Abstract. We show that reciprocity laws for generalized Dedekind sums can be deduced from a concrete realization of the bounded Euler class for lattices in $\text{SL}_2(\mathbb{R})$.

Asai observed that the reciprocity law for Dedekind sums follows from a certain "splitting" of the central extension
\[ 0 \to \mathbb{Z} \to \widetilde{\text{SL}(2, \mathbb{R})} \to \text{SL}(2, \mathbb{Z}) \to 1. \]
We develop and extend this point of view. First, to highlight that the splitting is provided by the explicit construction of a bounded representative of the Euler class. Then, to deduce the reciprocity of Dedekind symbols, which are natural generalizations of the Dedekind sums for lattices in $\text{SL}_2(\mathbb{R})$. In doing so, we demonstrate that the reciprocity of Dedekind sums goes well beyond any arithmetical aspect of the associated group. As an application of our results, we prove combinatorial formulas for the Dedekind symbol for Hecke triangle groups.

Let $\lfloor \cdot \rceil : \mathbb{R} \to \left( -\frac{1}{2}, \frac{1}{2} \right)$ be the sawtooth function defined by
\[ \lfloor x \rceil = \begin{cases} x - \lfloor x \rfloor & x \notin \mathbb{Z}, \\ 0 & x \in \mathbb{Z}, \end{cases} \]
where $\lfloor x \rfloor$ is the largest integer $\leq x$. Dedekind [De892] introduced the arithmetic sums
\[ s(d, c) = \sum_{k=1}^{c-1} \left( \left\lfloor \frac{k}{c} \right\rfloor \right) \left( \left\lfloor \frac{kd}{c} \right\rfloor \right), \tag{1} \]
where $c \in \mathbb{N}$, $(c, d) = 1$, in connection to the transformation of the logarithm of the Dedekind $\eta$-function, and deduced from that transformation the beautiful identity
\[ s(d, c) + s(c, d) = \frac{1}{12} \left( \frac{d}{c} + \frac{1}{dc} + \frac{c}{d} \right) - \frac{1}{4}. \tag{2} \]
Beyond being esthetically appealing, the reciprocity law (2) has an immediate practical purpose; together with the more obvious observations
\[ s(0, 1) = 0 \quad \text{and} \quad s(d', c) = s(d, c) \quad \text{if} \quad d' \equiv d \mod c, \]
(2) allows for the fast computation of values of Dedekind sums via the Euclidean algorithm. (In terms of applications, this is all the more significant when considering, say, the various interactions between Dedekind-type sums and pseudo-random number generation.)
Dedekind’s reciprocity law (2) can be seen as a special case of the more general formula deduced by Dieter [Die57] – again from the transformation of \( \log \eta \) –,

\[
s(d_1, c_1) + s(d_2, c_2) + s(d_3, c_3) = \frac{1}{12} \left( \frac{c_1}{c_3 c_2} + \frac{c_2}{c_1 c_3} + \frac{c_3}{c_2 c_1} \right) - \frac{1}{4}
\]

for \( c_i, d_i \) given by the relation

\[
\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

where each matrix is an element of \( \text{SL}_2(\mathbb{Z}) \).

Following an idea of Kubota, Asai [Asa70] argues that these reciprocity formulas are consequences of a deeper mechanism underlying the relation between Dedekind sums and the transformation of \( \log \eta \). More particularly, he shows that (3) can be derived without relying explicitly on the theory of modular forms, but rather by a careful investigation of the "splitting" of the central extension

\[
0 \to \mathbb{Z} \to \tilde{\text{SL}}_2(\mathbb{R}) \to \text{SL}_2(\mathbb{Z}) \to 1,
\]

where \( \tilde{\text{SL}}_2(\mathbb{R}) \) denotes the universal covering group of \( \text{SL}_2(\mathbb{R}) \).

It is a standard fact that isomorphism classes of central extensions are classified by cohomology in degree 2 (see e.g. [Bro82]), and that the second cohomology group \( H^2(\text{SL}_2(\mathbb{R})) \) is generated by the Euler class. In these terms, the reciprocity law (3) can be tracked down to two features of a concrete (bounded) 2-cocycle representative \( \omega \) of the Euler class:

(I) the Dedekind sums are determined by a function \( \rho : \text{SL}_2(\mathbb{Z}) \to \mathbb{R} \) satisfying

\[
\rho(\gamma \tau) - \rho(\gamma) - \rho(\tau) = \omega(\gamma, \tau);
\]

(II) the values of \( \omega \) can be easily computed.

