POSITIVE MAPS ON OPERATOR ALGEBRAS AND QUANTUM MARKOV SEMIGROUPS

An introductory course with application in quantum mechanics

Eric A. Carlen[1]

Department of Mathematics, Hill Center,
Rutgers University, 110 Frelinghuysen Road Piscataway NJ 08854-8019 USA

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1 Fundamentals of the Theory of Banach Algebras

1.1 Basic definitions and notation

1.1 DEFINITION (Banach algebra)**.** A Banach algebra is an algebra $\mathscr{A}$ over the complex numbers equipped with a norm $\|\cdot\|$ under which it is complete as a metric space, and such that

$$\|AB\| \le \|A\|\|B\| \quad \text{for all} \quad A, B \in \mathscr{A} \ . \tag{1.1}$$

1.2 EXAMPLE. For a locally compact Hausdorff space X, we write $\mathscr{C}_0(X)$ to denote the set of continuous complex valued functions on X that vanish at infinity. We equip it with the supremum norm and the usual algebraic structure of pointwise addition and multiplication. Then $\mathscr{A} = \mathscr{C}_0(X)$ is a commutative Banach algebra. There is a multiplicative identity if and only if X is compact.

1.3 EXAMPLE. Equip $\mathbb{R}^n$ with Lebesgue measure, and let $\mathscr{A}$ be the Banach space $L^1(\mathbb{R}^n)$ further equipped with the convolution product

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y \ .$$

Then $\mathscr{A}$ is a commutative Banach algebra that does not have an identity.

1.4 EXAMPLE. Let $\mathscr{H}$ be a Hilbert space, and let $\mathscr{A} = \mathscr{B}(\mathscr{H})$, the set of all continuous linear transformations from $\mathscr{H}$ to $\mathscr{H}$, equipped with the composition product and the operator norm

$$\begin{aligned} \|A\| &= \sup\{ \|A\psi\|_{\mathscr{H}} \ : \ \psi \in \mathscr{H}, \|\psi\|_{\mathscr{H}} = 1 \} \\ &= \sup\{ \Re(\langle \varphi, A\psi\rangle_{\mathscr{H}}) \ : \ \varphi, \psi \in \mathscr{H}, \|\varphi\|_{\mathscr{H}}, \|\psi\|_{\mathscr{H}} = 1 \} \ , \end{aligned} \tag{1.2}$$

where $\|\cdot\|_{\mathscr{H}}$ is the norm on $\mathscr{H}$, and $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ is the inner product in $\mathscr{H}$. This is a canonical example of a non-commutative Banach algebra.

1.5 EXAMPLE. Let $\mathscr{A}$ be the algebra of $n \times n$ matrices. The *Frobenius norm*, or *Hilbert-Schmidt norm*, on $\mathscr{A}$ is the norm $\|\cdot\|_2$ given by $\|A\|_2 = \left(\sum_{i,j=1}^{n} |A_{i,j}|^2\right)^{1/2}$ where $A_{i,j}$ denotes the i,jth entry of A. By the Cauchy-Schwarz inequality, for all $A, B \in \mathscr{A}$,

$$\|AB\|_2 = \left(\sum_{i,j=1}^{n} \left|\sum_{k=1}^{n} A_{i,k}B_{k,j}\right|^2\right)^{1/2} \leq \left(\sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} |A_{i,k}|^2\right)\left(\sum_{k=1}^{n} |B_{k,j}|^2\right)\right)^{1/2} = \|A\|_2\|B\|_2 \ ,$$

and thus (1.1) is satisfied. Note that the algebra of $n \times n$ matrices with the operator norm is the special case of Example 1.4 in which $\mathscr{H} = \mathbb{C}^n$, but the Frobenius norm is not the operator norm.

1.2 The spectrum and the resolvent set

A Banach algebra $\mathscr{A}$ that has a multiplicative identity 1 is said to be *unital*. Otherwise, $\mathscr{A}$ is *non-unital*.

1.6 DEFINITION (Canonical unital extension)**.** Let $\mathscr{A}$ be any Banach algebra, with or without a unit. Define $\widetilde{\mathscr{A}}$ to be $\mathbb{C} \oplus \mathscr{A}$ with the multiplication

$$(\lambda, A)(\mu, B) = (\lambda\mu, \lambda B + \mu A + AB) \ , \tag{1.3}$$

and the norm

$$\|(\lambda, A)\| = |\lambda| + \|A\| \ . \tag{1.4}$$

By the definitions,

$$\begin{aligned} \|(\lambda, A)(\mu, B)\| = \|(\lambda\mu, \lambda B + \mu A + AB)\| &= |\lambda\mu| + \|\lambda B + \mu A + AB\| \\ &\leq |\lambda||\mu| + |\lambda|\|B\| + |\mu|\|A\| + \|A\|\|B\| \\ &= (|\lambda| + \|A\|)(|\mu| + \|B\|) = \|(\lambda, A)\|\|(\mu, B)\| \ . \end{aligned}$$

This shows that (1.1) is satisfied, and hence that $\widetilde{\mathscr{A}}$ is a Banach algebra. Now define $\mathbb{1} := (1, 0) \in \widetilde{\mathscr{A}}$. Then $(1, 0)(\lambda, A) = (\lambda, A)(1, 0) = (\lambda, A)$ so that $\mathbb{1}$ is the identity in $\widetilde{\mathscr{A}}$.

The map $A \mapsto (0, A)$ is an isometric embedding of $\mathscr{A}$ into $\widetilde{\mathscr{A}}$. None of the elements $(0, A)$ are invertible in $\widetilde{\mathscr{A}}$, even when $\mathscr{A}$ itself has an identity. Indeed, if (λ, A) has an inverse (μ, B), then

$$(1, 0) = (\lambda, A)(\mu, B) = (\lambda\mu, \lambda B + \mu A + AB) \ ,$$

and this is impossible if $\lambda = 0$. However, it will be important in what follows that if $\mathscr{A}$ has a unit $\mathbb{1}$, then $\mathbb{1} - A$ is invertible in $\mathscr{A}$ if and only if $(1, -A)$ is invertible in $\widetilde{\mathscr{A}}$.

1.7 PROPOSITION. *Let $\mathscr{A}$ be a Banach algebra with unit 1. Then $1 + A$ is invertible in $\mathscr{A}$ if and only if there exists $B \in \mathscr{A}$ such that*

$$A + B + AB = A + B + BA = 0 \ . \tag{1.5}$$

Consequently, $1 + A$ is invertible in $\mathscr{A}$ if and only if $(1, A)$ is invertible in $\widetilde{\mathscr{A}}$.

Proof. Suppose that there exists $B \in \mathscr{A}$ such that (1.5) is true. Then

$$(1+B)(1+A) = 1 + B + A + AB = 1 \quad \text{and} \quad (1+A)(1+B) = 1 + B + A + BA = 1 \ .$$

Therefore, $(1+A)$ is invertible, and $(1+B) = (1+A)^{-1}$.

Suppose that $1+A$ is invertible. Define $B := (1+A)^{-1} - 1$. Then $(1+A)B = 1-(1+A) = -A$, and hence $A + B + AB = 0$. The proof of $A + B + BA = 0$ is similar. This proves the first part.

Next, $(1, A)$ is invertible in $\widetilde{\mathscr{A}}$ if and only if there exists $B \in \mathscr{A}$ such that $(1, A)(1, B) = (1, B)(1, A) = (1, 0)$, and by the definition of the product in $\widetilde{\mathscr{A}}$, this is the same as (1.5). □

1.8 DEFINITION (Spectrum and resolvent set)**.** Let $\mathscr{A}$ be a Banach algebra, and let $A \in \mathscr{A}$. If $\mathscr{A}$ has a unit, the *spectrum of A in $\mathscr{A}$*, $\sigma_{\mathscr{A}}(A)$ is defined to be the set of all $\lambda \in \mathbb{C}$ such that $\lambda 1 - A$ is not invertible. If $\mathscr{A}$ does not have a unit, then $\sigma_{\mathscr{A}}(A)$ is defined to be the spectrum of $(0, A) \in \widetilde{\mathscr{A}}$. The resolvent set of A in $\mathscr{A}$, $\rho_{\mathscr{A}}(A)$ is defined to be the complement of $\sigma_{\mathscr{A}}(A)$.

It is useful to have an intrinsic characterization of the spectrum for non-unital $\mathscr{A}$. By Proposition 1.7, for any given $A \in \mathscr{A}$, there is at most one $X \in \mathscr{A}$ such that

$$A + X + AX = A + X + XA = 0 \ , \tag{1.6}$$

and there is one solution exactly when $(1, A)$ is invertible in $\widetilde{\mathscr{A}}$, and in that case $(1, X) = (1, A)^{-1}$ in $\widetilde{\mathscr{A}}$. This justifies the following definition:

1.9 DEFINITION. Let $\mathscr{A}$ be a non-unital Banach algebra. Define a map $\circ : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ by

$$A \circ B = A + B + AB \ , \tag{1.7}$$

$A \in \mathscr{A}$ is *quasi regular* in case there exists $A' \in \mathscr{A}$, necessarily unique, such that $A \circ A' = A' \circ A = 0$. In this case, A' is called the *quasi inverse* of A. We reserve the prime to denote the quasi inverse of a quasi regular element of a non-unital Banach algebra.

The definition has been made so that $A \circ B = B \circ A = 0$ if and only if $(1, A)(1, B) = (1, B)(1, A) = \mathbb{1}$.

1.10 LEMMA. *Let $\mathscr{A}$ be a non-unital Banach algebra. Let $A, B \in \mathscr{A}$ be quasi regular. Then $A \circ B$ is quasi regular and*

$$(A \circ B)' = B' \circ A' \ , \tag{1.8}$$

and for $\lambda \in \mathbb{C} \backslash \{0\}$,

$$\lambda \in \rho_{\mathscr{A}}(A) \iff -\lambda^{-1} A \text{ is quasi regular.} \tag{1.9}$$

In particular.

$$\rho_{\mathscr{A}}(A) = \{\lambda \in \mathbb{C} \backslash \{0\} \ : \ -\lambda^{-1} A \text{ is quasi regular } \} \ . \tag{1.10}$$

Proof. By Proposition 1.7, $A, B \in \mathscr{A}$ are quasi regular exactly when $(1, A), (1, B)$ are invertible in $\widetilde{\mathscr{A}}$, and in this case

$$((1, A)(1, B))^{-1} = (1, B)^{-1}(1, A)^{-1} = (1, B')(1, A') = 1 + B' \circ A' \ .$$

Therefore, $A \circ B$ is quasi regular, and since $(1, A)(1, B) = (1, A \circ B)$, $B' \circ A'$ is the pseudo inverse of $A \circ B$. Next, for $\lambda \in \mathbb{C} \backslash \{0\}$, $(-\lambda, A)$ is invertible in $\widetilde{\mathscr{A}}$ if and only if $(1, -\lambda^{-1} A)$ is invertible in $\widetilde{\mathscr{A}}$, which is the case if and only if $-\lambda^{-1} A$ is quasi regular. □

Let $\mathscr{A}$ be a Banach algebra with identity 1. Then we can still carry out the process of adjoining an identity to form $\widetilde{\mathscr{A}}$, and can regard each $A \in \mathscr{A}$ also as an element of $\widetilde{\mathscr{A}}$. Since no element of $\mathscr{A}$ is invertible in $\widetilde{\mathscr{A}}$, $0 \in \sigma_{\widetilde{\mathscr{A}}}(A)$ for all $A \in \mathscr{A}$. However, for $\lambda \neq 0$, $\lambda 1 - A$ is invertible if and only if $1 - \lambda^{-1}A$ is invertible. Likewise, $(\lambda, -A)$ is invertible if and only if $(1, -\lambda^{-1}A)$ is invertible. Then by Proposition 1.7, $\lambda 1 - A$ is invertible in $\mathscr{A}$ if and only if $(1,0) - (0, \lambda^{-1}A)$ is invertible in $\widetilde{\mathscr{A}}$. This shows that for $\lambda \neq 0$, $\lambda \in \sigma_{\mathscr{A}}(A) \iff \lambda \in \sigma_{\widetilde{\mathscr{A}}}((0,A))$. We summarize:

$$\{0\} \cup \sigma_{\mathscr{A}}(A) = \sigma_{\widetilde{\mathscr{A}}}((0,A)) \ . \tag{1.11}$$

1.11 LEMMA (Spectral Mapping Lemma)**.** *Let $\mathscr{A}$ be a Banach algebra, and let p be a polynomial. In case $\mathscr{A}$ has no identity, we suppose that p has no constant term. Then for all $A \in \mathscr{A}$,*

$$p(\sigma_{\mathscr{A}}(A)) = \sigma_{\mathscr{A}}(p(A)) \ .$$

Proof. We may suppose that p is not identically constant. We first suppose that $\mathscr{A}$ has an identity. Fix $\lambda \in \sigma_{\mathscr{A}}(A)$. We shall show that $p(\lambda)1 - p(A)$ is not invertible. The polynomial $p(\lambda) - p(z)$ has a root at $z = \lambda$, and hence $p(\lambda) - p(z) = (\lambda - z)q(z)$ for some polynomial $q(z)$. Replacing z by A,

$$p(\lambda)1 - p(A) = (\lambda - A)q(A) \ .$$

Were $p(\lambda)1 - p(A)$ invertible, we would have $1 = (\lambda - A)[q(A)(p(\lambda) - p(A))^{-1}]$, and then since polynomials in A commute, $1 = [q(A)(p(\lambda) - p(A))^{-1}](\lambda - A)$. This would mean that $\lambda 1 - A$ is invertible, contradicting our hypothesis that $\lambda \in \sigma_{\mathscr{A}}(A)$. Hence $p(\lambda) - p(A)$ is not invertible, and hence $p(\lambda) \in \sigma_{\mathscr{A}}(p(A))$. This shows that $p(\sigma_{\mathscr{A}}(A)) \subset \sigma_{\mathscr{A}}(p(A))$.

Next, fix $\mu \in \sigma_{\mathscr{A}}(p(A))$, and factor $\mu - p(z) = \alpha(\lambda_1 - z)\cdots(\lambda_n - z)$ where $\alpha \neq 0$ and $n \geq 1$. For each j, $\mu = p(\lambda_j)$. We have

$$\mu 1 - p(A) = \alpha(\lambda_1 1 - A)\cdots(\lambda_n 1 - A)$$

and if each $\lambda_j 1 - A$ were invertible, then $\mu 1 - p(A)$ would be invertible, but this is not the case. Hence for some j, $\lambda_j \in \sigma_{\mathscr{A}}(A)$, and $\mu = p(\lambda_j) \in \sigma_{\mathscr{A}}(p(A))$. This shows that $\sigma_{\mathscr{A}}(p(A)) \subset p(\sigma_{\mathscr{A}}(A))$, and completes th proof when $\mathscr{A}$ has an identity. The general case now follows by adjoining an identity and then appealing to (1.11). $\square$

1.3 Properties of the inverse function

Now let $\mathscr{A}$ be a Banach algebra with an identity 1. Let $A \in \mathscr{A}$ be such that $\|1 - A\| = r < 1$. By (1.1), $\|(1-A)^n\| \leq r^n$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, define $S_n = \sum_{j=0}^{n}(1-A)^j$ where, as usual, we interpret $(1-A)^0 = 1$. For all $n > m$, by the triangle inequality and (1.1),

$$\|S_n - S_m\| \leq \sum_{j=m+1}^{n} \|(1-A)^j\| \leq \sum_{j=m+1}^{n} r^j = \frac{r^m - r^n}{r-1} \ .$$

Hence $\{S_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{A}$, and by the metric completeness of $\mathscr{A}$, there exists $B \in \mathscr{A}$ such that $\lim_{n\to\infty}\|B - S_n\| = 0$ and $\|B\| \leq (1-r)^{-1}$. Then

$$BA = \lim_{n\to\infty} S_n A = \lim_{n\to\infty} S_n(1 - (1-A)) = \lim_{n\to\infty}(1 - (1-A)^{n+1}) = 1 \ .$$

The same reasoning shows that $AB = 1$, and so A is invertible, and $A^{-1} = B$, so that $\|A^{-1}\| \leq (1-r)^{-1}$. . Let Ω denote the set of invertible elements in $\mathscr{A}$. This brings us to:

1.12 LEMMA. *Let $\mathscr{A}$ be a unital Banach algebra. Let Ω be the set of invertible elements of $\mathscr{A}$. Then Ω contains every $A \in \mathscr{A}$ such that $\|1 - A\| < 1$, and in this case A^{-1} is given by the convergent series*

$$A^{-1} = \sum_{j=0}^{\infty}(1-A)^j \quad \text{and} \quad \|A^{-1}\| \leq \frac{1}{1 - \|1-A\|} \ . \tag{1.12}$$

Moreover, if $|\lambda| > \|A\|$, then $\lambda 1 - A$ is invertible. In particular, $\sigma_{\mathscr{A}}(A)$ is contained in the centered closed disk in $\mathbb{C}$ of radius $\|A\|$.

