bounded subset of $\mathbb{R}$, then so is $f(D)$.
(5) If $D$ is a closed and bounded subset of $\mathbb{R}$, then so is $f(D)$.

Exercise 2
(Reprinted from Ex. 36 p. 106 in [Gaughan]).
Let $K_1, \ldots, K_n$ be $n$ compact subsets of $\mathbb{R}$. Show that the finite union $\bigcup_{i=1}^{n} K_i$ is compact.

Exercise 3
Let $a < b$ be two real numbers, and $f : [a, b] \to \mathbb{R}$ be a continuous function. The purpose of this exercise is to show that:

$$\sup \{ f(x), a < x < b \} = \sup \{ f(x), a \leq x \leq b \}. $$

(1) Why do both quantities exist?
(2) Show the inequality:

$$\sup \{ f(x), a < x < b \} \leq \sup \{ f(x), a \leq x \leq b \}. $$

Do you need the continuity of $f$ for this to hold?

We now focus on the proof of the converse inequality:

$$\sup \{ f(x), a < x < b \} \geq \sup \{ f(x), a \leq x \leq b \}. $$

To this end, let $M = \sup \{ f(x), a \leq x \leq b \}$.

(3) Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = M$.
(4) In this question, and only in this question, we assume that $x_0 \in (a, b)$. Conclude that, in this case, $\sup \{ f(x), a < x < b \} \geq \sup \{ f(x), a \leq x \leq b \}$.
(5) In this question, and only in this question, we assume that $x_0 = a$. Show that, in this case also $\sup \{ f(x), a < x < b \} \geq \sup \{ f(x), a \leq x \leq b \}$.

[Hint: you may need to introduce the sequence $\{ x_n \}_{n \in \mathbb{N}}$, defined by $x_n = a + \frac{b-a}{n}$.]
(6) Admit now that the same result holds in the case that $x_0 = b$ (the proof being the same as in Question (5)). Conclude as for the main goal of the exercise.
(7) Does the conclusion hold if $f$ is no longer continuous? If your answer is yes, prove it. Else, supply a counterexample.

Exercise 4
Let $f, g : [0, 1] \to \mathbb{R}$ be two continuous functions such that:

$$\forall x \in [0, 1], \ f(x) < g(x).$$

Show that there exists a real number $m > 0$ such that:

$$\forall x \in [0, 1], \ f(x) + m < g(x).$$

[Hint: Consider the continuous function $h(x) = g(x) - f(x)$ on the compact $[0, 1]$.]

Exercise 5
(Reprinted from Ex. 41 p. 106 in [Gaughan]).
Find an interval of length 1 that contains a solution of the equation:

\[ xe^x = 1. \]

**Exercise 6**

Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function.

1. Show that, if the image \( \text{Im}(f) \) of \( f \) is contained in \( \{0, 1\} \), then \( f \) is constant.
2. Show that, if the image \( \text{Im}(f) \) of \( f \) is contained in \( \mathbb{Q} \), then \( f \) is constant.