Consequently, Asai suggests that (5) be named the generalized reciprocity law.

In this framework, we will deduce the reciprocity law (3) for Dedekind symbols attached to lattices in \( \text{SL}_2(\mathbb{R}) \). We conclude that, while (1), (2) and (3) are arithmetic formulas, and can as well be proven purely arithmetically (see [RaG72]), their underlying mechanism is not only independent of arithmetic – that was already clear from Dedekind’s original proof of (2) – but also independent of any arithmetic properties of the group at large.

**Presentation of main results.** If \( \Gamma < \text{SL}_2(\mathbb{R}) \) is a lattice with cusp(s), a construction of maps \( \rho : \Gamma \to \mathbb{R} \) satisfying (5) has appeared in different places in the automorphic forms literature, with connections to the Kronecker first limit formula and to the theory of multiplier systems [Gol73, Gol74, Hej83, ?]. Using this construction, the author introduced in [Bu015] the analogue of Dedekind sums for \( \Gamma \). We recall its definition below.

Let \( \Gamma_\alpha \) be the isotropy subgroup for a cusp \( \alpha \) for \( \Gamma \). Fix a scaling \( \sigma_\alpha \). Each non-trivial double coset \( \gamma \in \Gamma \), yields the Dedekind symbol

\[
S_\alpha ([\gamma]) = \frac{\text{vol}(\Gamma \\backslash \mathbb{H})}{4\pi} \frac{a_\alpha + d_\alpha}{c_\gamma} - \rho_\alpha \left( \begin{array}{cc} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{array} \right) - \frac{1}{4} \text{sign}(c_\gamma)
\]

where \( \left( \begin{array}{cc} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{array} \right) = \sigma_\alpha^{-1} \gamma \sigma_\alpha \) and where \( \rho_\alpha \) verifies (5). We note that the definition of Dedekind symbol does not actually depend on the particular choice of scaling \( \sigma_\alpha \). Alternatively, the

\[1\] Asai’s notion of splitting differs from the usual definition; his is given by the functional equation (5) below.
double coset structure yields an equivalent definition of the Dedekind symbol as a periodic function on a restricted subset of the orbit \( O = \Gamma.a \).

Our main result expresses the reciprocity law (3) for the Dedekind symbols.

**Theorem 1.** In the notation introduced above,

\[
S_\alpha(\gamma) + S_\alpha(\tau) - S_\alpha(\gamma \tau) = \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{4\pi} \left( \frac{c_\gamma}{c_\gamma c_\tau} + \frac{c_\tau}{c_\gamma c_\tau} + \frac{c_{\gamma \tau}}{c_\gamma c_\tau} \right) - \frac{1}{4}\text{sign}(c_\gamma c_\tau c_{\gamma \tau}),
\]

for all \( \gamma, \tau \in \Gamma \) such that \( [\gamma], [\tau], [\gamma \tau] \) are non-trivial double cosets in \( \Gamma \backslash \Gamma/\Gamma_a \).

On the other hand, the generalization of Dedekind’s reciprocity law (2) only makes sense for groups that contain the involution \( S = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \). In the second part of this article, we restrict our attention to the Hecke triangle groups \( G_q \). We recall that \( G_q \) is a discrete triangle group of type \((2, q, \infty)\) for \( q \geq 3 \). This means that \( G_q \) is generated by reflections on the sides of a triangle with interior angles \((\pi/2, \pi/q, 0)\). Algebraically, \( G_q \) is generated by the involution \( S \) together with the translation \( T_q = \left( \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix} \right) \), where \( \lambda_q = 2 \cos(\pi/q) \). Since there is only one cusp, hence one Dedekind symbol, we will write \( S \) instead of \( S_\infty \).

**Theorem 2.** For each \( \gamma = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix} \in G_q \), the following reciprocity law holds

\[
S([\gamma]) - S([\gamma S]) = \frac{1 - 2/q}{8 \cos(\pi/q)} \left( \frac{d}{c} + \frac{1}{d/c} + \frac{c}{d} \right) - \frac{1}{4}\text{sign}(cd)
\]

whenever \([\gamma], [\gamma S]\) are non-trivial.