Proof. If $|\lambda| > \|A\|$, then $\lambda 1 - A = \lambda(1 - \lambda^{-1}A)$ and $\|1 - (1 - \lambda^{-1}A)\| = |\lambda|^{-1}\|A\| < 1$, so that $(1 - \lambda^{-1}A)$ is invertible. The rest has been proved above. □

1.13 LEMMA. *Let $A_0 \in \Omega$ and let $A \in \mathscr{A}$ satisfy $\|A - A_0\| < \|A_0^{-1}\|^{-1}$. Then $A \in \Omega$ and*

$$\|A^{-1}\| \leq \frac{1}{\|A_0^{-1}\|^{-1} - \|A - A_0\|} \ . \tag{1.13}$$

In particular, Ω is open.

Proof. Note that

$$\|1 - AA_0^{-1}\| = \|(A_0 - A)A_0^{-1}\| \leq \|A - A_0\|\|A_0^{-1}\|, \tag{1.14}$$

Therefore, if $\|A - A_0\| < \|A_0^{-1}\|^{-1}$, then $\|1 - AA_0^{-1}\| < 1$, and by Lemma 1.12, $AA_0^{-1} \in \Omega$. Then since Ω is closed under multiplication, $A = (AA_0^{-1})A_0 \in \Omega$, and moreover, $A^{-1} = A_0^{-1}(AA_0^{-1})^{-1}$. By the bound in (1.12) and then the bound in (1.14)

$$\|(AA_0^{-1})^{-1}\| \leq \frac{1}{1 - \|1 - AA_0^{-1}\|} \leq \frac{1}{1 - \|A - A_0\|\|A_0^{-1}\|} = \frac{1}{\|A_0^{-1}\|}\frac{1}{\|A_0^{-1}\|^{-1} - \|A - A_0\|} \ .$$

Then since $\|A^{-1}\| \leq \|A_0^{-1}\|\|(AA_0^{-1})^{-1}\|$, (1.12) is proved. □

Since Ω is open, for all $A \in \mathscr{A}$, $\rho_{\mathscr{A}}(A)$ is open, and hence that $\sigma_{\mathscr{A}}(A)$ is closed. The following simple identity will prove useful in what follows.

1.14 LEMMA (Resolvent identity)**.** *Let $\mathscr{A}$ be a unital Banach algebra. For all $A, B \in \Omega$,*

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} \ . \tag{1.15}$$

Proof. $A^{-1} - B^{-1} = A^{-1}(BB^{-1}) - (A^{-1}A)B^{-1} = A^{-1}(B - A)B^{-1}$. □

1.15 COROLLARY. *For A, A_0 invertible in a unital Banach algebra $\mathscr{A}$,*

$$\|A - A_0\| \leq \tfrac{1}{2}\min\{\|A^{-1}\|^{-1}|, \|A_0^{-1}\|^{-1}\} \quad \Rightarrow \quad \|A^{-1} - A_0^{-1}\| \leq 2\|A_0^{-1}\|^2\|A - A_0\| \ .$$

In particular, the map $A \mapsto A^{-1}$ is coniinuous on Ω.

Proof. By (1.15), $\|A^{-1} - A_0^{-1}\| \leq \|A^{-1}\|\|A_0 - A\|\|A_0^{-1}\|$, and by the hypothesis $\|A - A_0\| \leq \frac{1}{2}\|A_0^{-1}\|^{-1}$, (1.13) is valid, and $\|A^{-1}\| \leq 2\|A_0^{-1}\|$. □

For a non-unital Banach algebra $\mathscr{A}$, 0 is in the spectrum of every element A of $\mathscr{A}$. For unital $\mathscr{A}$, we have not yet shown that $\sigma_{\mathscr{A}}(A)$ is non-empty for all $A \in \mathscr{A}$. We now prove this fundamental fact:

1.16 THEOREM. *Let $\mathscr{A}$ be a Banach algebra. Then for all $A \in \mathscr{A}$, $\sigma_{\mathscr{A}}(A)$ is a nonempty closed set contained in the closed disc of radius $\|A\|$ centered at 0 in $\mathbb{C}$.*

Proof. Suppose that $\sigma_{\mathscr{A}}(A) = \emptyset$. Then A is invertible, and $(1 - \zeta A)$ is invertible for all $\zeta \in \mathbb{C}$. Then $A = 0$: By the Hahn-Banach Theorem, there is a linear functional $\varphi \in \mathscr{A}^*$ such that $\|\varphi\| = 1$ and $\varphi(A) = \|A\|$.

Define a complex valued function g on $\mathbb{C}$ by $g(\zeta) = \varphi((1 - \zeta A)^{-1})$. For any ζ and $\zeta + \eta$, by (1.15), $g(\zeta + \eta) - g(\zeta) = -\eta\varphi[(1 - (\zeta + \eta)A)^{-1}A(1 - \zeta A)^{-1}]$. From this identity and the continuity of the inverse function, it follows that

$$\lim_{\eta \to 0} \frac{g(\zeta + \eta) - g(\zeta)}{\eta} = -\varphi[A(1 - \zeta A)^{-2}] \ ,$$

which shows that g is an entire analytic function, and $g'(0) = \varphi(A) = \|A\|$.

For $\zeta \neq 0$, $\|(1 - \zeta A)^{-1}\| = |\zeta|^{-1}\|(\zeta^{-1}1 - A)^{-1}\|$ For $|\zeta|^{-1} \leq \frac{1}{2}\|A^{-1}\|^{-1}$, and then by (1.13), $\|(\zeta^{-1}1 - A)^{-1}\| \leq 2\|A^{-1}\|$. Hence $\lim_{\zeta \to \infty} g(\zeta) = 0$, and g is bounded.

By Liouville's Theorem, g is constant. Hence $\|A\| = g'(0) = 0$. That is, if $A \neq 0$, there is some ζ, necessarily non-zero, such that $(1 - \zeta A)$ is not invertible and then $\zeta^{-1} \in \sigma_{\mathscr{A}}(A)$. Of course if $A = 0$, $0 \in \sigma_{\mathscr{A}}(A)$. Hence it is impossible that $\sigma_{\mathscr{A}}(A) = \emptyset$. We have already seen that $\sigma_{\mathscr{A}}(A)$ is closed and bounded, hence compact. □

The fact that in a Banach algebra $\mathscr{A}$, every element has a non-empty spectrum has a corollary that we shall make use of shortly:

1.17 COROLLARY (Gelfand-Mazur Theorem)**.** *Let $\mathscr{A}$ be a unital Banach algebra. If $\mathscr{A}$ is a division algebra, then $\mathscr{A}$ is isomorphic to $\mathbb{C}$. More specifically, each element A of $\mathscr{A}$ satisfies $A = \lambda 1$ for some necessarily unique $\lambda \in \mathbb{C}$, and $A \mapsto \lambda$ is an isomorphism with $\mathbb{C}$.*

Proof. Suppose that $\mathscr{A}$ is a division algebra. By Theorem 1.16, there exists $\lambda \in \sigma_{\mathscr{A}}(A)$. Thus $\lambda 1 - A$ is not invertible. Since the only non-invertible element in a division algebra is 0, $A = \lambda 1$. □

1.18 DEFINITION (Spectral radius)**.** The *spectral radius* of $A \in \mathscr{A}$, $\mathscr{A}$ a Banach algebra, is

$$\nu(A) = \max\{ \ |\lambda| \ : \ \lambda \in \sigma_{\mathscr{A}}(A) \ \} \ . \tag{1.16}$$

1.19 THEOREM (The Beurling-Gelfand Formula)**.** *The spectral radius of an element A of a Banach algebra $\mathscr{A}$ is given by*

$$\nu(A) = \lim_{n \to \infty} \|A^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|A^n\|^{1/n} \ . \tag{1.17}$$

In particular, the limit exists.

Proof. Suppose that $\lambda \in \sigma_{\mathscr{A}}(A)$. By the Spectral Mapping Lemma, $\lambda^n \in \sigma_{\mathscr{A}}(A^n)$, and then by Theorem 1.16, $|\lambda|^n \leq \|A^n\|$. Taking the nth root, we obtain $\nu(A) \leq \|A^n\|^{1/n}$ for all $n \in \mathbb{N}$.

Therefore, the second equality in (1.17) will follow from the first, and it remains to show that

$$\limsup_{n\to\infty} \|A^n\|^{1/n} \le \nu(A) \ . \tag{1.18}$$

To this end, pick λ with $|\lambda| > \nu(A)$ so that $\lambda \in \rho_{\mathscr{A}}(A)$. Let φ be any continuous linear functional on $\mathscr{A}$. Then, as in the proof of Theorem 1.16, the function g defined by $g(\zeta) = \varphi((1-\zeta A)^{-1})$ is analytic on the open set where $(1-\zeta A)$ is invertible; i.e., $\{0\} \cup \{\zeta \ : \ \zeta^{-1} \in \rho_{\mathscr{A}}(A)\}$. Hence g is analytic on the open disc about 0 with radius $1/\nu(A)$.

For ζ with $|\zeta| < \|A\|^{-1}$, $(1-\zeta A)^{-1}$ has the convergent power series $(1-\zeta A)^{-1} = \sum_{n=0}^{\infty} \zeta^n A^n$.

Therefore, by the uniqueness of the power series representation, $g(\zeta) = \sum_{n=0}^{\infty} \zeta^n \varphi(A^n)$ is a convergent power series for all ζ with $|\zeta| \le 1/\nu(A)$. It follows that for all such ζ, $\lim_{n\to\infty} \zeta^n \varphi(A^n) = 0$. In particular, there exists a finite constant C_φ such that

$$|\zeta^n \varphi(A^n)| \le C_\varphi \quad \text{for all} \quad n \in \mathbb{N} \ . \tag{1.19}$$

For each $n \in \mathbb{N}$ define a linear functional Λ_n on $\mathscr{A}^*$, the Banach space dual of $\mathscr{A}$, by $\Lambda_n(\varphi) = \zeta^n \varphi(A^n)$. Then (1.19) says that $\sup_{n\in\mathbb{N}}\{ \, |\Lambda_n(\varphi)|\} \le C_\varphi$. The Uniform Boundedness Principle then implies that there exists a finite constant M such that $\|\Lambda_n\| \le M$ for all n, and hence for all $\varphi \in \mathscr{A}^*$ with $\|\varphi\| = 1$,

$$|\zeta|^n |\varphi(A^n)| \le M \quad \text{for all} \quad n \in \mathbb{N}$$

. The Hahn-Banach Theorem provides $\varphi \in \mathscr{A}^*$ with $\|\varphi\| = 1$ such that $\varphi(A^n) = \|A^n\|$. Hence $|\zeta|^n \|A^n\| \le M$, and then, $|\zeta| \|A^n\|^{1/n} \le M^{1/n}$. This proves that $|\zeta| \limsup_{n\to\infty} \|A^n\|^{1/n} \le 1$. However, ζ was any complex number with $|\zeta| < 1/\nu(A)$, this proves (1.18), and hence the first equality in (1.17). □

The following result is trivial for commutative Banach algebras, and familiar for the algebra of $n \times n$ matrices:

1.20 THEOREM (Spectrum of AB and BA)**.** *If $\mathscr{A}$ is a Banach algebra, then for all $A, B \in \mathscr{A}$,*

$$\{0\} \cup \sigma_{\mathscr{A}}(AB) = \{0\} \cup \sigma_{\mathscr{A}}(BA) \ . \tag{1.20}$$

Proof. Passing to $\widetilde{\mathscr{A}}$, we may suppose that $\mathscr{A}$ has an identity. For each $\lambda \ne 0$, we must show that $(\lambda 1 - AB)$ is invertible if and only if $(\lambda 1 - BA)$ is invertible. Dividing through by λ, we may take $\lambda = 1$. Therefore, suppose that $(1-AB)$ is invertible, and let $Z = (1-AB)^{-1}$. Then

$$\begin{aligned}(1-BA)(1+BZA) &= 1 - BA + BZA - BABZA \\ &= 1 - BA + B(1-AB)ZA = 1 - BA + BA = 1 \ .\end{aligned}$$

Likewise, $(1+BZA)(1-BA) = 1$, and so $(1-BA)$ is invertible with inverse $(1+BZA)$. □

1.21 THEOREM (Spectral Contraction Theorem)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras, and let $\pi : \mathscr{A} \to \mathscr{B}$ be a homomorphism. Then for all $A \in \mathscr{A}$,*

$$\sigma_{\mathscr{B}}(\pi(A)) \subset \{0\} \cup \sigma_{\mathscr{A}}(A) \ . \tag{1.21}$$

Proof. Adjoin identities to $\mathscr{A}$ and $\mathscr{B}$, and define $\widetilde{\pi} : \widetilde{\mathscr{A}} \to \widetilde{\mathscr{B}}$ by $\widetilde{\pi}((1, A)) = (1, \pi(A))$. This is a homomorphism, and takes the identity in $\widetilde{\mathscr{A}}$ to the identity in $\widetilde{\mathscr{B}}$. Since adjoining an identity had no effect on non-zero spectrum, we may assume that $\mathscr{A}$ and $\mathscr{B}$ have identities $1_{\mathscr{A}}$ and $1_{\mathscr{B}}$ respectively, and that $\pi(1_{\mathscr{A}}) = 1_{\mathscr{B}}$.

Now suppose that $\lambda \in \rho_{\mathscr{A}}(A)$. Then $1_{\mathscr{A}} = (\lambda 1_{\mathscr{A}} - A)(\lambda 1_{\mathscr{A}} - A)^{-1}$. Since π is a homomorphism,

$$1_{\mathscr{B}} = \pi(1_{\mathscr{A}}) = (\lambda 1_{\mathscr{B}} - \pi(A))\pi((\lambda 1_{\mathscr{A}} - A)^{-1}) \ .$$

Thus $\pi((\lambda 1_{\mathscr{A}} - A)^{-1})$ is a right inverse of $\lambda 1_{\mathscr{B}} - \pi(A)$, and the same reasoning shows it is also a left inverse. Hence $\lambda \in \rho_{\mathscr{B}}(\pi(A))$. This shows that $\rho_{\mathscr{A}}(A) \subset \rho_{\mathscr{B}}(\pi(A))$, which is equivalent to the statement $\sigma_{\mathscr{B}}(\pi(A)) \subset \sigma_{\mathscr{A}}(A)$, and even shows that when $\mathscr{A}$ and $\mathscr{B}$ have identities and π takes the identity in $\mathscr{A}$ to that in $\mathscr{B}$, it is not necessary to adjoin $\{0\}$ on the right side in (1.21) □

The next theorem gives a useful continuity property of the spectrum.

1.22 THEOREM (Newburgh's Theorem)**.** *Let $\mathscr{A}$ be a Banach algebra and $A \in \mathscr{A}$. Let U be an open subset of $\mathbb{C}$ with $\sigma_{\mathscr{A}}(A) \subset U$. Then there exists $\delta > 0$ such that if $\|B - A\| \leq \delta$,*

$$\sigma_{\mathscr{A}}(B) \subset U \ .$$

Proof. Adjoining a unit if needed, we may assume that $\mathscr{A}$ is unital. First note that for all $B \in \mathscr{A}$ with $\|B - A\| < 1$, $\|B\| < \|A\| + 1$, and hence for all $\lambda \in \mathbb{C}$ with $\lambda \geq \|A\| + 1$, $\lambda \in \rho_{\mathscr{A}}(B)$. Hence when $\|B - A\| \leq 1$, $\sigma_{\mathscr{A}}(B)$ is contained in the closed centered disc of radius $\|A\| + 1$.

Let $K = U^c \cap \{ \ \lambda \ : \ |\lambda| \leq \|A\| + 1 \ \}$ which is a compact subset of $\rho_{\mathscr{A}}(A)$. It suffices to show that there is an $r > 0$ so that for all $\mu \in K$, $(\mu 1 - B)$ is invertible whenever $\|B - A\| < r$.

Let $\lambda \in K$. Then $\lambda \in \rho_{\mathscr{A}}(A)$, and for all $B \in \mathscr{A}$ and $\mu \in \mathbb{C}$ with

$$|\mu - \lambda| + \|B - A\| < \|(\lambda 1 - A)^{-1}\|^{-1} \quad \Rightarrow \quad \|(\mu 1 - B) - (\lambda 1 - A)\| \leq \|(\lambda 1 - A)^{-1}\|^{-1} \ ,$$

and hence $\mu \in \rho_{\mathscr{A}}(B)$. For each $\lambda \in K$, define $U_\lambda = \{\mu \ : |\mu - \lambda| < \frac{1}{2}\|(\lambda 1 - A)^{-1}\|^{-1} \ \}$. Since K is compact, there exists a finite sub-cover $\{U_{\lambda_1}, \dots, U_{\lambda_n}\}$. Define

$$r = \min\{\frac{1}{2}\|(\lambda_1 1 - A)^{-1}\|^{-1} \ , \dots , \ \frac{1}{2}\|(\lambda_n 1 - A)^{-1}\|^{-1} \ \} \ .$$

Then for any B with $\|B - A\| < r$ and any $\mu \in K$, $\mu \in U_{\lambda_j}$ for some $j = 1, \dots, n$, and then

$$\|(\mu 1 - B) - (\lambda_j 1 - A)\| \leq |\mu - \lambda_j| + \|B - A\| < \|(\lambda_j 1 - A)^{-1}\|^{-1} \ .$$

Therefore, $(\mu 1 - B)$ is invertible. Thus, for all $\mu \in K$, whenever $\|B - A\| < r$, $\mu \in \rho_{\mathscr{A}}(B)$. □

1.4 Characters and the Gelfand Transform

1.23 DEFINITION (Characters)**.** A *character* of a Banach algebra $\mathscr{A}$ is a non-zero algebraic homomorphism from $\mathscr{A}$ to $\mathbb{C}$. The set of characters of $\mathscr{A}$ is denoted $\Delta(\mathscr{A})$, and the set $\{0\} \cup \Delta(\mathscr{A})$ is denoted $\Delta'(\mathscr{A})$.

