Exercise 1 (Reprinted from Ex. 7 p. 55 in [Gaughan]). Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of real numbers. Show that \((a_n)\) converges to a real number \(a\) if and only if the sequence with general term \((a_n - a)\) converges to 0.

Exercise 2 (Reprinted from Ex. 8 p. 55 in [Gaughan]). Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of real numbers such that \(a_n \to a\), for some \(a \in \mathbb{R}\), and define the new sequence \((b_n)_{n \in \mathbb{N}}\) by:
\[
\forall n \in \mathbb{N}, \quad b_n = \frac{1}{2}(a_n + a_{n+1}).
\]
Show that \(b_n \to a\).

Exercise 3 (Partially reprinted from Ex. 32 p. 56 in [Gaughan]). In each of the following cases, find the limit of the sequence \((a_n)\):
1. \(a_n = \frac{n^2 + 2n}{n^2 - 8}\).
2. \(a_n = \cos(n)\).
3. \(a_n = \frac{\sin(n^2)}{\sqrt{n}}\).
4. \(a_n = \frac{n}{3n^2 + 2}\).
5. \(a_n = \left(\sqrt{4 - \frac{1}{n}} - 2\right)n\).
6. \(a_n = (-1)^n \frac{\sqrt{n}}{n^2}\).
7. \(a_n = \sqrt{n^2 + 1} - n\).

Exercise 4 (Reprinted from Ex. 25 p. 56 in [Gaughan]). Let \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) be two sequences of real numbers. Assume that \(a_n \to a\), where \(a\) is a real number different from 0, and that the product sequence \((a_nb_n)\) converges. Show that \((b_n)\) is a convergent sequence.

Exercise 5 (Reprinted from Ex. 10 p. 55 in [Gaughan]). Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of real numbers.
1. Show that, if \((a_n)\) converges towards a real number \(a\), then the sequence with general term \(|a_n|\) converges to \(|a|\).
2. Is the converse property true? If your answer is yes, prove it; else, find a counterexample.

Exercise 6 (Square roots and limits)
Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of nonnegative real numbers (i.e. \(a_n \geq 0\) for \(n \in \mathbb{N}\)) which converges to \(a \in \mathbb{R}\).
1. By using a Theorem of the lectures, justify that the limit \(a\) satisfies: \(a \geq 0\).
2. In this question, we assume that \(a = 0\). By using the \(\varepsilon\)-definition of the limit, show that \(\sqrt{a_n} \to 0\).
3. In this question, we assume that \(a > 0\). Show that:
\[
\forall n \in \mathbb{N}, \quad |\sqrt{a_n} - \sqrt{a}| \leq \frac{1}{\sqrt{a}}|a_n - a|.
\]
[Hint: you may consider using a trick already presented during the lectures.]
4. Let now \((b_n)_{n \in \mathbb{N}}\) be a sequence of real numbers such that \(b_n^2 \to \ell\), for some real number \(\ell\). Is it true that \((b_n)\) necessarily converges? If your answer is yes, prove it; else, show a counterexample.