For the Hecke triangle group \( G_q \), the Dedekind symbol \( S \) can equivalently be seen as a function on \( O' = \{ \gamma, \infty \bmod \lambda_q : \gamma \in G_q \} \). In this setting, \( S \) can be expressed in terms of \( \lambda_q \)-continued fractions. In fact, each element \( \gamma \in G_q \) can be expressed as a word in the group generators \( S_q = \left( \begin{pmatrix} 1 & 1 \\ \lambda_q & 1 \end{pmatrix} \right) \) and \( T_q = \left( \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix} \right) \), and

\[
(S_q^{a_1} T_q^{a_2} \cdots T_q^{a_{n-1}} S_q^{a_n}) \cdot \infty = \frac{1}{a_1 \lambda_q + \frac{1}{a_2 \lambda_q + \frac{1}{\ldots}}}
\]

By applying the reciprocity law recursively, we obtain an explicit formula for the Dedekind symbol \( S \) in terms of \( \lambda_q \)-continued fractions, which generalizes a theorem of Hickerson for the Dedekind sums \([\text{Hic}77, \text{Thm. 1}]\).

**Theorem 3.** For each \( \gamma \in G_q \),

\[
[\gamma] = \begin{cases} 
[S_q^{a_1} T_q^{a_2} \cdots T_q^{a_{n-1}} S_q^{a_n}] & \text{if } n \text{ is odd}, \\
[S_q^{a_1} T_q^{a_2} \cdots T_q^{a_n} S] & \text{if } n \text{ is even},
\end{cases}
\]

for a uniquely determined finite sequence \((a_n) \subset \mathbb{N}\). Then

\[
S([\gamma]) = \frac{1 - 2/q}{8 \cos(\pi/q)} \left( [0 : a_1, \ldots, a_n] + (-1)^{n+1} [0 : a_n, \ldots, a_1] - \sum_{j=1}^{n} (-1)^j a_j \lambda_q \right) - \frac{1 - (-1)^n}{8}.
\]
0. Notation and terminology

We let \( \mathbb{H} \) denote the hyperbolic upper half-plane, and recall that \( \text{SL}_2(\mathbb{R}) \) acts on \( \mathbb{H} \) by fractional linear transformations, and that this action factors through \( \text{PSL}_2(\mathbb{R}) \). Throughout the article, \( \Gamma \) will denote a cofinite Fuchsian group. That is, a discrete subgroup of \( \text{SL}_2(\mathbb{R}) \) of finite covolume \( V = \text{vol}(\Gamma \backslash \mathbb{H}) < \infty \), containing at least one parabolic element. We recall that \( \gamma \in \Gamma \) is parabolic if \( |\text{tr}(\gamma)| = 2 \) or, equivalently, if the action of \( \gamma \) on \( \mathbb{H} = \mathbb{H} \cup \partial \mathbb{H} \) fixes a single point and that this point is in \( \partial \mathbb{H} = \mathbb{R} \cup \{ \infty \} \). Such a point is referred to as a cusp and will be denoted \( \mathfrak{a} \). For each cusp \( \mathfrak{a} \), the isotropy subgroup of elements of \( \Gamma \) fixing \( \mathfrak{a} \) is denoted by \( \Gamma_\mathfrak{a} \). It is, up to sign, an infinite cyclic subgroup of \( \Gamma \). A matrix \( \sigma_\mathfrak{a} \in \text{SL}(2, \mathbb{R}) \) is called a scaling, if it verifies \( \sigma_\mathfrak{a}(\infty) = \mathfrak{a} \) and

\[
\sigma_\mathfrak{a}^{-1} \Gamma_\mathfrak{a} \sigma_\mathfrak{a} = (\sigma_\mathfrak{a}^{-1} \Gamma_\mathfrak{a} \sigma_\mathfrak{a})_\infty = \pm \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}.
\]

These two conditions do not determine uniquely \( \sigma_\mathfrak{a} \) but up to right multiplication by any element of the group \( \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\} \). There is a one-to-one correspondence between subgroups of \( \text{PSL}_2(\mathbb{R}) \) and subgroups of \( \text{SL}_2(\mathbb{R}) \) that contain \( -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \). We shall always assume that \( -I \in \Gamma \).