Though characters are defined with respect to the algebraic structure alone, they are necessarily continuous:

1.24 LEMMA. *If $\mathscr{A}$ is a Banach algebra and φ is a character of $\mathscr{A}$, then for all $A \in \mathscr{A}$ $\varphi(A) \in \sigma_{\mathscr{A}}(A)$, and $|\varphi(A)| \leq \|A\|$. Moreover, if $\mathscr{A}$ has an identity 1, then $\varphi(1) = 1$.*

Proof. Suppose first that $\mathscr{A}$ contains an identity 1. Since $\varphi(1) = \varphi(1^2) = (\varphi(1))^2$, $\varphi(1)$ solves $\zeta - \zeta^2 = 0$, so either $\varphi(1) = 0$ or $\varphi(1) = 1$. But if $\varphi(1) = 0$, then for all $A \in \mathscr{A}$, $\varphi(A) = \varphi(1A) = \varphi(1)\varphi(A) = 0$, and this is excluded by the definition. Hence $\varphi(1) = 1$.

Next, if $\varphi(A) = 0$, A is not invertible since if A has an inverse, $1 = \varphi(1) = \varphi(A)\varphi(A^{-1})$ which is incompatible with $\varphi(A) = 0$. However, for any $A \in \mathscr{A}$, $\varphi(\varphi(A)1 - A) = 0$, and hence $\varphi(A)1 - A$ is not invertible. Hence $\varphi(A) \in \sigma_{\mathscr{A}}(A)$, and therefore $|\varphi(A)| \leq \|A\|$.

Now suppose that $\mathscr{A}$ lacks a unit. Let $\widetilde{\mathscr{A}}$ be the algebra obtained by adjoining an identity, and let $\widetilde{\varphi}$ be the character on $\widetilde{\mathscr{A}}$ given by

$$\widetilde{\varphi}((\lambda, A)) = \lambda + \varphi(A) ,$$

which is easily seen to be a character. Since $\sigma_{\mathscr{A}}(A) = \sigma_{\widetilde{\mathscr{A}}}((0, A))$ by definition, and $\widetilde{\varphi}((0, A)) = \varphi(A)$, it follows from the above that $\varphi(A) \in \sigma(A)$, and then that $|\varphi(A)| \leq \|(0, A)\| = \|A\|$. $\square$

Note that if $\varphi \in \Delta(\mathscr{A})$, then for all $A, B \in \mathscr{A}$,

$$\varphi(AB) = \varphi(A)\varphi(B) = \varphi(B)\varphi(A) = \varphi(BA) .$$

Consequently, any character φ must satisfy $\varphi(AB - BA) = 0$ for all A, B. When the algebra $\mathscr{A}$ is not commutative, this can be a stringent constraint, and *there may not exist any characters at all.*

1.25 EXAMPLE. Let $\mathscr{A}$ be the algebra of 2×2 matrices. The Pauli matrices are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{and} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

Then with $[A, B]$ denoting the commutator $AB - BA$,

$$[\sigma_1, \sigma_2] = i2\sigma_3 , \qquad [\sigma_2, \sigma_3] = i2\sigma_1 \quad \text{and} \quad [\sigma_3, \sigma_1] = i2\sigma_2 .$$

It follows that for any homomorphism φ of $\mathscr{A}$ into $\mathbb{C}$, $\varphi(\sigma_j) = 0$ for $j = 1, 2, 3$. Next, the identity matrix I satisfies $I = \sigma_1^2$, and so $\varphi(I) = (\varphi(\sigma_1))^2 = 0$. Thus, for all $(z_0, z_1, z_2, z_3) \in \mathbb{C}^4$,

$$\varphi(z_0 I + z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3) = 0 .$$

Since evidently $\{I, \sigma_1, \sigma_2, \sigma_3\}$ is linearly independent and $\mathscr{A}$ is 4 dimensional, $\mathscr{A}$ is the span of $\{I, \sigma_1, \sigma_2, \sigma_3\}$, and hence φ vanishes identically on $\mathscr{A}$. Thus, if $\mathscr{A}$ is the algebra of 2×2 matrices, $\Delta(\mathscr{A}) = \emptyset$ and $\Delta'(\mathscr{A})$ is the one-point space $\{0\}$.

In the rest of this section, commutativity will not play any role in the proofs, and so we shall state the results without reference to commutativity. However, one should keep in mind that without commutativity, $\Delta(\mathscr{A})$ may be empty and $\Delta'(\mathscr{A})$ may be a one-point space.

1.26 DEFINITION (Gelfand topology)**.** For a Banach algebra $\mathscr{A}$, the *Gelfand topology* on $\Delta'(\mathscr{A})$ is the relative weak-$*$ topology on $\Delta'(\mathscr{A})$ considered as a subset of $\mathscr{A}^*$, the Banach space dual to $\mathscr{A}$. That is, the Gelfand topology is the weakest topology on $\Delta'(\mathscr{A})$ that makes the functions $\varphi \mapsto \varphi(A)$ continuous for all $A \in \mathscr{A}$.

1.27 LEMMA. *Let $\mathscr{A}$ be a Banach algebra. Then $\Delta'(\mathscr{A})$, equipped with the Gelfand topology is a compact Hausdorff space. If $\mathscr{A}$ does not have an identity, then with the Gelfand topology, $\Delta(\mathscr{A})$ is a locally compact Hausdorff space, and $\Delta'(\mathscr{A})$ is its one-point compactification. If $\mathscr{A}$ has an identity, $\Delta(\mathscr{A})$ itself is compact and 0 is an isolated point in $\Delta'(\mathscr{A})$.*

Proof. Equip $\mathscr{A}^*$ with the weak-$*$ topology; i.e., the weakest topology making all of functions $\varphi \mapsto \varphi(A)$ continuous for all $A \in \mathscr{A}$. The Banach-Alaoglu Theorem asserts that the unit ball in $\mathscr{A}^*$ is compact in the weak-$*$ topology. For each $A, B \in \mathscr{A}$, define a function $f_{A,B}$ on $\mathscr{A}^*$ by

$$f_{A,B}(\varphi) = \varphi(AB) - \varphi(A)\varphi(B) \ .$$

This is evidently continuous for the weak-$*$ topology. Now note that

$$\Delta'(\mathscr{A}) = \bigcap_{A,B\in\mathscr{A}} \{\varphi \in \mathscr{A}^* \ : \ f_{A,B}(\varphi) = 0 \ \} \ .$$

This displays $\Delta'(\mathscr{A})$ as an intersection of closed sets. Hence $\Delta'(\mathscr{A})$ is a closed subset of the unit ball in $\mathscr{A}^*$, and hence is compact.

For $\varphi_1, \varphi_2 \in \Delta'(\mathscr{A})$ with $\varphi_1 \neq \varphi_2$, there exists $A \in \mathscr{A}$ such that $\varphi_1(A) \neq \varphi_2(A)$. Let U_1 and U_2 be disjoint open sets in $\mathbb{C}$ that contain $\varphi_1(A)$ and $\varphi_2(A)$ respectively. Then

$$\{\psi \in \Delta'(\mathscr{A}) \ : \ \psi(A) \in U_1 \ \} \quad \text{and} \quad \{\psi \in \Delta'(\mathscr{A}) \ : \ \psi(A) \in U_2 \ \}$$

are disjoint open sets in $\Delta'(\mathscr{A})$ that contain φ_1 and φ_2 respectively. In particular, for each $\varphi \in \Delta(\mathscr{A})$, there exist disjoint open neighborhoods V_1 of φ and V_2 of 0, and then since $V_1 \subset V_2^c$, V_2^c is a compact neighborhood of φ. Thus, $\Delta(\mathscr{A})$ is locally compact and Haussdorf. If $\mathscr{A}$ has an identity 1, $\varphi(1) = 1$ for all $\varphi \in \Delta(\mathscr{A})$, while $0(1) = 0$. Consequently, the zero homomorphism is an isolated point of $\Delta'(\mathscr{A})$ in this case. □

1.28 DEFINITION (Gelfand transform)**.** Let $\mathscr{A}$ be a Banach algebra. The *Gelfand transform* is the map γ from $\mathscr{A}$ to $\mathscr{C}(\Delta'(\mathscr{A}))$ given by

$$(\gamma(A))[\varphi] = \varphi(A) \ . \tag{1.22}$$

That is, $\gamma(A)$ is the function of evaluation at A, and it is continuous by the definition of the Gelfand topology.

1.29 THEOREM. *Let $\mathscr{A}$ be a Banach algebra. The Gelfand transform is a norm reducing homomorphism from $\mathscr{A}$ to $\mathscr{C}((\Delta'(\mathscr{A}))$. That is, the Gelfand transform is a homomorphism of algebras, and for all $A \in \mathscr{A}$,*

$$\|\gamma(A)\|_{\mathscr{C}(\Delta'(\mathscr{A}))} \leq \|A\| \ .$$

Proof. The homomorphism property is evident since for all $A, B \in \mathscr{A}$ and all $\varphi \in \Delta'(\mathscr{A})$,

$$(\gamma(AB))[\varphi] = \varphi(AB) = \varphi(A)\varphi(B) = (\gamma(A))[\varphi](\gamma(B))[\varphi] \ .$$

By Lemma 1.24, $|(\gamma(A))[\varphi]| = |\varphi(A)| \leq \|A\|$, proving $\|\gamma(A)\|_{\mathscr{C}(\Delta'(\mathscr{A}))} \leq \|A\|$. □

This result, as it stands, does not take us far at all. Even in the commutative case, there may be so few characters that the transform may be a trivial homomorphism into a trivial algebra.

1.30 EXAMPLE. Let A_0 be the $n \times n$ matrix, $n > 1$, with $A_{i,j} = \begin{cases} 1 & j = i+1 \\ 0 & j \neq i+1 \end{cases}$, and note that $A_0^n = 0$. Let $\mathscr{A}$ denote the subalgebra of the $n \times n$ matrices that are polynomials in A_0. That is, every $A \in \mathscr{A}$ has the form

$$A = \sum_{j=0}^{n-1} p_j A_0^j \ . \tag{1.23}$$

This is a commutative algebra with an identity. Let $\varphi \in \Delta'(\mathscr{A})$. Then $0 = \varphi(A_0^n) = (\varphi(A_0))^n$ so that $\varphi(A_0) = 0$. Then for A given by (1.23), $\varphi(A) = p_0\varphi(I) = p_0$. Thus, the only candidate for a character on $\mathscr{A}$ is the map φ_0 given by $\varphi_0\left(\sum_{j=0}^{n-1} p_j A_0^j\right) = p_0$. It is readily checked that this is indeed a homomorphism and it is non-zero. Hence $\Delta(\mathscr{A}) = \{\varphi_0\}$ and $\Delta'(\mathscr{A}) = \{\varphi_0\} \cup \{0\}$. Since $\Delta'(\mathscr{A})$ consists of two isolated points, we may identify $\mathscr{C}(\Delta'(\mathscr{A}))$ with $\mathbb{C}^2$ in the usual way, and then we may write the Gelfand transform as

$$\gamma\left(\sum_{j=0}^{n-1} p_j A_0^j\right) = (p_0, 0) \ ,$$

which is indeed a norm reducing homomorphism, but not very interesting.

Before leaving this example, we note that for elements of $\mathscr{A}$, the spectrum is as trivial as Theorem 1.16 allows: For all $A \in \mathscr{A}$, $\sigma_{\mathscr{A}}(A)$ consists of a single point: $\sigma_{\mathscr{A}}(A) = \{\varphi_0(A)\}$. This is true since when A is given by (1.23), then A is invertible if and only if $p_0 \neq 0$.

1.5 Characters and spectrum in commutative Banach algebras

We have seen in Example 1.30 that in a commutative Banach Algebra $\mathscr{A}$ with an identity, $\Delta'(\mathscr{A})$ may consist of only two isolated points. However, this happens only when for each $A \in \mathscr{A}$, $\sigma_{\mathscr{A}}(A)$ consists of a single point. We now show when $\mathscr{A}$ contains elements with a more interesting spectrum, then $\Delta'(\mathscr{A})$ is also more interesting.

The Hahn-Banach Theorem, which provides the existence of continuous linear functionals on a Banach space, may be viewed as a theorem asserting the existence of maximal closed subspaces containing a given subspace. In the Banach algebra setting, the kernel of a homomorphism of a Banach algebra $\mathscr{A}$ to $\mathbb{C}$ is not only a closed subspace, it is a closed *ideal*, as we now explain, and consideration of *maximal ideals* leads to a Banach algebra version of the Hahn-Banach Theorem for commutative Banach algebras. Much of what is introduced here is also useful without assuming that $\mathscr{A}$ is commutative. We therefore start in general, and shall be clear about the key point when commutativity enters.

1.31 DEFINITION. Let $\mathscr{A}$ be a Banach algebra. An *ideal* of $\mathscr{A}$ is a subspace of $\mathscr{A}$ such that for all $B \in \mathscr{J}$ and $A \in \mathscr{A}$, $BA \in \mathscr{J}$ and $AB \in \mathscr{J}$. An ideal of $\mathscr{A}$ is *proper* in case it is not equal to $\mathscr{A}$ itself. An ideal of $\mathscr{A}$ is *closed* in case it is topologically closed as a subset of $\mathscr{A}$. If $\mathscr{J}$ is an ideal in $\mathscr{A}$, and $\overline{\mathscr{J}}$ is the norm closure of $\mathscr{J}$, then $\overline{\mathscr{J}}$ is also an ideal in $\mathscr{A}$. If $\mathscr{J}$ is an ideal, an element U of $\mathscr{A}$ is called a *unit mod* $\mathscr{J}$ in case

$$AU - A \in \mathscr{J} \quad \text{and} \quad UA - A \in \mathscr{J} \quad \text{for all} \quad A \in \mathscr{A} \ . \tag{1.24}$$

An ideal $\mathscr{J}$ is called a *modular ideal* in case there exists a unit mod $\mathscr{J}$.

Given a Banach algebra $\mathscr{A}$ and an ideal $\mathscr{J}$, there is a natural equivalence relation $\sim$ on $\mathscr{A}$ given by

$$A \sim B \iff A - B \in \mathscr{J} \ .$$

Let $\{A\}$ and $\{B\}$ denote the equivalence classes of A and B respectively. Let $\tilde{A}$ and $\tilde{B}$ be any other representative of $\{A\}$ and $\{B\}$ respectively. Then for some $X, Y \in \mathscr{J}$, $\tilde{A} = A + X$ and $\tilde{B} = B + Y$. Then

$$\tilde{A}\tilde{B} = (A + X)(B + Y) = AB + (AY + XB + XY) \sim AB \ .$$

Even more simply one sees that $\tilde{A} + \tilde{B} \sim A + B$ and for all $\lambda \in \mathbb{C}$, $\lambda\tilde{A} \sim \lambda A$. Hence $\mathscr{A}/\mathscr{J}$, the set of equivalence classes in $\mathscr{A}$, equipped with the operations

$$\{A\}\{B\} = \{AB\} \quad \text{and} \quad \{A\} + \{B\} = \{A + B\} \quad \text{and} \quad \lambda\{A\} = \{\lambda A\}$$

is an algebra, and $A \mapsto \{A\}$ is a homomorphism of $\mathscr{A}$ onto $\mathscr{A}/\mathscr{J}$.

Now introduce a norm on $\mathscr{A}/\mathscr{J}$ by

$$\|\{A\}\| = \inf\{ \ \|\tilde{A}\| \ : \ \tilde{A} \sim A \ \} = \inf\{ \ \|A - B\| \ : \ B \in \mathscr{J} \ \} \ .$$

Note that $\|\{A\}\| \leq \|A\|$. To see that

$$\|\{A\}\{B\}\| \leq \|\{A\}\|\|\{B\}\| \tag{1.25}$$

for all $\{A\}, \{B\} \in \mathscr{A}/\mathscr{J}$, let $0 < \epsilon < \min\{\|\{A\}\|, \|\{B\}\|\}$, and pick $\tilde{A} \in \{A\}$ and $\tilde{B} \in \{B\}$ so that $\|\{A\}\| > \|\tilde{A}\| - \epsilon$ and $\|\{B\}\| > \|\tilde{B}\| - \epsilon$. Then

$$\|\{A\}\{B\}\| = \|\{\tilde{A}\tilde{B}\}\| \leq \|\tilde{A}\tilde{B}\| \leq \|\tilde{A}\|\|\tilde{B}\| \leq (\|\{A\}\| + \epsilon)(\|\{B\}\| + \epsilon) \ .$$

Since ϵ can be taken arbitrarily small, (1.25) is proved.

Therefore, $\mathscr{A}/\mathscr{J}$ will be a Banach algebra with this norm provided it is complete in this norm. Consider a Cauchy sequence $\{\{A\}_n\}_{n\in\mathbb{N}}$ in $\mathscr{A}/\mathscr{J}$. A standard argument shows that this sequence always has a limit if $\mathscr{J}$ is closed. Thus, when $\mathscr{J}$ is a closed ideal, $\mathscr{A}/\mathscr{J}$ is a Banach algebra, and the map $A \mapsto \{A\}$ is a contractive homomorphism of $\mathscr{A}$ onto $\mathscr{A}/\mathscr{J}$. This homomorphism is called the *natural homomorphism of $\mathscr{A}$ onto $\mathscr{A}/\mathscr{J}$*.