1. A concrete realization of the bounded Euler class

Petersson’s cocycle. We review the classical construction investigated by Petersson [Pet30, Pet38], which provides the explicit bounded Euler class representative \( \omega \). For any \( z \in \mathbb{H} \), set

\[
\omega(g, h) = \frac{1}{2\pi i} (\log j(g, hz) + \log j(h, z) - \log j(gh, z)),
\]

where \( \log \) denotes the principal branch of the logarithm, that is \( \log(cz + d) = \ln |cz + d| + i \arg(cz + d) \) for \( -\pi < \arg(cz + d) \leq \pi \), and where \( j(g, z) = cz + d, g = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \), is the usual automorphy factor, which satisfies

\[
j(gh, z) = j(g, hz) j(h, z).
\]

Lemma 1.1. The function \( \omega \) defines a bounded 2-cocycle representative of the Euler class.

Proof. Observe that (10) implies that the LHS of (9) is real-valued. Since it is also holomorphic in \( z \), it must be constant; \( \omega \) is indeed independent of the choice of \( z \). By definition, \( \omega \) can only take integer values. Since moreover \( |\omega(g, h)| \leq \frac{\pi}{4} \), we conclude that \( \omega \) only takes values in \( \{-1, 0, 1\} \). Finally, one can check by direct computation that

\[
\omega(g_1, g_2) + \omega(g_1 g_2, g_3) = \omega(g_1, g_2 g_3) + \omega(g_2, g_3).
\]

Hence \( [\omega] \in H^2(\text{SL}_2(\mathbb{R})) \) and we can conclude from the previous observations that \( [\omega] \) coincides with the Euler class.

Computing values of \( \omega \). The 2-cocycle (9) can be computed on explicit elements. Asai provides a clean formula to do so [Asa70, Thm. 2], simplifying the laborious presentation of Petersson [Pet38]. However, his formula is obtained under the unusual choice \( -\pi \leq \arg(cz + d) < \pi \). By a simple reparametrization, we recover that formula for the principal branch of the logarithm.
Theorem 1.2. Set
\[c(-d) = \begin{cases} \frac{c}{d} \text{ if } c \neq 0, \\ c = 0, \end{cases}\] and \[\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}\]

The explicit values of \(\omega(g, h)\) are given by the following table.
\[
\begin{array}{ccc}
\text{sign } (c_g(-d_g)) & \text{sign } (c_h(-d_h)) & \text{sign } (c_{gh}(-d_{gh})) \\
1 & 1 & 1 \\
-1 & -1 & 1 \\
\text{otherwise} & \text{1} & 0
\end{array}
\] (11)

Proof. One can check directly the validity of the identity
\[
\log(cz + d) = \log \left( \frac{cz + d}{i \operatorname{sign}(c(-d))} \right) + i \frac{\pi}{2} \operatorname{sign}(c(-d)).
\]

Showing by inspection that
\[
\log \left( \frac{j(g, hz)}{i \operatorname{sign}(c_g(-d_g))} \right) + \log \left( \frac{j(h, z)}{i \operatorname{sign}(c_h(-d_h))} \right) - \log \left( \frac{j(gh, z)}{i \operatorname{sign}(c_{gh}(-d_{gh}))} \right)
\]

we obtain the formula
\[
\omega(g, h) = \frac{1}{4} \left( \operatorname{sign}(c_g(-d_g)) + \operatorname{sign}(c_h(-d_h)) - \operatorname{sign}(c_{gh}(-d_{gh})) - \operatorname{sign}(c_g(-d_g)c_r(-d_h)c_{gh}(-d_{gh})) \right)
\]

from which it is easy to complete table (11). \qed

2. Reciprocity of Dedekind symbols

Definition of Dedekind symbols. Let \(\Gamma\) be a cofinite Fuchsian group with a cusp \(a\) and fix a scaling \(\sigma_a\). In [Bu015], we introduced the Dedekind symbol \(S_a\), which we recall is defined on non-trivial double cosets of \(\Gamma_a \setminus \Gamma / \Gamma_a\) by
\[S_a([\gamma]) = \frac{\operatorname{vol}(\Gamma \setminus \mathbb{H})}{4\pi} a_{\gamma} + d_{\gamma} - \rho_a \left( \begin{array}{cc} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{array} \right) - \frac{1}{4} \operatorname{sign}(c_{\gamma})\]

for \(\left( \begin{array}{cc} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{array} \right) = \sigma_a^{-1} \gamma \sigma_a\), and \(\rho_a : \sigma_a^{-1} \Gamma \sigma_a \rightarrow \mathbb{R}\) satisfying
\[\rho_a(\gamma \tau) - \rho_a(\gamma) - \rho_a(\tau) = \omega(\gamma, \tau),\] (12)

where \(\omega\) is the bounded 2-cocycle given by (9), and the definition does not depend on the particular scaling \(\sigma_a\) [Bu015, Thm. 2]. We will not review here the construction of \(\rho_a\) or the relation of Dedekind symbols to automorphic forms; we refer the reader to [Bu015].

The most important characteristic of this definition is that the Dedekind symbols factor through (non-trivial) double cosets in \(\Gamma_a \setminus \Gamma / \Gamma_a\). By conjugation with \(\sigma_a\), a double coset representative \(\gamma\) of \([\gamma]\) gets sent to
\[\left( \sigma_a^{-1} \Gamma_a \sigma_a \right) \left( \begin{array}{cc} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{array} \right) \left( \sigma_a^{-1} \Gamma_a \sigma_a \right) = \pm \left( \begin{array}{cc} a_{\gamma} + c_{\gamma} Z & * \\ c_{\gamma} & d_{\gamma} + c_{\gamma} Z \end{array} \right).
\] (13)

In consequence, we observe that \([\gamma]\) is non-trivial if and only if \(c_{\gamma} \neq 0\). Moreover, one can always choose a representative \(\gamma\) of \([\gamma]\) such that \(c_{\gamma} > 0\), and \(0 \leq a_{\gamma}, d_{\gamma} < c_{\gamma}\). These

\[\text{In the form of generalized log } \eta\text{-functions.}\]
two simple observations are primordial in establishing the Dedekind symbol as the natural generalization of the Dedekind sums. (On SL₂(ℤ), the two definitions coincide.)

Alternatively (and equivalently), the Dedekind symbol Sₐ can be defined as a periodic function on the orbit (or cusp set) \( \mathcal{O}' = \Gamma.a \setminus \{a\} \). Define an equivalence relation on \( \mathcal{O} \) that identifies \( x, y \in \mathcal{O} \) if there exists some \( \gamma \in \Gamma.a \) such that \( \gamma x = y \).

**Theorem 2.1.** For any \( \gamma \in \Gamma \) such that \([\gamma]\) is non-trivial, \( Sₐ([\gamma,a]) = Sₐ([\gamma]) \).

**Proof.** Consider the double coset on the LHS of (13). Let \( \left( \frac{a_\gamma b}{c_\gamma d}, \frac{a'_\gamma b'}{c'_\gamma d'} \right) \) be two other representative of \([\gamma]\). Then

\[
\left( \frac{a_\gamma b}{c_\gamma d} \right)^{-1} \left( \frac{a_\gamma b}{c_\gamma d} \right) \left( \frac{a_\gamma b}{c_\gamma d} \right)^{-1} = \left( \frac{1}{1} \right) \in (\sigmaₐ^{-1}\Gammaₐ\sigmaₐ)\).
\]

That is, any double coset representative of \([\gamma]\) is completely determined by either its first column or its first row. Therefore, the double cosets \( \left[ \left( \frac{a_\gamma}{c_\gamma} \right) \right] \in (\sigmaₐ^{-1}\Gammaₐ\sigmaₐ)\) are in one-to-one correspondence with the cusp points \( a_\gamma/c_\gamma \) mod 1 \( \in (\sigmaₐ^{-1}\Gammaₐ\sigmaₐ)\).

**Selected properties.**

**Lemma 2.2.** For any non-trivial double coset \([\gamma]\), \( Sₐ([-\gamma]) = Sₐ([\gamma]) \).

**Proof.** Following the discussion above, we may assume that \( c > 0 \). Using (12),

\[
\rhoₐ \left( - \left( \frac{a}{c} \right) \right) \cdot \frac{1}{4} \text{sign}(c) = \rhoₐ(-I) + \frac{1}{4} \left( I, \left( \frac{a}{c} \right) \right) + \frac{1}{4} \text{.}
\]

With (11), we can check that

\[
\rhoₐ(I) = -\omega(I, I) = 0,
\]

\[
2\rhoₐ(-I) = \rhoₐ(I) - \omega(-I, -I) = -1,
\]

and \( \omega \left( -I, \left( \frac{a}{c} \right) \right) = 0 \).

The statement follows.