It is possible for $\mathscr{A}/\mathscr{J}$ to have an identity even when $\mathscr{A}$ does not. Suppose that $\mathscr{J}$ is modular, and that U is a unit mod $\mathscr{J}$. Then for all $A \in \mathscr{A}$, $\{U\}\{A\} = \{UA\} = \{A\}$ and $\{A\}\{U\} = \{AU\} = \{A\}$. Thus, $\{U\}$ is a multiplicative identity in $\mathscr{A}/\mathscr{J}$. Clearly if $\mathscr{A}$ has an identity 1, 1 is a unit mod $\mathscr{J}$.

There is a close connection between closed ideals and kernels of continuous homomorphisms of Banach algebras.

1.32 PROPOSITION. *Let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras, and let $\pi : \mathscr{A} \to \mathscr{B}$ be a continuous homomorphism. Then $\mathscr{J} = \ker(\mathscr{A})$ is a closed ideal in $\mathscr{A}$. Conversely, if $\mathscr{J}$ is a closed ideal in $\mathscr{A}$, then the map $A \mapsto \{A\}$, sending A to its equivalence class mod $\mathscr{J}$, is a homomorphism with kernel $\mathscr{J}$ of $\mathscr{A}$ onto $\mathscr{A}/\mathscr{J}$.*

Proof. Suppose that $\pi : \mathscr{A} \to \mathscr{B}$ be a continuous homomorphism. Then evidently $\mathscr{J} = \ker(\mathscr{A})$ is closed by the continuity of π, and it is a subspace by the linearity of π. Next, for all $X \in \mathscr{J}$ and $A, B \in \mathscr{A}$, $\pi(AXB) = \pi(A)\pi(X)\pi(B) = \pi(A)\pi(B) = 0$. Hence $AXB \in \ker(\pi)$, and so $\mathscr{J}$ is an ideal. The converse is clear from the construction of $\mathscr{A}/\mathscr{J}$ described above. □

Now consider a *commutative* Banach algebra $\mathscr{A}$. For any $X_0 \in \mathscr{A}$, we can define $\mathscr{J}(X_0)$ to be the subset of $\mathscr{A}$ given by

$$\mathscr{J}(X_0) = \{ \, YX_0 \; : \; Y \in \mathscr{A} \, \} \; . \tag{1.26}$$

Then for all $YX_0 \in \mathscr{J}(X_0)$ and all $A, B \in \mathscr{A}$, $AYX_0B = (AYB)X_0 \in \mathscr{J}(x_0)$, and evidently $\mathscr{A}$ is a subspace of $\mathscr{A}$. Hence $\mathscr{J}(X_0)$ is an ideal, and it is called *the ideal generated by* X_0.

Let X_0 be a non-invertible element of $\mathscr{A}$. Let $\mathscr{J}(X_0)$ be the ideal generated by X_0. Then no element of $\mathscr{J}(X_0)$ is invertible. Indeed, if YX_0 were invertible, there would exist $Z \in \mathscr{A}$ such that $(YX_0)Z = (YZ)X_0 = 1$, and then (since $\mathscr{A}$ is commutative), YZ would be an inverse of X_0, which is not possible. Hence, for all non-invertible X_0, $\mathscr{J}(X_0)$ consists entirely of non-invertible elements. Since the open unit ball about the identity consists of invertible elements, $\mathscr{J}(X_0)$ does not intersect the open unit ball about the identity 1. In particular, 1 does not belong to $\overline{\mathscr{J}(X_0)}$, the closure of $\mathscr{J}(X_0)$.

Now we come to a crucial construction of characters in a commutative Banach algebra:

1.33 THEOREM. *Let $\mathscr{A}$ be a commutative Banach algebra with identity* 1*. Then for all non-invertible $X_0 \in \mathscr{A}$, there exists a character $\varphi \in \Delta(\mathscr{A})$ such that $\varphi(X_0) = 0$.*

Proof. Since X_0 is not invertible, $\mathscr{J}(X_0)$ is a proper ideal in $\mathscr{A}$, and in fact, as explained above, the open unit ball about 1 does not intersect $\mathscr{J}(X_0)$. Now consider any chain of proper ideals in $\mathscr{A}$, ordered by inclusion. Since no proper ideal contains the identity, the union of this chain is again a proper ideal. Hence by Zorn's Lemma, there exists a maximal proper ideal $\mathscr{M}$ containing $\mathscr{J}(X_0)$. Since no proper ideal can contain any invertible elements, this ideal does not intersect the open unit ball about 1. Hence its closure $\overline{\mathscr{M}}$ also contains $\mathscr{J}(X_0)$ and is proper. Since $\mathscr{M}$ is maximal among such proper ideals, $\mathscr{M} = \overline{\mathscr{M}}$. Hence in a commutative Banach algebra $\mathscr{A}$ with identity 1, for each non-invertible $X_0 \in \mathscr{A}$, there exists a closed proper ideal $\mathscr{M}$ that contains any ideal in $\mathscr{A}$ that contains $\mathscr{J}(X_0)$.

We now claim that the Banach algebra $\mathscr{B} = \mathscr{A}/\mathscr{M}$ is a division algebra. Suppose not. Then it contains a non-zero, non-invertible element $\{Y_0\}_{\mathscr{M}}$. Let $\mathscr{N}$ be the closure of the ideal in $\mathscr{B}$ generated by $\{Y_0\}_{\mathscr{M}}$. Let π_1 be the natural homomorphism of $\mathscr{A}$ onto $\mathscr{B}$, and let π_2 be the natural homomorphism of $\mathscr{B}$ onto $\mathscr{B}/\mathscr{N}$. Then $\pi_2 \circ \pi_1$ is a homomorphism of $\mathscr{A}$ onto $\mathscr{B}/\mathscr{N}$. By Proposition 1.32, $\ker(\pi_2 \circ \pi_1)$ is a closed ideal that contains $\mathscr{M} = \ker(\pi_1)$. The containment is proper since $\pi_2 \circ \pi_1(Y_0) = 0$, but $Y_0 \notin \mathscr{M}$ since $\{Y_0\}_{\mathscr{M}} \neq 0$. Finally, $1 \notin \ker(\pi_1 \circ \pi_1)$ since $\{1\}_{\mathscr{M}}$ is a unit $\mathscr{B} = \mathscr{A}/\mathscr{M}$, and $\mathscr{N}$ does not contain any invertible elements, so $\pi_2(\{1\}_{\mathscr{M}}) = \pi_2(\pi_1(1)) \neq 0$. Thus, $\ker(\pi_2 \circ \pi_1)$ is a closed proper ideal that strictly contains $\mathscr{M}$, which is impossible. Hence the hypothesis that $\mathscr{B} = \mathscr{A}/\mathscr{M}$ contains a non-zero, non-invertible element is false. This shows that $\mathscr{B} = \mathscr{A}/\mathscr{M}$ is a division algebra, and then the Gelfand-Mazur Theorem tells us that $\mathscr{B} = \mathscr{A}/\mathscr{M}$ is canonically isomorphic to $\mathbb{C}$. Hence π_1 may be regarded as a character of $\mathscr{A}$, and by construction $X_0 \in \mathscr{J}(X_0) \subset \mathscr{M} = \ker(\pi_1)$. □

This theorem has the following important consequence:

1.34 COROLLARY. *Let $\mathscr{A}$ be a commutative Banach algebra. For every non-zero $\lambda \in \sigma_{\mathscr{A}}(A)$, there exists $\varphi \in \Delta(\mathscr{A})$ such that $\varphi(A) = \lambda$. In particular, the spectral radius $\nu(A)$ of A is given by*

$$\nu(A) = \sup\{\ |\varphi(A)|\ :\ \varphi \in \Delta(\mathscr{A})\ \}\ . \tag{1.27}$$

Proof. Adjoining an identity has no effect on the non-zero spectrum; we may assume that $\mathscr{A}$ is unital. For $\lambda \in \sigma_{\mathscr{A}}(A)$, $\lambda 1 - A$ is not invertible. By Theorem 1.33, there exists $\varphi \in \Delta(\mathscr{A})$ such that $\varphi(\lambda 1 - A) = 0$. But $\varphi(\lambda 1 - A) = \lambda\varphi(1) - \varphi(A) = \lambda - \varphi(A)$. □

2 Fundamentals of the Theory of C^* Algebras

2.1 C^* algebras

2.1 DEFINITION (Banach $*$-algebra)**.** A Banach $*$-algebra is a Banach algebra $\mathscr{A}$ equipped with a map $* : \mathscr{A} \to \mathscr{A}$, the action of which is written as $A \mapsto A^*$, and which satisfies:

(i) The map $*$ is conjugate linear. That is, for all $A, B \in \mathscr{A}$ and all $z \in \mathbb{C}$, $(zA + B)^* = \overline{z}A^* + B^*$.

(ii) For all $A, B \in \mathscr{A}$, $(AB)^* = B^*A^*$.

(iii) The $*$ map is an involution; for all $A \in \mathscr{A}$, $A^{**} = A$.

The map $A \mapsto A^*$ is called *the involution* in $\mathscr{A}$.

2.2 DEFINITION (C^*-algebra)**.** A C^* *algebra* is a Banach $*$ algebra for which the involution and the norm are related by the requirement that: *(iv)* For all $a \in \mathscr{A}$,

$$\|AA^*\| = \|A\|\|A^*\|\ . \tag{2.1}$$

The identity (2.1) is called *the C^* algebra identity.*

Definition 2.2 comes from a seminal 1943 paper of Gelfand and Neumark [15]. The term C^* algebra was introduced by Segal [39] in 1947. Segal originally used the term [39, p. 75] to describe norm closed self adjoint subalgebras of the algebra of bounded linear operator on a Hilbert space; see Example 2.4 below. These later became known as "concrete C^*-algebras", and the *a priori* more general class of C^* algebras specified in Definition 2.2 became known as "abstract C^*-algebras". The main result of [15] is that, under some additional assumptions (see below), every unital abstract C^* algebra is isometric and $*$-isomorphic with a concrete C^*- algebra. The additional assumptions, and the requirement that $\mathscr{A}$ be unital, have since been shown to be unnecessary, as is explained below, and the fact that every abstract C^* algebra can be realized as a concrete C^* algebra of operators on a Hilbert space will then bring a wide range of new tools into the theory, especially a range of useful topologies weaker than the norm topology.

In their paper, Gelfand and Neumark not only assumed the existence of a multiplicative identity in $\mathscr{A}$; they also imposed two other hypotheses that they conjectured to be consequences of *(i)* - *(iii)* and (2.1) which was their *(iv)*, but they were unable to show this. The additional hypotheses that they imposed were that

(v) The $*$ map is an isometry; for all $A \in \mathscr{A}$, $\|A^*\| = \|A\|$.

(vi) For all $A \in \mathscr{A}$, $1 + A^*A$ is invertible.

In the second footnote of their paper, they conjecture that *(i)* - *(iv)* imply *(v)* and *(vi)*, remarking that the authors "have not succeeded in proof of this fact". They then note that postulates *(iv)* and *(v)* may be replaced by

$$\|A^*A\| = \|A\|^2 \tag{2.2}$$

for all $A \in \mathscr{A}$, since then $\|A\|^2 = \|A^*A\| \leq \|A^*\|\|A\|$, and hence $\|A\| \leq \|A^*\|$. Then by *(iii)*, the involution must be an isometry, so that (2.2) and *(iii)* imply *(v)*. Conversely, *(iv)* and *(v)* imply (2.2). It took 10 years for the conjecture of Gelfand and Neumark to be verified for *(vi)* in work of Fukamiya [12] and Kaplansky (See [38] for an account of Kaplansky's unpublished contribution), and 17 years for *(v)* in work of Glimm and Kadison [16], and then only for unital C^* algebras. Finally in 1967, Vowden [46] proved that every C^* algebra $\mathscr{A}$ can be isometrically and $*$-isomorphically embedded in a unital C^* algebra $\mathscr{A}_1$. As Vowden noted, it is an immediate consequence of this result that in any Banach $*$-algebra in which (2.1) is valid for all $A \in \mathscr{A}$, it is also the case that (2.2) is valid for all $A \in \mathscr{A}$.

In 1946, Rickart defined a B^**-algebra* to be a Banach $*$-algebra $\mathscr{A}$ in which the norm and the involutions satisfy (2.2) for all $A \in \mathscr{A}$. By what Gelfand and Neumark observed in the second footnote, every B^* algebra is C^* algebra. In 1967, Vowden [46] completed the proof of the fact that every C^* algebra is also a B^* algebra, and the latter term is now rarely used.

In the more recent literature on the subject, many authors simply define a C^*-algebra to be a Banach $*$-algebra $\mathscr{A}$ in which the norm and the involution satisfy (2.2) for all $A \in \mathscr{A}$; see e.g., [1]. This has some advantages since, for instance, the isometry property of the involution is an immediate consequence of the other postulates. Here we shall have to work harder for this since we start from the original postulates *(i)* through *(iv)*, *(iv)* being (2.1), as originally proposed by Gelfand and Neumark, and present the proof of their conjecture on the redundancy of *(v)* and *(vi)* in the general non-unital case. While the proof of this conjecture (in its non-unital form) had to wait for two dozen years and the contributions of a number of mathematicians, the insights through which this was achieved are quite simple, clean and clear. Like many simple, clean and clear ideas, they will turn out to have other uses, and this more than justifies our choice of starting point.

2.3 EXAMPLE. Let $\mathscr{A} = \mathscr{C}_0(X)$ with the structure specified in Example 1.2, together with the involution $*$ defined by $f^*(x) = \overline{f(x)}$ for all $x \in X$ and all $f \in \mathscr{A}$. Then one readily checks that $\mathscr{A}$ is a commutative C^* algebra. A theorem of Gelfand and Neumark to be proved in this chapter says that up to isometric isomorphism, every commutative C^* algebra is of this form.

2.4 EXAMPLE. Let $\mathscr{H}$ be a Hilbert space, and let $\mathscr{A} = \mathscr{B}(\mathscr{H})$ equipped with the structure specified in Example 1.4, together with the involution $*$ defined by taking A^* to be the Hermitian adjoint of A. That is, A^* is the unique operator in $\mathscr{B}(\mathscr{H})$ such that

$$\langle A^*\phi, \psi\rangle = \langle \phi, A\psi\rangle \tag{2.3}$$

for all $\phi, \psi \in \mathscr{H}$. Then properties *(i)*, *(ii)* and *(iii)* are evidently satisfied. To see that *(iv)* is satisfied, choose $\epsilon > 0$ and a unit vector $\psi \in \mathscr{H}$ such that $(1-\epsilon)\|A\| \leq \|A\psi\|$. Then

$$\langle \psi, A^*A\psi\rangle = \langle A\psi, A\psi\rangle = \|A\psi\|^2 \geq (1-\epsilon)^2\|A\|^2 \ .$$

Since $\epsilon > 0$ is arbitrary, it follows that $\|A^*A\| \geq \|A\|^2$. Since (2.3) says that $|\langle \psi, A^*\phi\rangle| = |\langle \phi, A\psi\rangle|$, it is evident that $\|A^*\| = \|A\|$ and hence $\|A^*A\| \geq \|A\|^2$ can be written as $\|A^*A\| \geq \|A^*\|\|A\|$. Since $\mathscr{B}(\mathscr{H})$ is a Banach algebra, we have $\|A^*A\| \leq \|A^*\|\|A\|$, and hence *(iv)* is proved.

We have already proved the isometry property *(v)* in this example on our way to *(iv)*. It is also not hard to give a direct proof of *(vi)* in this context; see Remark 3.6 and the discussion just above it. That proof of *(vi)*, just as this proof of *(iv)* and *(v)*, makes use of the vectors in the space on which the elements of $\mathscr{A}$ operate. In the abstract setting of Definition (2.2), the elements of $\mathscr{A}$ need not be operators on a Hilbert space, and such arguments are not immediately available. However, the fundamental theorem in the 1943 paper of Gelfand and Neumark – as extended over the next two dozen years – says that every C^* algebra is isometrically $*$-isomorphic to a C^* algebra of operators in a Hilbert space $\mathscr{H}$.

A difficulty that arises when studying non-unital C^*-algebras is that our standard construction for adjoining a unit does not in general yield a norm satisfying *(iv)*. However, we can always embed a Banach $*$-algebra $\mathscr{A}$ in a unital Banach $*$-algebra $\widetilde{\mathscr{A}}$ by a small extension of our standard construction: Let $\mathscr{A}$ be a non-unital Banach $*$-algebra and let $\widetilde{\mathscr{A}} := \mathbb{C} \oplus \mathscr{A}$ with the product defined in (1.3) and the norm defined in (1.4). For $(\lambda, A) \in \widetilde{\mathscr{A}}$, define $(\lambda, A)^* := (\overline{\lambda}, A^*)$. It is easy to see that $\widetilde{\mathscr{A}}$ satisfies *(i)*, *(ii)* and *(iii)* of Definition 2.1. However, *(iv)* will not in general be satisfied. When $\mathscr{A}$ is a Banach $*$-algebra, $\widetilde{\mathscr{A}}$ *always* denotes the unital Banach $*$-algebra obtained using this construction.