**Lemma 2.3.** For any non-trivial double coset \([\gamma]\), \( Sₐ([\gamma]) = -Sₐ([\gamma^{-1}]) \).

**Proof.** It is enough to show that \( \rhoₐ \left( \frac{a_\gamma b_\gamma}{c_\gamma d_\gamma} \right)^{-1} = -\rhoₐ \left( \frac{a_\gamma b_\gamma}{c_\gamma d_\gamma} \right) \). By (12), \( \rhoₐ(g) + \rhoₐ(g^{-1}) = \omega(g, g^{-1}) \) and (11) yields \( \omega(g, g^{-1}) = 0 \).

With this last identity, we can see that the reciprocity law (7) expressed in Theorem 1 corresponds exactly to (3). In exactly the same fashion, we deduce (7) from (12).

**Generalized reciprocity law for Dedekind symbols.**

**Proof of Theorem 1.** For \( c_\gamma c_τ c_τ \neq 0 \),

\[
\frac{a_\gamma + d_\gamma}{c_\gamma} + \frac{a_τ + d_τ}{c_τ} - \frac{a_\gamma c_τ + d_\gamma c_τ}{c_\gamma c_τ} = \frac{a_\gamma + d_\gamma}{c_\gamma} + \frac{a_τ + d_τ}{c_τ} - \frac{(a_\gamma a_τ + b_\gamma c_τ) + (c_\gamma b_τ + d_\gamma d_τ)}{c_\gamma a_τ + d_τ c_τ} = \frac{c_γ^2 + c_τ^2 + (c_γ a_τ + d_τ c_τ)^2}{c_γ c_τ (c_γ a_τ + d_τ c_τ)} = \frac{c_γ^2 + c_τ^2 + c_τ^2}{c_γ c_τ (c_γ a_τ + d_τ c_τ)}
\]
and thus, by (12),
\[ S_a([\gamma]) + S_a([\tau]) - S_a([\gamma \tau]) = \frac{V}{4\pi} \left( \frac{c_\gamma}{c_\gamma c_\tau} + \frac{c_\tau}{c_\gamma c_\tau} + \frac{c_\gamma \tau}{c_\gamma c_\tau} \right) + \omega(\gamma, \tau) + \frac{1}{4} (\text{sign}(c_\gamma) - \text{sign}(c_\gamma) - \text{sign}(c_\tau)) \]

as in the proof of Theorem 1.2.

3. Explicit formulas for Hecke triangle groups

Parametrization. Fix the scaling \( \sigma_\infty = \left( \sqrt{q}, \frac{1}{\sqrt{q}} \right) \) and set
\[
\begin{align*}
    s &= \sigma_\infty^{-1} S \sigma_\infty = \left( \lambda_q, -1/\lambda_q \right), \\
    u &= \sigma_\infty^{-1} T_n \sigma_\infty = \left( 1, n \right), \\
    v &= \sigma_\infty^{-1} S_n \sigma_\infty = \left( 1, n\lambda_q^2 \right), \\
    \gamma_q &= \sigma_\infty^{-1} \gamma \sigma_\infty = \left( a, b/\lambda_q \right), \\
    \end{align*}
\]

for any \( n \in \mathbb{Z} \) and any \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G_q \). We record the following formulas for later computations. Let \( V_q = \text{vol}(G_q \backslash \mathbb{H}) \). Using the Gauss–Bonnet formula, \( V_q = \pi(1 - 2/q) \).

Lemma 3.1. For each \( n \in \mathbb{Z} \),
\[ \rho(u^n) = \frac{V_q n}{4\pi}. \]

Proof. This is Lemma 4.2 in [Bu015].

Lemma 3.2. Let \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{R}) \). Then
\[
\omega(g, s) = \begin{cases} 1 & \text{if } c > 0 \text{ and } d < 0, \\ 0 & \text{otherwise}; \end{cases} \\
\omega(g, u^n) = 0 & \text{for all } n \in \mathbb{Z} \\
\rho(s) = -1/4 \\
\rho(v^n) = -\rho(u^n) & \text{for all } n \in \mathbb{Z}. \]

Proof. Follows from Theorem 1.2.

Dedekind’s reciprocity law for Hecke triangle groups.