2.5 LEMMA. *Let $\mathscr{A}$ be a unital Banach $*$-algebra with unit 1. Then $1^* = 1$, and $A \in \mathscr{A}$ is invertible if and only if A^* is invertible.*

Proof. Evidently $1^* = 1^*1$, and then by *(ii)* and *(iii)*, $1 = 1^*1$. Next, suppose that A is invertible and let B be its inverse. Then $1 = AB = BA$. By *(ii)* and *(iii)*, and the first part of the proof, $1 = 1^* = B^*A^* = A^*B^*$. Thus, A^* is invertible, and B^* is the inverse. By *(iii)* once more, when A^* is invertible, A is invertible. □

2.6 LEMMA. *Let $\mathscr{A}$ be a Banach $*$-algebra Then for all $A \in \mathscr{A}$,*

$$\sigma_{\mathscr{A}}(A^*) = \overline{\sigma_{\mathscr{A}}(A)} \ . \tag{2.4}$$

In particular, $\nu(A^*) = \nu(A)$.

Proof. Suppose $\mathscr{A}$ is non-unital. Let $\widetilde{\mathscr{A}}$ be unital Banach $*$-algebra obtained from our standard construction. By definition, $\sigma_{\mathscr{A}}(A) = \sigma_{\widetilde{\mathscr{A}}}((0, A))$. By Lemma 2.5, $(\lambda, -A)$ is invertible if and only if $(\lambda, -A)^* = (\overline{\lambda}, -A^*)$ is invertible, and this proves (2.4). The unital case is even more direct. The final statement follows from the definition of the spectral radius. □

2.7 DEFINITION (Self-adjoint, normal)**.** Let $\mathscr{A}$ be a C^* algebra. Then:

(1) $A \in \mathscr{A}$ is *self-adjoint* in case $A = A^*$.

(2) $A \in \mathscr{A}$ is *normal* in case $AA^* = A^*A$.

We write $\mathscr{A}_{\text{s.a.}}$ to denote the set of self-adjoint elements in $\mathscr{A}$.

2.8 LEMMA. *Let $\mathscr{A}$ be a C^* algebra. Then for all normal $A \in \mathscr{A}$,*

$$\|A\| = \|A^*\| = \nu(A) \ . \tag{2.5}$$

Proof. Suppose first that A is self-adjoint. By *(iv)*, $\|A^2\| = \|A\|^2$, and by an obvious induction, for all $m \in \mathbb{N}$, $\|A^{2^m}\| = \|A\|^{2^m}$. By the Beurling-Gelfand Formula,

$$\nu(A) = \lim_{m\to\infty} (\|A^{2^m}\|)^{1/2^m} = \|A\|.$$

This proves (2.5) when A is self-adjoint.

Now observe that when A and B commute, $(AB)^n = A^n B^n$ and hence $\|(AB)^n\|^{1/n} \leq \|A^n\|^{1/n}\|B^n\|^{1/n}$. By the Beurling-Gelfand Formula, $\nu(AB) \leq \nu(A)\nu(B)$. Also, by Lemma 2.6, $\nu(A^*) = \nu(A)$. Hence when A is normal,

$$\|A\|^2 \geq \nu(A)^2 = \nu(A^*)\nu(A) \geq \nu(A^*A) = \|A^*A\| = \|A^*\|\|A\| \ , \tag{2.6}$$

where in the last two steps we have used (2.5) for the self-adjoint operator A^*A, and then *(iv)*. We conclude that when A is normal, $\|A^*\| = \|A\|$, and then (2.6) implies that $\|A\|^2 \geq \nu(A)^2 \geq \|A\|^2$, which proves (2.5) for all normal A. □

2.9 THEOREM. *Let $\mathscr{A}$ be a C^* algebra, and let $A \in \mathscr{A}_{\text{s.a.}}$. Then $\sigma_{\mathscr{A}}(A) \subset \mathbb{R}$.*

Proof. If $\mathscr{A}$ has no unit, we adjoin a unit to obtain $\widetilde{\mathscr{A}}$ using our standard construction. Then $\widetilde{\mathscr{A}}$ is a Banach $*$-algebra, but not necessarily a C^* algebra. We proceed by contradiction, and suppose that A is a self-adjoint element of $\mathscr{A}$, and there exists some $\lambda \in \sigma_{\mathscr{A}}(A)$ such that $\lambda \notin \mathbb{R}$. Taking an appropriate real multiple of A (still self adjoint), we may suppose that $|e^{i\lambda}| = 4$ for some $\lambda \in \sigma_{\mathscr{A}}(A)$.

For each $n \in \mathbb{N}$, define the polynomial $p_n(\zeta) = \sum_{j=0}^n (i\zeta)^j/j!$. By the Spectral Mapping Lemma, for each n, $p_n(\lambda) \in \sigma_{\mathscr{A}}(p_n(A))$. For $n > m$,

$$\|p_n(A) - p_m(A)\| \leq \sum_{j=m+1}^{n} \|A\|^j/j! \ .$$

By standard estimates on the exponential power series in $\mathbb{C}$, $\{p_n(A)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\widetilde{\mathscr{A}}$. Therefore, there exists $U \in \widetilde{\mathscr{A}}$ with $U = \lim_{n\to\infty} p_n(A)$. Evidently, for all n, $(p_n(A))^* = p_n(-A)$, so that once again, by standard estimates for the exponential power series, $U^*U = UU^* = 1$. Now define $W = U - 1$, and note that W is in $\mathscr{A}$. From $U^*U = UU^* = 1$ in $\widetilde{\mathscr{A}}$, we have

$$W + W^* + W^*W = W + W^* + WW^* = 0 \ , \tag{2.7}$$

and in particular, W is normal. By (2.7) once more, $\|W^*W\| \leq \|W\| + \|W^*\|$, and then by Lemma 2.8, $\|W\|^2 \leq 2\|W\|$ and hence $\|W\| \leq 2$. Then in $\widetilde{\mathscr{A}}$,

$$\lim_{n\to\infty} (p_n(\lambda)1 - p_n(A)) = e^{i\lambda}1 - 1 - W = e^{i\lambda}(1 - e^{-i\lambda}(1 + W)) \ .$$

Since $|e^{-i\lambda}| = \frac{1}{4}$, and $\|W\| \leq 2$, $\|e^{-i\lambda}(1+W)\| \leq \frac{3}{4}$, and $1 - e^{-i\lambda}(1+W)$ is invertible. However, for each n, $p_n(\lambda)1 - p_n(A)$ is non-invertible. Since the set of invertible elements is open, it cannot be that a sequence of non-invertible elements converges to an invertible element. Thus, the hypothesis that for $A \in \mathscr{A}_{\text{s.a.}}$, $\sigma_{\mathscr{A}}(A)$ is not contianed in $\mathbb{R}$ leads to contradiction. □

2.10 THEOREM. *In a Banach $*$-algebra $\mathscr{A}$, the involution is continuous if and only if the set $\mathscr{A}_{\text{s.a}}$ of self adjoint elements is closed.*

Proof. If the involution is continous, then as the inverse image of $\{0\}$ under the continuous function $A \mapsto A - A^*$, $\mathscr{A}_{\mathrm{s.a.}}$ is closed.

For the converse, we apply the Closed Graph Theorem, which is also valid for conjugate linear maps. Consider a sequence $\{A_n\}_{n\in\mathbb{N}}$ such that for some $B, C \in \mathscr{A}$, $\lim_{n\to\infty} A_n = B$ and $\lim_{n\to\infty} A_n^* = C$. We must show that $C^* = B$. Note that

$$\lim_{n\to\infty}\left(\frac{A_n + A_n^*}{2}\right) = \frac{B+C}{2} \quad \text{and} \quad \lim_{n\to\infty}\left(\frac{A_n - A_n^*}{2i}\right) = \frac{B-C}{2i} \ .$$

Since $\mathscr{A}_{\mathrm{s.a.}}$ is closed, both $\frac{B+C}{2}$ and $\frac{B-C}{2i}$ are self-adjoint, and hence

$$B + C = B^* + C^* \quad \text{and} \quad B - C = C^* - B^* \ ,$$

and thus $2B = (B+C) + (B-C) = (B^* + C^*) + (C^* - B^*) = 2C^*$, as was to be shown. □

2.11 THEOREM. *Let $\mathscr{A}$ be a C^* algebra. Then $\mathscr{A}_{\mathrm{s.a.}}$ is closed, and the involution is continuous.*

Proof. By Theorem 2.10, it suffices to prove that $\mathscr{A}_{\mathrm{s.a.}}$ is closed. Let $\{H_n\}_{n\in\mathbb{N}}$ be a convergent sequence in $\mathscr{A}_{\mathrm{s.a.}}$ with $\lim_{n\to\infty} H_n = H + iK$, $H, K \in \mathscr{A}_{\mathrm{s.a.}}$. Subtracting H from each H_n, we may suppose that $H = 0$. Also without loss of generality, we may suppose that $\|H_n\| \leq 1$ for all n, with the consequence that $\|K\| \leq 1$. By the Spectral Mapping Theorem, $\sigma_{\mathscr{A}}(H_n^2 - H_n^4) = \{\lambda^2 - \lambda^4 : \lambda \in \sigma_{\mathscr{A}}(H_n)\}$. Then by Lemma 2.5 and Theorem 2.9,

$$\|H_n^2 - H_n^4\| = \sup\{\lambda^2 - \lambda^4 : \lambda \in \sigma_{\mathscr{A}}(H_n)\} \leq \sup\{\lambda^2 : \lambda \in \sigma_{\mathscr{A}}(H_n)\} = \|H_n\|^2 \ ,$$

where we have used the fact that $\sigma_{\mathscr{A}}(H_n) \subset [-1,1]$ by Lemma 2.5 once more. Taking the limit $n \to \infty$, we obtain $\|K^2 + K^4\| \leq \|K\|^2$. Choose $\lambda_1 \in \sigma_{\mathscr{A}}(K)$ so that $\lambda_1 = \nu(K^2) = \|K\|^2$. Then $\lambda_1 + \lambda_1^2 \leq \lambda_1$ so $\lambda_1 = 0$ and $K = 0$. □

2.2 Commutative C^* algebras

2.12 DEFINITION (Hermitian character)**.** Let $\mathscr{A}$ be a C^* algebra. A character φ of $\mathscr{A}$ is *Hermitian* in case for all $A \in \mathscr{A}$,

$$\varphi(A^*) = \overline{\varphi(A)} \ .$$

2.13 LEMMA. *All characters of a C^* algebra are Hermitian.*

Proof. For any $A \in \mathscr{A}$ define $X = \dfrac{1}{2}(A + A^*)$ and $Y = \dfrac{1}{2i}(A - A^*)$. Then X and Y are self-adjoint, and $A = X + iY$. For any character φ of $\mathscr{A}$,

$$\varphi(A) = \varphi(X + iY) = \varphi(X) + i\varphi(Y) \quad \text{and} \quad \varphi(A^*) = \varphi(X - iY) = \varphi(X) - i\varphi(Y) \ .$$

By Lemma 1.24 and Theorem 2.9, $\varphi(X), \varphi(Y) \in \mathbb{R}$, and hence $\varphi(A^*) = \overline{\varphi(A)}$. □

2.14 THEOREM (Commutative Gelfand-Neumark Theorem)**.** *Let $\mathscr{A}$ be a commutative C^*-algebra. Then the Gelfand transform is an isometric isomorphism of $\mathscr{A}$ onto $\mathscr{C}_0(\Delta(\mathscr{A}))$.*

Proof. The Gelfand transform is a $*$-homomorphism on account of Lemma 2.13: For all $A \in \mathscr{A}$ and all $\varphi \in \Delta(\mathscr{A})$:

$$\gamma(A^*)[\varphi] = \varphi(A^*) = \overline{(\varphi(A))} = \gamma(A)^*[\varphi] \ .$$

By the easy Lemma 1.24, $|\gamma(A)[\varphi]| = |\varphi(A)| \leq \|A\|$, so that $\|\gamma(A)\| \leq \|A\|$. To show that $\|\gamma(A)\| \geq \|A\|$, note that in a commutative C^* algebra, *every* element is normal, and hence $\|A\| = \nu(A)$ for all A. By Corollary 1.34 of Theorem 1.33, there exists a character φ with $\nu(A) = |\varphi(A)| = |\gamma(A)(\varphi)|$. Hence $\|\gamma(A)\| \geq \nu(A) = \|A\|$.

This proves that the Gelfand transform is an isometry, and hence is injective onto a subalgebra $\gamma(\mathscr{A})$ of $\mathscr{C}_0(\Delta(\mathscr{A}))$. However, $\gamma(\mathscr{A})$ separates points, and does not vanish at any $\varphi \in \Delta(\mathscr{A})$, and is closed under complex conjugation. Hence by the Stone-Wierstrass Theorem, and the fact that $\gamma(\mathscr{A})$ is closed, $\gamma(\mathscr{A}) = \mathscr{C}_0(\Delta(\mathscr{A}))$. □

2.3 Spectral invariance and the Abstract Spectral Theorem

Let $\mathscr{A}$ be a Banach algebra with identity, and let $\mathscr{B}$ be a Banach subalgebra. It can happen that some $B \in \mathscr{B}$ is not invertible in $\mathscr{B}$, but is invertible in $\mathscr{A}$.

2.15 EXAMPLE. Let D denote the closed unit disc in $\mathbb{C}$, and let S^1 denote its boundary, the unit circle. Let $\mathscr{A} = \mathscr{C}(S^1)$, the algebra of continuous functions on S^1. Let $\mathscr{B}$ denote the algebra of continuous functions on D that are holomorphic in the interior of D. These functions are determined by their values on S^1, and their maximum absolute value is attained on S^1. Therefore, restriction to S^1 is an isometric embedding of $\mathscr{B}$ in $\mathscr{A}$, so we may regard $\mathscr{B}$ as a subalgebra of $\mathscr{A}$.

Let B denote the function $f(e^{i\theta}) = e^{i\theta}$, which evidently belongs to $\mathscr{B}$. Then $1\lambda - B$ is invertible in $\mathscr{A}$ if and only if $\lambda \notin S^1$, in which case the inverse is the function $g(e^{i\theta}) = (\lambda - e^{i\theta})^{-1}$. However, for λ in the interior of D, $\zeta \mapsto (\lambda - \zeta)^{-1}$ is not holomorphic in the interior of D, and so the inverse of B in $\mathscr{A}$ does not belong to $\mathscr{B}$. That is $\sigma_{\mathscr{A}}(B) = S^1$, but $\sigma_{\mathscr{B}}(B) = D$.

Now we specialize to C^* algebras, first introducing certain minimal subalgebras:

2.16 DEFINITION. Let $\mathscr{A}$ be a C^* algebra, and S a subset of $\mathscr{A}$. Then $C(S)$ is the smallest C^* subalgebra of $\mathscr{A}$ that contains S.

2.17 THEOREM. *Let $\mathscr{A}$ be a C^* algebra, and let $\mathscr{B}$ be a C^* subalgebra of $\mathscr{A}$. Suppose either that $\mathscr{A}$ is unital with unit 1 and that $1 \in \mathscr{B}$ or else that neither $\mathscr{A}$ nor $\mathscr{B}$ is unital. Then for all $B \in \mathscr{B}$,*

$$\sigma_{\mathscr{B}}(B) = \sigma_{\mathscr{A}}(B) \ . \tag{2.8}$$

Proof. It is evident that $\sigma_{\mathscr{A}}(B) \subset \sigma_{\mathscr{B}}(B)$, and we need only prove that $\sigma_{\mathscr{B}}(B) \subset \sigma_{\mathscr{A}}(B)$

Suppose first that $\mathscr{A}$ is unital with unit 1 and that $1 \in \mathscr{B}$. We first prove (2.8) for B self adjoint: Suppose that B is self adjoint and invertible in $\mathscr{A}$. By Theorem 2.9, $\sigma_{C(\{1,B\})}(B) \subset \mathbb{R}$, and consequently, for all $n \in \mathbb{N}$, $B - (i/n)1$ is invertible in $C(\{1,B\})$. Since $\lim_{n\to\infty}(B - (i/n)1) = B$ in $\mathscr{A}$, and the inverse is continuous, $\lim_{n\to\infty}(B - (i/n)1)^{-1} = B^{-1}$ in $\mathscr{A}$. But since $(B - (i/n)1)^{-1} \in C(\{1,B\})$ for all n, and since $C(\{1,B\})$ is closed,

$$B^{-1} = \lim_{n\to\infty}(B - (i/n)1)^{-1} \in C(\{1,B\}) \ .$$

Hence, B is invertible within $C(\{1,B\})$.

Now let B be any invertible element of $\mathscr{A}$. Then B^* and B^*B are invertible in $\mathscr{A}$, and also belong to $C(\{1,B\})$. Since B^*B is self adjoint, what we have proved above says that $(B^*B)^{-1} \in C(\{1,B\})$. Define $X = (B^*B)^{-1}B^* \in C(\{1,B\})$. Evidently $XB = 1$. Thus, B has a left inverse in $C(\{1,B\})$. The same argument shows that $Y = B^*(BB^*)^{-1}$ is a well defined right inverse of B in $C(\{1,B\})$, and then $X = X(BY) = (XB)Y = Y$ so $X = Y$ is the inverse of B in $C(\{1,B\})$. In particular, for all $\lambda \in \mathbb{C}$, $\lambda 1 - B$ is invertible in $C(\{1,B\})$ if and only if it is invertible in $\mathscr{A}$. Thus, $\lambda 1 - B$ is invertible in $\mathscr{A}$ if and only if it is invertible in $C(\{1,B\})$, and this proves (2.8).