Proof of Theorem 2. Choose a double coset representative \( \gamma \) such that \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) has matrix entries \( c > 0 \) and \( 0 < d < c \). Note that
\[
\gamma_q s = \left( \begin{array}{cc} b & -a/\lambda_q \\ d\lambda_q & -c \end{array} \right). \]
Proceeding as in the proof of Theorem 1, we obtain
\[
S([\gamma]) - S([\gamma S]) = \frac{V_q}{4\pi \lambda_q} \left( \frac{d}{c} + \frac{1}{cd} + \frac{c}{d} \right) + \rho(s) + \omega(\gamma_q, s) - \frac{1}{4}(\text{sign}(c) - \text{sign}(d))
\]
\[
= \frac{1 - 2/q}{8\cos(\pi/q)} \left( \frac{d}{c} + \frac{1}{cd} + \frac{c}{d} \right) - \frac{1}{4}\text{sign}(cd).
\]

\[\square\]

A formula for the Dedekind symbol attached to a Hecke triangle group.

**Proof of Theorem 3.** We proceed by induction. If \( n = 1 \), then, by definition,
\[
S([S_q^{a_1}]) = \frac{V_q}{4\pi \lambda_q} \left( \frac{a + d}{c \lambda_q} - \rho(q^{a_1}) - \frac{1}{4} \right) = \frac{V_q}{4\pi \lambda_q} \frac{a + d}{c \lambda_q} + a_1 - \frac{1}{4},
\]
and, we can check with Lemma 3.2 that
\[
S([S_q^{a_1}T_q^{a_2}S]) = \frac{V_q}{4\pi \lambda_q} \left( \frac{a + d}{c \lambda_q} - \rho(q^{a_1}q^{a_2}s) - \frac{1}{4} \right) = \frac{V_q}{4\pi \lambda_q} \frac{a + d}{c \lambda_q} + a_1 - a_2
\]
for \( n = 2 \). For \( n \geq 3 \), let
\[
\gamma_n = \begin{cases} 
S_q^{a_1}T_q^{a_2} \cdots T_q^{a_{n-1}}S_q^{a_n} & \text{if } n \text{ is odd}, \\
S_q^{a_1}T_q^{a_2} \cdots T_q^{a_{n}} & \text{if } n \text{ is even},
\end{cases}
\]
and observe that
\[
[\gamma_{n-1}] = \begin{cases} 
[\gamma_nST_q^{a_n}] = [\gamma_nS] & \text{if } n \text{ is odd}, \\
[\gamma_nST_q^{-a_n}] = [\gamma_nS] & \text{if } n \text{ is even}.
\end{cases}
\]
By our induction hypothesis,
\[
S([\gamma_{n-1}]) = \frac{V_q}{4\pi \lambda_q} \left( \gamma_{n-1}\infty - \gamma_{n-1}^{-1}\infty - \sum_{j=1}^{n-1} (-1)^ja_j\lambda_q \right) - \frac{1 - (-1)^{n-1}}{8},
\]
where
\[
\gamma_{n-1}\infty - \gamma_{n-1}^{-1}\infty = (\gamma_nS)\infty - (T_q^{a_n}\gamma_{n-1})\infty = (\gamma_nS)\infty \pm a_n\lambda_q - (S\gamma_{n-1})\infty
\]
with \(+a_n\lambda_q\) if \( n \) is odd and \(-a_n\) otherwise. Applying the reciprocity law,
\[
S([\gamma_n]) = \frac{V_q}{4\pi \lambda_q} \left( -\gamma_{n-1}\infty + \frac{1}{cd} + S\gamma_{n-1}\infty \right) - \frac{1}{4}\text{sign}(cd) + S([\gamma_{n-1}])
\]
\[
= \frac{V_q}{4\pi \lambda_q} \left( \frac{d}{c} + \frac{1}{cd} + \frac{b}{d} - \sum_{j=1}^{n} (-1)^ja_j\lambda_q \right) - \frac{1 - (-1)^{n-1}}{8}
\]
\[
= \frac{V_q}{4\pi \lambda_q} \left( \frac{a + d}{c} - \sum_{j=1}^{n} (-1)^ja_j\lambda_q \right) - \frac{1 - (-1)^{n-1}}{8}
\]
and the identification with continued fraction expansions is immediate considering \( \frac{a}{c} = \gamma_n\infty \) and \( \frac{d}{c} = -\gamma_{n-1}\infty \).

\[\square\]
References


Dept. of Mathematics, Rutgers University, 110 Frelinghuysen Rd, Piscataway, NJ 08854

E-mail address: claire.burrin@math.ethz.ch