Next, consider the case in which neither $\mathscr{A}$ nor $\mathscr{B}$ are unital. Again, let $B \in \mathscr{B}$ be self adjoint. Then $\sigma_{\mathscr{B}}(B) \subset \mathbb{R}$. Let $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ be the Banach $*$ algebras obtained by using the standard construction to adjoin a unit; note that it is the same unit 1 that is adjoined to both $\mathscr{A}$ and $\mathscr{B}$. Although $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ need not be C^* algebras, this does not matter. Suppose that $(1,B)$ is invertible in $\widetilde{\mathscr{A}}$. Then since $\sigma_{\widetilde{\mathscr{B}}}((1,B))$ is real, $(1+it,B)$ is invertible in $C(\{(1,0),(1,B)\}$ for all $t \in \mathbb{R}$. Taking limits, we conclude as above that the inverse belongs to $C(\{(1,0),(1,B)\}$.

Now let $(1,B)$ be any invertible element of $\widetilde{\mathscr{A}}$. Then $(1,B)^*$ and $(1,B)^*(1,B)$ are invertible in $\widetilde{\mathscr{A}}$, and also belong to $C(\{(1,0),(1,B)\})$. By what we have proved above, $((1,B)^*(1,B))^{-1} \in C(\{(1,0),(1,B)\}$. The argument proceeds from here just as in the case in the unital case considered at the beginning. $\square$

2.18 LEMMA. *Let $\mathscr{A}$ be a unital C^* algebra, and let $A \in \mathscr{A}$ be normal. Then the map $\varphi \mapsto \varphi(A)$ is a homeomorphism of $\Delta(C(\{1,A\}))$ onto $\sigma_{\mathscr{A}}(A)$. Let $\mathscr{A}$ be a non-unital C^* algebra, and let $A \in \mathscr{A}$ be normal. Then the map $\varphi \mapsto \varphi(A)$ is a homeomorphism of $\Delta'(C(\{A\}))$ onto $\sigma_{\mathscr{A}}(A)$.*

Proof. First suppose that $\mathscr{A}$ is unital. Since A and A^* commute, the closure of the linear span of $\{\ A^m(A^*)^n \ : m,n \geq 0\ \}$ is a C^* algebra that contains 1 and A. Evidently, it is $C(\{1,A\})$. Hence if $\varphi \in \Delta(C(\{1,A\}))$, φ is determined by its values on A and A^*. In fact, since φ is necessarily Hermitian, φ is determined by its value on A. That is, for any $\varphi,\psi \in \Delta(C(\{1,A\}))$,

$$\varphi = \psi \qquad \Longleftrightarrow \qquad \varphi(A) = \psi(A)\ . \tag{2.9}$$

We have also seen that for all $\varphi \in \Delta(C(\{1,A\}))$, $\varphi(A) \in \sigma_{C(\{1,A\})}(A) = \sigma_{\mathscr{A}}(A)$, and for all $\lambda \in \sigma_{\mathscr{A}}(A) = \sigma_{C(\{1,A\})}(A)$, there is a $\varphi_\lambda \in \Delta(C(\{1,A\}))$ such that $\varphi_\lambda(A) = \lambda$. This shows that the map $\varphi \mapsto \varphi(A)$ is a one-to-one map of $\Delta(C(\{1,A\}))$ onto $\sigma_{\mathscr{A}}(A)$. This map is also continuous by the definition of the Gelfand topology, and continuous bijections between compact spaces are homeomorphisms.

The case in which $\mathscr{A}$ is non-unital is similar. Since A and A^* commute, the closure of the linear span of $\{\ A^m(A^*)^n \ : m,n \geq 0\ , m+n > 0\ \}$ is a C^* algebra that contains A. Evidently, it is $C(\{A\})$. As before, (2.9) is valid, and $\varphi(A) = 0$ if and only if φ is the zero character. Then, as before $\varphi \mapsto \varphi(A) \in \sigma_{C(\{A\})}(A)$ is a one to one continuous map from the compact set $\Delta'(C(\{A\}))$ to the compact set $\sigma_{C(\{A\})}(A) = \sigma_{\mathscr{A}}(A)$, and is a homeomorphism. $\square$

We now come to the Abstract Spectral Theorem:

2.19 THEOREM (Abstract Spectral Theorem)**.** *Let $\mathscr{A}$ be a C^* algebra with identity* 1*, and let $A \in \mathscr{A}$ be normal. Identifying $\mathscr{C}_{\Delta(C(\{1,A\}))}$ and $\mathscr{C}(\sigma_{\mathscr{A}}(A))$ through the homeomorphism provided by Lemma 2.18, we may regard the Gelfand transform as a homomorphism of $C(\{1,A\})$*

into $\mathscr{C}(\sigma_{\mathscr{A}}(A))$. Then the Gelfand transform γ is an isometric isomorphism of $C(\{1, A\})$ onto $\mathscr{C}(\sigma_{\mathscr{A}}(A))$. For all non-negative integers m, n, $\gamma(A^m(A^*)^n)$ is the function on $\sigma_{\mathscr{A}}(A)$ given by

$$\lambda \mapsto \lambda^m \overline{\lambda}^n \ . \tag{2.10}$$

In case $\mathscr{A}$ is non-unital, the homeomorphism provided by Lemma 2.18 identifies $C(\{A\})$ with $\mathscr{C}_0(\sigma_{\mathscr{A}}(A)\backslash\{0\})$; i.e., the C^* algebra of continuous functions f on $\sigma_{\mathscr{A}}(A)$ that vanish at 0, and the Gelfand transform γ is an isometric isomorphism of $C(\{A\})$ onto $\mathscr{C}_0(\sigma_{\mathscr{A}}(A)\backslash\{0\})$, and (2.10) is valid for all $m, n \geq 1$.

Proof. The Commutative Gelfand-Neumark Theorem says that γ is an isometric isomorphism, and if $\varphi \in \Delta(C(A))$,

$$\gamma(A^m(A^*)^n)[\varphi] = \varphi(A)^m((\varphi(A))^*)^n = \lambda^m\overline{\lambda}^n$$

for $\lambda = \varphi(A)$ so that under the identification provided by Lemma 2.18, $\gamma(A^m(A^*)^m)$ is indeed given by (2.10). □

2.20 DEFINITION. For a unital C^* algebra $\mathscr{A}$ and for any normal element A of $\mathscr{A}$, and any $f \in \mathscr{C}(\sigma_{\mathscr{A}}(A))$, $f(A)$ is defined by $\gamma^{-1}(f)$; i.e., $f(A)$ is the inverse image of f under the isometric isomorphism of $C(\{1, A\})$ onto $\mathscr{C}(\sigma_{\mathscr{A}}(A))$ that is provided by the Commutative Gelfand Neumark Theorem. Likewise, if $\mathscr{A}$ is not unital, and f is a continuous function of $\sigma_{\mathscr{A}}(A)$ with $f(0) = 0$, $f(A)$ is defined by $\gamma^{-1}(f)$.

2.21 THEOREM (Spectral Mapping Theorem)**.** *Let $\mathscr{A}$ be a C^* algebra, A a normal element of $\mathscr{A}$, and $f \in \mathscr{C}(\sigma_{\mathscr{A}}(a))$. Then*

$$\sigma_{\mathscr{A}}(f(A)) = f(\sigma_{\mathscr{A}}(A)) \ .$$

Proof. First suppose that $\mathscr{A}$ is unital. For all $\mu \in \mathbb{C}$, the function $\lambda \mapsto \mu - f(\lambda)$ is invertible in $\mathscr{C}(\sigma_{\mathscr{A}}(A))$ if and only if μ does not belong to the range of f, which is $f(\sigma_{\mathscr{A}}(A))$. Then, using the isomorphism provided by the Commutative Gelfand Neumark Theorem, we see that $\mu 1 - f(A)$ is invertible in $C(\{1, A\})$ if and only if $\mu \notin f(\sigma_{\mathscr{A}}(A))$, and hence $\sigma_{C(\{1,A\})}(f(A)) = f(\sigma_{\mathscr{A}}(A))$. Finally, by Theorem 2.17, the spectrum of $f(A)$ is the same in all C^* subalgebras of $\mathscr{A}$ that contain $f(A)$ and 1. In particular, $\sigma_{C(\{1,A\})}(f(A)) = \sigma_{\mathscr{A}}(f(A))$.

Their argument in the non-unital case is essentially the same, except we must use quasi inverses to characterize the spectrum. □

2.22 THEOREM. *Let $\mathscr{A}$ be a C^* algebra with identity 1. Let $U \subset \mathbb{C}$ be open with $\overline{U}$ compact. Let $\mathscr{N}_U$ be given by*

$$\mathscr{N}_U = \{\ A \in \mathscr{A}\ : AA^* = A^*A \quad \text{and} \quad \sigma_{\mathscr{A}}(A) \subset U\ \} \ . \tag{2.11}$$

Then $\mathscr{N}_U$ is an open subset of the normal elements of $\mathscr{A}$. Moreover, let f be a continuous complex valued function on $\overline{U}$, and for all $A \in \mathscr{N}_U$, define $f(A) \in \mathscr{A}$ using the Abstract Spectral Theorem. Then the map $A \mapsto f(A)$ is continuous on $\mathscr{N}_U$.

Proof. The first assertion is an immediate consequence of Newburg's Theorem, Theorem 1.22. For the second, first suppose that p is an monomial; i.e., $p(\lambda) = \lambda^n$. Then the telescoping sum identity and the obvious estimates yield $\|p(B) - p(A)\| \leq n(\max\{\|A\|, \|B\|\})^{n-1}\|B - A\|$. It follows that for any polynomial p, the map $A \mapsto p(A)$ is continuous. Now consider any sequence $\{p_n\}$ of polynomials converging uniformly to f on $\overline{U}$. Then for all $A \in \mathscr{N}_U$,

$$\|p_n(A) - f(A)\| \leq \sup_{\lambda \in \overline{U}} \{|p_n(\lambda) - f(\lambda)|\}.$$

That is,

$$\lim_{n \to \infty} \left(\sup_{A \in \mathscr{N}_U} \{ \ \|p_n(A) - f(A)\| \ \} \right) = 0 \ .$$

Thus, the function $A \mapsto f(A)$ is the norm limit of the continuous functions $A \mapsto p_n(A)$. □

2.23 COROLLARY. *Let A be a normal element of a C^* algebra $\mathscr{A}$. Let g be continuous on $\sigma_{\mathscr{A}}(A)$. Let f be continuous on a compact set K containing $\sigma_{\mathscr{A}}(g(A))$ in its interior. Then $f(g(A)) = f \circ g(A)$.*

Proof. Fix $\epsilon > 0$, and choose a polynomial q such that

$$\sup_{\lambda \in K} \{|f(\lambda) - q(\lambda)|\} < \epsilon \ . \tag{2.12}$$

Let $\{p_n\}_{n \in \mathbb{N}}$ be a sequence of polynomials converging uniformly to g on $\sigma_{\mathscr{A}}(A)$. By Theorem 2.22, for all n sufficiently large, $\sigma_{\mathscr{A}}(p_n(A)) \subset K^\circ$. It is evident that $q \circ p_n(A) = q(p_n(A))$. By (2.12), and this identity,

$$\|f(p_n(A)) - q(p_n(A))\| = \|f(p_n(A)) - q \circ p_n(A)\| \leq 2\epsilon$$

for all n sufficiently large, and then by Theorem 2.22, $\|f(g(A)) - q \circ g(A))\| < 2\epsilon$. By (2.12) once more, $\sup_{\lambda \in \sigma_{\mathscr{A}}(A)} \{|f \circ g(\lambda) - q \circ g(\lambda)|\} < \epsilon$, and hence $\|q \circ g(A) - f \circ g(A)\| < \epsilon$. Altogether, $\|f(g(A)) - f \circ g(A)\| < 3\epsilon$, and since $\epsilon > 0$ is arbitrary, the proof is complete. □

2.4 Positivity in C^* algebras

2.24 DEFINITION. Let $\mathscr{A}$ be a C^* algebra. The *positive* elements of $\mathscr{A}$ are the self adjoint elements A such that $\sigma_{\mathscr{A}}(A) \subset [0, \infty)$. The set of all positive elements of $\mathscr{A}$ is denoted $\mathscr{A}^+$.

If $A \in \mathscr{A}^+$, we may use the Abstract Spectral Theorem to define $\sqrt{A}$, and then $A = (\sqrt{A})^2 = (\sqrt{A})^*(\sqrt{A})$. One of the 1943 conjectures of Gelfand and Neumark was that for all $B \in \mathscr{A}$, B^*B is positive. This conjecture was finally proved in 1952 and 1953 through the contributions of Fukamiya and Kaplansky, for the case of unital C^* algebras. (Gelfand and Neumark only considered unital C^*-algebras.) The history is interesting: Kaplansky had managed to prove that *if* the sum of two positive elements is necessarily positive, then B^*B is necessarily positive. However, he was unable to show that $\mathscr{A}^+$ was closed under addition. He published nothing, but showed his proof to many people. When Fukamiya proved the closure of $\mathscr{A}^+$ under addition in 1952, the reviewer of Fukamiya's paper in Math Reviews knew of Kaplansky's proof, and Kaplansky gave him permission to publish it in the review. Finally, in 1956, Bonsall removed the hypothesis that $\mathscr{A}$ be unital.

2.25 THEOREM (Fukamiya's Theorem)**.** *Let $\mathscr{A}$ be a unital C^* algebra. Then $\mathscr{A}^+$ is a closed, pointed convex cone. That is:*

(1) For all $\lambda \in \mathbb{R}^+$, and all $A \in \mathscr{A}^+$, $\lambda A \in \mathscr{A}^+$, and for all $A, B \in \mathscr{A}^+$, $A + B \in \mathscr{A}^+$.

(2) $-\mathscr{A}^+ \cap \mathscr{A}^+ = \{0\}$.

(The first part says that $\mathscr{A}^+$ is a convex cone; the second part says that this cone is pointed.)

Proof. Let $B_{\mathscr{A}}$ denote the closed unit ball in $\mathscr{A}$. Fukamiya observed that $\mathscr{A}^+ \cap B_{\mathscr{A}}$ consists precisely of the self-adjoint elements A with both A and $1-A$ in $B_{\mathscr{A}}$. To see this suppose that $A \in \mathscr{A}^+ \cap B_{\mathscr{A}}$. Since A is self adjoint, $\nu(A) = \|A\| \le 1$, and so $\sigma_{\mathscr{A}}(A) \subset [0,1]$. By (an easy case of) the Spectral Mapping Lemma, $\sigma_{\mathscr{A}}(1-A) \subset [0,1]$, and hence $\|1-A\| = \nu(1-A) \le 1$.

Conversely, suppose A is self-adjoint and both A and $1-A$ are in $B_{\mathscr{A}}$. Since A is self adjoint and $\|A\| \le 1$, $\sigma_{\mathscr{A}}(A) \subset [-1,1]$. Then by the Spectral Mapping Lemma, $\sigma_{\mathscr{A}}(1-A) \subset [0,2]$. However, if $\|1-A\| \le 1$, then $\sigma_{\mathscr{A}}(1-A) \subset [-1,1]$, and altogether we have that $\sigma_{\mathscr{A}}(1-A) \subset [0,1]$, and then by the identity $A = 1-(1-A)$ and the Spectral Mapping Lemma once more, $\sigma_{\mathscr{A}}(A) \subset [0,1]$, so that $A \in \mathscr{A}^+$. That is,

$$\mathscr{A}^+ \cap B_{\mathscr{A}} = \{\ A \in \mathscr{A}_{\text{s.a.}}\ :\ A\ ,\ 1-A \in B_{\mathscr{A}}\ \}\ . \tag{2.13}$$

Let $A, B \in B_{\mathscr{A}}$. By Minkowski's inequality, $\|(A+B)/2\| \in B_{\mathscr{A}}$ and

$$\left\| 1 - \frac{A+B}{2} \right\| \le \frac{1}{2}(\|1-A\| + \|1-B\|)\ . \tag{2.14}$$

If furthermore, $A, B \in \mathscr{A}^+$, then we also have that $\|1-A\| \le 1$ and $\|1-B\| \le 1$, and then from (2.14), $\|1-(A+B)/2\| \le 1$. Thus, $C := (A+B)/2$ is self adjoint and both C and $1-C$ belong to $B_{\mathscr{A}}$. Therefore, $(A+B)/2 \in \mathscr{A}^+$.

Since the closure of $\mathscr{A}^+$ under positive multiples is clear, it then clear that $\mathscr{A}^+$ is a cone. If $A \in \mathscr{A}^+$ and $-A \in \mathscr{A}^+$, then $\sigma_{\mathscr{A}}(A) \subset (-\infty,0] \cap [0,\infty) = \{0\}$, so that $\|A\| = \nu(a) = 0$, and hence $A = 0$. The norm closure of $\mathscr{A}^+ \cap B_{\mathscr{A}}$ is evident from (2.13) together with Theorem 2.11, and then the closure of $\mathscr{A}^+$ follows by a simple scaling argument. □

2.26 THEOREM (Bonsall's Theorem)**.** *The condition in Fukamiya's Theorem that $\mathscr{A}$ be unital is superfluous.*

Proof. Let $\mathscr{A}$ be a non-unital C^* algebra. It suffices to show that $\mathscr{A}^+$ is closed under addition, and closed in norm, since the unital nature of $\mathscr{A}$ was used only in proving these fundamental parts. We begin with closure under addition. Since $A \in \mathscr{A}^+$ if and only if A is self adjoint and $(\lambda, -A)$ is invertible in $\widetilde{\mathscr{A}}$ for all $\lambda < 0$, $A \in \mathscr{A}^+$ if and only if A is self adjoint and tA is quasi regular for all $t > 0$. Now let $A, B \in \mathscr{A}^+$. Then $A+B$ is self adjoint, and to show that $A+B$ in $\mathscr{A}^+$, it suffices to show that for all $t > 0$, $t(A+B) = (tA+tB)$ is quasi regular. Hence it suffices to show that the sum of quasi regular elements in $\mathscr{A}^+$ is quasi regular.

Let $A \in \mathscr{A}^+$. We first show that A', the quasi inverse of A, satisfies $-A' \in \mathscr{A}^+$ and $\|A'\| < 1$. Indeed, by the Abstract Spectral Theorem in $C(\{A\})$, $A' = f(A)$ where

$$f(t) = \frac{-t}{1+t}\ ,$$

since in $\mathscr{C}_0(\sigma_{\mathscr{A}}(A)\backslash\{0\})$, $t + f(t) + tf(t) = t + t(f) + f(t)t = 0$. By the Spectral Mapping Theorem,

$$\sigma_{\mathscr{A}}(A') = \{-\lambda/(1+\lambda)\ :\ \lambda \in \sigma_{\mathscr{A}}(A)\} \subset (-1,0]\ .$$

Hence $\|A\| = \nu(A) < 1$, and $-A \in \mathscr{A}^+$.

Now let $A, B \in \mathscr{A}^+$. Then since $\|A'\|, \|B'\| < 1$, $\|A'B'\| < 1$ and hence $(1, -A'B')$ is invertible in $\widetilde{\mathscr{A}}$. That is, $-A'B'$ is quasi regular. Now a simple calculation shows that $A \circ (-A'B') \circ B = A+B$, and by Lemma 1.10, $A+B$ is quasi regular.

For the closure in norm, let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{A}^+$ converging to B. For each $\epsilon > 0$, $n \in \mathbb{N}$, (ϵ, A_n) is invertible in $\widetilde{\mathscr{A}}$, and by the above, with $f(t) = -t/(1+t)$, $(\epsilon, A_n)^{-1} = \frac{1}{\epsilon}\left(1, f(\epsilon^{-1}A_n)\right)$, and hence $\left\|(\epsilon, A_n)^{-1}\right\| \leq \dfrac{1 + \|f(\epsilon^{-1}A_n)\|}{\epsilon} \leq \dfrac{2}{\epsilon}$. Since $\|(\epsilon, B) - (\epsilon, A_n)\| = \|B - A_n\|$, by (1.13) and Lemma 1.13, since $\|B - A_n\| < \epsilon/2$ for all sufficiently large n, (ϵ, B) is invertible. Since $\epsilon > 0$ is arbitrary, $B \in \mathscr{A}^+$. □

2.27 THEOREM (Fukamiya-Kaplansky-Bonsall Theorem)**.** *Let $\mathscr{A}$ be a C^* algebra. For all $A \in \mathscr{A}$, $A^*A \in \mathscr{A}^+$.*

Proof. We first show that if $A^*A \in -\mathscr{A}^+$, then $A^*A = 0$. Since A^*A and AA^* have the same spectrum, Fukamiya's Theorem says that $A^*A + AA^* \in -\mathscr{A}^+$. However, writing $A = X + iY$ with X and Y self adjoint,

$$A^*A + AA^* = 2(X^2 + Y^2) \in \mathscr{A}^+ \ ,$$

where once again we have used Fukamiya's Theorem, and the Spectral Mapping Lemma. Since $\mathscr{A}^+$ is a pointed cone, this means that $A^*A + AA^* = 0$. But then $A^*A = (A^*A + AA^*) - AA^* = -AA^* \in \mathscr{A}^+$. Again since $\mathscr{A}^+$ is pointed, this means that $A^*A = 0$, as claimed.

Define continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ by $f(t) = \max\{t, 0\}$ and $g(t) = t - f(t)$. Our goal is to show that for all $B \in \mathscr{A}$, $Z := g(B^*B) = 0$, since then $B^*B = f(B^*B) \in \mathscr{A}^+$.

Note that $f(t)g(t) = 0$ for all t, and hence with $Y := f(B^*B)$, $YZ = 0$. Then

$$(BZ)^*(BZ) = ZB^*BZ = Z(Y + Z)Z = Z^3 \in -\mathscr{A}^+ \ .$$

The first part of the proof says that $(BZ)^*(BZ) = Z^3 = 0$, and hence $Z = 0$. □

2.28 DEFINITION. Let $\mathscr{A}$ and $\mathscr{B}$ be C^* algebras. A linear transformation $\Phi : \mathscr{A} \to \mathscr{B}$ is *positive* in case $\Phi(A) \in \mathscr{B}^+$ for all $A \in \mathscr{A}^+$. Define $\mathscr{L}^+(\mathscr{A}, \mathscr{B})$ to be the set of positive linear transformations from $\mathscr{A}$ to $\mathscr{B}$. The term *positive map* is a synonym for *positive linear transformation.*

2.29 COROLLARY. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^* algebras. Then $\mathscr{L}^+(\mathscr{A}, \mathscr{B})$ is a pointed cone in $\mathscr{L}(\mathscr{A}, \mathscr{B})$, the space of linear transformations from $\mathscr{A}$ to $\mathscr{B}$.*

Proof. It is evident that $\mathscr{L}^+(\mathscr{A}, \mathscr{B})$ is closed under multiplication by non-negative scalars, and it is closed under addition as an immediate consequence of Theorem 2.27. If $\Phi \in \mathscr{L}^+(\mathscr{A}, \mathscr{B})$ and $-\Phi \in \mathscr{L}^+(\mathscr{A}, \mathscr{B})$, then for all $A \in \mathscr{A}^+$, $\Phi(A) \in \mathscr{B}^+ \cap (-\mathscr{B}^+) = \{0\}$ by Theorem 2.25 and Theorem 2.26. Since every element of $\mathscr{A}$ is a linear combination of at most 4 elements of $\mathscr{A}^+$, $\Phi(A) = 0$ for all $A \in \mathscr{A}$. This proves that $\mathscr{L}^+(\mathscr{A}, \mathscr{B})$ is a pointed cone. □

The next corollary of Theorem 2.27 gives us an important class of positive maps.

2.30 COROLLARY. *Let $\mathscr{A}$ be a C^* algebra, and let $B \in \mathscr{A}$. Then the linear transformation $\Phi : \mathscr{A} \to \mathscr{A}$ given by $\Phi(A) := B^*AB$ is positive.*

Proof. Let $A \in \mathscr{A}^+$. Then since $f(t) := \sqrt{t}$ is continuous and vanishes at 0, we may use the Abstract Spectral Theorem to define $A^{1/2} \in \mathscr{A}^+$ with $A = A^{1/2}A^{1/2}$. Then

$$\Phi(A) = B^*A^{1/2}A^{1/2}B = (A^{1/2}B)^*(A^{1/2}B) \in \mathscr{A}^+ \ ,$$

where we used Theorem 2.27 in the final step. □

In the next section we shall make use of the following lemma:

2.31 LEMMA (Vowden's Lemma). *Let $\mathscr{A}$ be a C^* algebra. For all $A, B \in \mathscr{A}^+$ such that $B - A \in \mathscr{A}^+$, $\|B\| \geq \|A\|$.*

When $\mathscr{A}$ has a unit, this is an easy consequence of Fukamiya observation that $\mathscr{A}^+ \cap B_{\mathscr{A}}$ consists precisely of the self-adjoint elements X with both X and $1 - X$ in $B_{\mathscr{A}}$: We may suppose without loss of generality that $\|B\| = 1$. Then $1 - B$ is positive. By Theorem 2.27, $1 - A = (1 - B) + (B - A)$ is positive, but if $\|A\| > 1$, there exists $\lambda \in \sigma_{\mathscr{A}}(S)$ with $\lambda > 1$, and then $1 - A$ is not positive. Hence $\|A\| \leq 1 = \|B\|$.

Vowden has given a proof of the same result without assuming the existence of an identity; his proof makes use of the Abstract Spectral Theorem and the main theorems of this section:

Proof of Lemma 2.31. Assume $\|B\| = 1$. We must show that $\|A\| \leq 1$. If we show that $A^2 - A^3 \geq 0$, then since for $\lambda > 1$, $\lambda^2 - \lambda^3 < 0$, no such λ can belong to $\sigma_{\mathscr{A}}(A)$, and then $\sigma_{\mathscr{A}}(A) \subset [0, 1]$, and hence $\|A\| \leq 1$.

We now show that $A^2 - A^3 \in \mathscr{A}^+$. For any $C \in \mathscr{A}_{\text{s.a.}}$, define $X = A - CA$. Then $X^*X = A^2 + A(C^2 - 2C)A$. If we can choose C so that $C^2 - 2C + B = 0$, then

$$X^*X = A^2 - ABA = A^2 - A^3 - A(B - A)A \ .$$

By Corollary 2.30, and Theorem 2.25, $A^2 - A^3 = X^*X + A(B - A)A \in \mathscr{A}^+$,

To produce C, note that the quadratic polynomial $x^2 - 2x + b = 0$ has the roots $x = 1 \pm \sqrt{1 - b}$. Therefore, define the function $f(t)$ on $[0, 1]$ by$f(t) = 1 - \sqrt{1 - t}$. Although the formula for f makes explicit reference to 1, f is continuous and $f(0) = 0$. Therefore, $f(B)$ is defined for any $B \in \mathscr{A}$ with $\sigma_{\mathscr{A}}(B) \subset [0, 1]$, even when $\mathscr{A}$ is not unital. Now define $C = f(B)$, and then $C^2 - 2C + B = 0$. $\square$

2.5 Adjoining a unit to a non-unital C^* algebra

We will make use of the following approximation lemma. Since it has many uses, we state it in a more general form than we need at present to avoid repeating ourselves later on.

2.32 THEOREM. *Let $\mathscr{A}$ be a C^* algebra, and let $\{A_1, \ldots, A_m\}$ be any finite subset of $\mathscr{A}$. Then there is a sequence $\{E_n\}_{n\in\mathbb{N}} \subset B_{\mathscr{A}} \cap \mathscr{A}^+$ such that for each $j = 1, \ldots, m$,*

$$\lim_{n\to\infty} \|A_j E_n - A_j\| = 0 \ . \tag{2.15}$$

Furthermore:

(1) Let $\mathscr{J}$ be a closed ideal in $\mathscr{A}$, and suppose that $\{A_1, \ldots, A_m\} \subset \mathscr{J}$. Then we may take $\{E_n\}_{n\in\mathbb{N}} \subset \mathscr{J}$, and still have that (2.15) *is valid for each $j = 1, \ldots, m$.*

(2) If $\{A_1, \ldots, A_m\}$ is closed under the involution, we may choose $\{E_n\}_{n\in N}$ to have the additional property that both (2.15) *and*

$$\lim_{n\to\infty} \|E_n A_j - A_j\| = 0 \tag{2.16}$$

are valid for $j = 1, \ldots, m$.

2.33 REMARK. *Theorem 2.32 provides a kind of "approximate identity" in a C^* algebra, and one can make precise sense of this notion. However, in applications, Theorem 2.32 can be applied directly without going through this (short) detour.*

Proof of Theorem 2.32. Define $X := \sum_{j=1}^m A_j^* A_j \in \mathscr{A}^+$. Consider the sequence of continuous functions $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f_n(t) = \min\{nt, 1\}$. Note that

$$g_n(t) := t(1 - f_n(t))^2 = \begin{cases} t(1-nt)^2 & t \leq 1/n \\ 0 & t > 1/n \ . \end{cases}$$

Moreover, g_n is continuous, $g_n(0) = 0$ and $\sup_{t \geq 0} |g_n(t)| \leq 1/n$. By the Abstract Spectral Theorem, $\|g_n(X)\| \leq 1/n$. For each $j = 1, \dots, m$,

$$(A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j) = f_n(X) A_j^* A_j f_n(X) + A_j^* A_j - A_j^* A_j f_n(X) - f_n(X) A_j^* A_j \ .$$

Summing on j, $\sum_{j=1}^m (A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j) = g_n(X)$, and therefore,

$$\left\| \sum_{j=1}^m (A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j) \right\| = \|g_n(X)\| \leq \frac{1}{n} \ .$$

Since for each $j = 1, \dots, m$,

$$g_n(X) - (A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j) = \sum_{k \neq j, k=1}^n (A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j)$$

is a sum of terms in $\mathscr{A}^+$, Theorem 2.26 implies that $g_n(X) - (A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j) \in \mathscr{A}^+$. Lemma 2.31 then yields

$$\frac{1}{n} \geq \max_{j=1,\dots,m} \left\{ \|(A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j)\| \right\} \ .$$

By the C^* algebra identity, $\|(A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j)\| = \|(A_j f_n(X) - A_j)^*\| \|A_j f_n(X) - A\|$, and then by the continuity of the involution, there is a $c > 0$ such that for each $j = 1, \dots, m$,

$$\frac{1}{n} \geq c \|A_j f_n(X) - A_j\|^2 \ .$$

Thus, $\lim_{n \to \infty} \|A_j f_n(X) - A_j\| = 0$. By the Abstract Spectral Theorem, $\|f_n(X)\| \leq 1$ and $f_n(X) \in \mathscr{A}^+$ for all n. Hence we may take $E_n = f_n(X)$, and then we have proved that $\lim_{n \to \infty} \|A_j E_n - A_j\| = 0$. This proves the first statement. To prove *(1)*, simply note that if $\{A_1, \dots, A_m\} \subset \mathscr{J}$, then $X \in \mathscr{J}$, and $E_n = f_n(X)$ is the norm limit of polynomials in $\mathscr{J}$, and then since $\mathscr{J}$ is closed, $E_n \in \mathscr{J}$.

To prove *(2)*, suppose that for each $j \in \{1, \dots, m\}$, there is some $k \in \{1, \dots, m\}$ so that $A_j^* = A_k$. Then, $\|E_n A_j - A_j\| = \|E_n A_k^* - A_k^*\| = \|(A_k E_n - A_k)^*\|$. Since the involution is continuous, (2.16) then follows from (2.15). □

2.34 LEMMA. *Let $\mathscr{A}$ be a C^* algebra. For all $A \in \mathscr{A}$,*

$$\sup_{\|B\| \leq 1} \{\|AB\|\} = \|A\| = \sup_{\|B\| \leq 1} \{\|BA\|\} \tag{2.17}$$

Proof. For $\|B\| \leq 1$, $\|AB\| \leq \|A\|$. For $A \neq 0$, define $B := \|A^*\|^{-1}A^*$. Then $\|B\| = 1$ and

$$\|AB\| = \frac{\|AA^*\|}{\|A^*\|} = \frac{\|A\|\|A^*\|}{\|A^*\|} = \|A\| \ .$$

Altogether, $\|A\| = \sup_{\|B\|\leq 1}\{\|AB\|\}$. The proof of the second identity is essentially the same. □

Now suppose that $\mathscr{A}$ is not unital, and let $\mathscr{B}(\mathscr{A})$ be the Banach space of bounded linear transformations on $\mathscr{A}$. We embed $\mathbb{C} \oplus \mathscr{A}$ into $\mathscr{B}(\mathscr{A})$ as follows: For $(\lambda, A) \in \mathbb{C} \oplus \mathscr{A}$, define $T_{(\lambda,A)} \in \mathscr{B}(\mathscr{A})$ by

$$T_{(\lambda,A)}B = \lambda B + AB$$

for all $B \in \mathscr{A}$. To see that $(\lambda, A) \mapsto T_{(\lambda,A)}$ is one-to-one, suppose that for some $(\lambda, A) \in \mathbb{C} \oplus \mathscr{A}$, $T_{(\lambda,A)} = 0$. If $\lambda = 0$, this is impossible unless also $A = 0$ since if $T_{(0,A)} = 0$, then $AB = 0$ for all B and then by Lemma 2.34, $A = 0$. However, if $T_{(\lambda,A)} = 0$ and $\lambda \neq 0$, replacing A by $-\lambda^{-1}A$, we have that $T_{(-1,A)} = 0$, but then for all $B \in \mathscr{A}$, $B = AB$. Taking $B = A^*$, $A^* = AA^*$, and hence A is self-adjoint. Then $B^* = B^*A$ for all $B \in \mathscr{A}$, and hence we have that $B = AB = BA$ for all $B \in \mathscr{A}$. This is impossible since $\mathscr{A}$ lacks a multiplicative identity. Hence $(\lambda, A) \mapsto T_{(\lambda,A)}$ is one-to-one from $\mathbb{C} \oplus \mathscr{A}$ into $\mathscr{B}(\mathscr{A})$. Define a norm $|\!|\!| \cdot |\!|\!|$ on $\mathbb{C} \oplus \mathscr{A}$ by $|\!|\!|(\lambda, A)|\!|\!| = \|T_{(\lambda,A)}\|_{\mathscr{B}(\mathscr{A})}$. That is,

$$|\!|\!|(\lambda, A)|\!|\!| = \sup_{\|B\|\leq 1} \{\|\lambda B + AB\|\} \ . \tag{2.18}$$

By Lemma 2.34.

$$|\!|\!|(0, A)|\!|\!| = \sup_{\|B\|\leq 1} \{\|AB\|\} = \|A\| \ , \tag{2.19}$$

and so $A \mapsto T_{(0,A)}$ is an isometry from $\mathscr{A}$ into $\mathscr{B}(\mathscr{A})$. In particular, the image of $\mathscr{A}$ under this embedding is closed. The image of $\mathbb{C} \oplus \mathscr{A}$ under this embedding is the sum of the image of $\mathscr{A}$ and a one dimensional subspace. Since the sum of a closed subspace and a finite dimensional subspace is always closed, the image of $\mathbb{C} \oplus \mathscr{A}$ under this embedding is closed in $\mathscr{B}(\mathscr{A})$, and hence complete since $\mathscr{B}(\mathscr{A})$ is a Banach space. It follows that $\mathbb{C} \oplus \mathscr{A}$ is complete in the norm $|\!|\!| \cdot |\!|\!|$. Since $\mathscr{B}(\mathscr{A})$ is a Banach algebra, $\mathbb{C} \oplus \mathscr{A}$ is a Banach algebra in this norm; it inherits the multiplicative property from $\mathscr{B}(\mathscr{A})$, and with the standard involution on $\mathbb{C} \oplus \mathscr{A}$, it is a Banach $*$-algebra. We have proved:

2.35 LEMMA. *Let $\mathscr{A}$ be a non-unital C^* algebra. Then $\mathbb{C} \oplus \mathscr{A}$, equipped with the standard multiplication defined in* (1.3)*, and the norm $|\!|\!| \cdot |\!|\!|$ defined in* (2.18) *is a Banach algebra. If we equip $\mathbb{C} \oplus \mathscr{A}$ with the involution $(\lambda, A)^* = (\overline{\lambda}, A^*)$, then it becomes a Banach $*$ algebra, and the map $A \mapsto (0, A)$ is an isometric $*$-isomorphism of $\mathscr{A}$ into this Banach $*$-algebra.*

2.36 LEMMA. *In the notation introduced above, we have the alternate formula*

$$|\!|\!|(\lambda, A)|\!|\!| = \sup_{\|C\|\leq 1} \{\|\lambda C + CA\|\} \ . \tag{2.20}$$

Proof. Choose $\epsilon > 0$, and choose $B \in \mathscr{A}$, $\|B\| \leq 1$ such that

$$(1 - \epsilon)|\!|\!|(\lambda, A)|\!|\!| \leq \|\lambda B + AB\| \ .$$

Then by the second identity in (2.17), there exists $C \in \mathscr{A}$, $\|C\| \leq 1$, so that

$$(1 - \epsilon)\|\lambda B + AB\| \leq \|(\lambda C + CA)B\| \leq \|\lambda C + CA\| \ .$$

Since $\epsilon > 0$ is arbitrary, $\sup_{\|C\|\le 1}\{\|\lambda C + CA\|\} \ge |\!|\!|(\lambda, A)|\!|\!| = \sup_{\|B\|\le 1}\{\|\lambda B + AB\|\}$. The same sort of reasoning then shows that $\displaystyle\sup_{\|B\|\le 1}\{\|\lambda B + AB\|\} \le \sup_{\|C\|\le 1}\{\|\lambda C + CA\|\}$. □

We are now ready to prove:

2.37 THEOREM (Vowden's Theorem)**.** *Let $\mathscr{A}$ be a non-unital C^* algebra. Then $\mathbb{C}\oplus\mathscr{A}$, equipped with the standard multiplication defined in* (1.3)*, the norm $|\!|\!|\cdot|\!|\!|$ defined in* (2.18) *and the involution $(\lambda, A)^* = (\overline{\lambda}, A^*)$, is a unital C^* algebra, and the map $A \mapsto (0, A)$ is an isometric $*$-isomorphism of $\mathscr{A}$ into this unital C^*-algebra. In particular, every C^* algebra $\mathscr{A}$ is isometrically and $*$-isomorphically embedded in a unital C^* algebra.*

Proof. We need only prove the C^* algebra identity $|\!|\!|(\lambda, A)^*(\lambda, A)|\!|\!| = |\!|\!|(\lambda, A)^*|\!|\!||\!|\!|(\lambda, A)|\!|\!|$, and since we already know that $|\!|\!|(\lambda, A)^*(\lambda, A)|\!|\!| \le |\!|\!|(\lambda, A)^*|\!|\!||\!|\!|(\lambda, A)|\!|\!|$, it remains to prove that

$$|\!|\!|(\lambda, A)^*(\lambda, A)|\!|\!| \ge |\!|\!|(\lambda, A)^*|\!|\!||\!|\!|(\lambda, A)|\!|\!| \ . \tag{2.21}$$

Choose $\epsilon > 0$. By Lemma 2.36, we may choose $B, C \in \mathscr{A}$ with $\|B\|, \|C\| \le 1$ such that

$$|\!|\!|(\lambda, A)^*|\!|\!| \le (1+\epsilon)\|\overline{\lambda}B + BA^*\| \quad \text{and} \quad |\!|\!|(\lambda, A)|\!|\!| \le (1+\epsilon)\|\lambda C + AC\| \ .$$

By Theorem 2.32 applied to $\{B, C\}$, for every $\delta > 0$, there exists $E \in \mathscr{A}^+$, with $\|E\| \le 1$ such that $\|BE - B\|, \|EC - C\| \le \delta$. For appropriate δ we then have

$$|\!|\!|(\lambda, A)|\!|\!| \le (1+\epsilon)^2\|(\lambda E + AE)C\| \le (1+\epsilon)^2\|\lambda E + AE\| \ .$$

and

$$|\!|\!|(\lambda, A)^*|\!|\!| \le (1+\epsilon)^2\|B(\overline{\lambda}E + EA^*)\| \le (1+\epsilon)^2\|(\lambda E + AE)^*\| \ .$$

Altogether we have

$$|\!|\!|(\lambda, A)^*|\!|\!||\!|\!|(\lambda, A)|\!|\!| \le (1+\epsilon)^4\|(\lambda E + AE)^*\|\|\lambda E + AE\| \ . \tag{2.22}$$

Then by the C^* algebra identity in $\mathscr{A}$,

$$\begin{aligned}
\|(\lambda E + AE)^*\|\|\lambda E + AE\| &= \|(\lambda E + AE)^*(\lambda E + AE)\| \\
&= \|(\overline{\lambda}E + EA^*)(\lambda E + AE)\| \\
&= \|E(|\lambda|^2 + (\overline{\lambda}A + \lambda A^* + A^*A))E\| \\
&\le \||\lambda|^2E + (\overline{\lambda}A + \lambda A^* + A^*A)E\| \\
&\le |\!|\!|(\lambda, A)^*(\lambda, A)|\!|\!| \ .
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, combining this with (2.22) yields the result. □

Vowden's Theorem has a number of consequences. Our first application of it, in the next section, will be the one that motivated Vowden – to finally prove that the involution is an isometry in full generality. This had been done earlier by Glimm and Kadison in the unital case, and Vowden's theorem immediately extends this to the general case.

2.6 The Russo-Dye Theorem and the Isometry Property

2.38 DEFINITION (Unitary)**.** Let $\mathscr{A}$ be a unital C^* algebra with multiplicative identity 1. Then $U \in \mathscr{A}$ is unitary in case

$$U^*U = UU^* = I \ . \tag{2.23}$$

Note in particular that every unitary U is normal. Therefore, by Lemma 2.8, $\|U^*\| = \|U\|$, and by the C^* algebra norm identity, $1 = \|1\| = \|U^*U\| = \|U^*\|\|U\| = \|U\|^2$. Therefore, $\|U\| = 1$ for all unitaries.

Noe let $A \in B_{\mathscr{A}}$. Define $(A^*A)^{1/2}$ using the Abstract Spectral Theorem, If A is invertible, so are A^* and A^*A, and then $(A^*A)^{-1}(A^*A)^{1/2}$ is the inverse of $(A^*A)^{1/2}$ which we denote by $(A^*A)^{-1/2}$. For such A, define

$$|A| := (A^*A)^{1/2} \qquad \text{and} \qquad V := A(A^*A)^{-1/2} \ . \tag{2.24}$$

Then $V^*V = (A^*A)^{-1/2}A^*A(A^*A)^{-1/2} = 1$. Since V is the product of invertible elements of $\mathscr{A}$, V is invertible, and hence its left inverse V^* is also its inverse. That is $V^*V = VV^* = 1$. In particular, V is unitary. This proves the greater part of the following lemma:

2.39 LEMMA. *Let $\mathscr{A}$ be a unital C^* algebra, and let $A \in \mathscr{A}$. If A is invertible, then there is a unitary V and a positive operator H such that $A = VH$. Moreover, H and V are uniquely determined, and are given by $H = |A| := (A^*A)^{1/2}$ and $V = A|A|^{-1}$.*

Proof. It remains to prove the uniqueness. Suppose that $A = WK$ where $K \in \mathscr{A}^+$ and W is unitary. Then $A^*A = K^2$, and hence $K = (A^*A)^{1/2} = |A|$, and then $W = V$ follows immediately. □

The factorization $A = VH$ provided by Lemma 2.39 is called the *polar factorization of A* .

2.40 LEMMA. *Let $\mathscr{A}$ be a unital C^* algebra. Then for all invertible $A \in \mathscr{A}$, $\|A^*\| = \|A\|$.*

Proof. Let A be invertible in $\mathscr{A}$, and let $A = VH$ be its polar factorization. Then

$$\|A\| = \|VH\| \leq \|V\|\|H\| = \|H\| = \|(A^*A)^{1/2}\| = \|A^*A\|^{1/2} = (\|A^*\|\|A\|)^{1/2} \ .$$

It follows that $\|A\| \leq \|A^*\|$. Since A^* is also invertible, the same reasoning yields $\|A^*\| \leq \|A\|$. □

2.41 LEMMA (Gardner's Lemma)**.** *Let $\mathscr{A}$ be a unital C^* algebra. For all invertible A in $B_{\mathscr{A}}$, the unit ball of A, there exist two unitaries U_1 and U_2 such that*

$$A = \frac{1}{2}(U_1 + U_2) \ . \tag{2.25}$$

Proof. Let $A = VH$ be the polar factorization of A. Then since A is invertible $\|H\| = \|(A^*A)^{1/2}\| = \|A^*A\|^{1/2} = \|A\| \leq 1$ by Lemma 2.40. By the Spectral Mapping Theorem, $\sigma_{\mathscr{A}}(H^2) \subset [0,1]$. Then we may define, using the Abstract Spectral Theorem, $W := H + i\sqrt{1 - H^2}$. Then

$$W^*W = WW^* = H^2 + (1 - H^2) = 1 \ ,$$

and hence W and W^* are unitary. Evidently, $H = \frac{1}{2}(W + W^*)$ and then with $U_1 = VW$ and $U_2 = VW^*$, U_1 and U_2 are unitary and (2.25) is satisfied. □

2.42 THEOREM (Kadison-Pederson)**.** *Let $n \in \mathbb{N}$, $n \geq 2$. If $A \in \mathscr{A}$ satisfies $\|A\| \leq 1 - \frac{2}{n}$, then there are unitaries $\{U_1, \dots, U_n\}$ such that*

$$A = \frac{1}{n}\sum_{j=1}^{n} U_j \ . \tag{2.26}$$

Proof. Let $B \in \mathscr{A}$, $\|B\| < 1$ and let V be unitary. Then $\|V^*B\| \leq \|V^*\|\|B\| < 1$. Therefore $1 + V^*B$ is invertible, with the consequence that $V + B$ is invertible. That is, for all B with $\|B\| < 1$ and all unitaries V, $V + B$ is invertible. Then $\frac{1}{2}(V + B)$ is an invertible element of $B_{\mathscr{A}}$, and by Gardner's Lemma, there are unitaries U_1 and U_2 such that $\frac{1}{2}(V + B) = \frac{1}{2}(U_1 + U_2)$. That is, given any $B \in \mathscr{A}$ with $\|B\| < 1$, and any unitary V, there are two other unitaries U_1 and U_2 such that

$$B + V = U_1 + U_2. \tag{2.27}$$

We now claim that for any $n > 2$, there are unitaries $U_1, \dots, U_n$ such that

$$\frac{n-2}{n}B = \frac{1}{n}\sum_{j=1}^{n} U_j \tag{2.28}$$

We get the $n = 3$ case of (2.28) from (2.27) if we define $U_3 = -V$. Make the inductive hypothesis that for some $n \geq 3$, (2.28) is valid. Then for any unitary V, using both (2.27) and (2.28) after multiplying the latter through by n,

$$(n-1)B = B + (n-2)B = B + \sum_{j=1}^{n} U_j = \sum_{j=1}^{n-1} U_j + B + U_n = \sum_{j=1}^{n-1} U_j + V_1 + V_2$$

with $U_1, \dots, U_n, V_1, V_2$ unitary. Dividing through by $n + 1$, (2.28) is proved. If $\|A\| < (1 - \frac{2}{n})$, $B := (1 - \frac{2}{n})^{-1}A$ satsifies $\|B\| < 1$. Expressing (2.28) for this B in terms of A yields (2.26) □

2.43 COROLLARY (Russo-Dye Theorem)**.** *Let $\mathscr{A}$ be a unital C^* algebra. Then $B_{\mathscr{A}}$, the closed unit ball of $\mathscr{A}$, is the norm closure of the convex hull of the unitary elements of $B_{\mathscr{A}}$.*

2.44 THEOREM (Isometry property of the involution)**.** *Let $\mathscr{A}$ be a C^* algebra, not necessarily unital. Then for all $A \in \mathscr{A}$, $\|A^*\| = \|A\|$, and hence $\|A^*A\| = \|A\|^2$.*

Proof. First suppose that $\mathscr{A}$ is unital. For $A \in \mathscr{A}$ with $\|A^*\|\|A\| = 1$ if $\|A\| \neq \|A^*\|$, then either $\|A\| < 1$, or $\|A^*\| < 1$. Replacing A by A^* if needed, we may suppose that $\|A\| < 1$ and then $\|A^*\| > 1$. Pick $n \in \mathbb{N}$, so that $\|A\| \leq (1 - 2/n)$. Then there are unitaries $U_1, \dots, U_n$ such that $A = \frac{1}{n}\sum_{j=1}^{n} U_j$. Then

$$\|A^*\| = \left\| \frac{1}{n}\sum_{j=1}^{n} U_j^* \right\| \leq \frac{1}{n}\sum_{j=1}^{n} \|U_j^n\| = 1 \ .$$

This contradiction, and a simple scaling argument, proves the result in this case.

If $\mathscr{A}$ is not unital, Vowden's Theorem says that $\mathscr{A}$ is a C^* subalgebra of a unital C^*-algebra $\mathscr{A}_1$. By what we just proved, the involution is an isometry in $\mathscr{A}_1$, and hence in $\mathscr{A}$. □

With the isometry property finally in hand, we may prove that the norm in a C^* algebra $\mathscr{A}$ is uniquely determined by the $*$-algebra itself.

2.45 THEOREM (Uniqueness of the norm). *Let $\mathscr{A}$ be a C^* algebra. Then for all $A \in \mathscr{A}$,*

$$\|A\|^2 = \nu(A^*A) \ .$$

Proof. By Theorem 2.44, $\|A^*A\| = \|A^*\|\|A\| = \|A\|^2$, and by Lemma 2.8, $\nu(A^*A) = \|A^*A\|$. □

2.7 Extreme points on the unit ball in a C^* algebra algebra

Let K be a convex subset of a linear space. A point $x \in K$ is *extreme* in case whenver $x = (1-t)y + tz$ with $t \in (0,1)$ and $x, y \in K$, then $z = y = x$.

Let $\mathscr{A}$ be a C^* algebra. Then $B_{\mathscr{A}}$ may contain no extreme points at all. For example, let $\mathscr{A} = \mathscr{C}_0(X)$, where X is a locally compact Haussdorf space that is not compact, so that $\mathscr{A}$ is not unital. Let $f \in B_{\mathscr{A}}$, and let K be a compact set outside of which $|f(x)| \leq 1/2$. Let $g \in B_{\mathscr{A}}$ have support in K^c, and note that $f = \frac{1}{2}(f + fg) + \frac{1}{2}(f - fg)$ and $f \pm fg \in B_{\mathscr{A}}$.

2.46 LEMMA. *In any unital C^* algebra, every unitary U is extreme in $B_{\mathscr{A}}$.*

Proof. We first show that the identity 1 is extreme. Suppose that for some $t \in (0,1)$ and $X, Y \in \mathscr{A}$, $1 = (1-t)X + tY$. If $t < \frac{1}{2}$, $1 = \frac{1}{2}X + \frac{1}{2}W$ where $W = (1-2t)X + 2tY \in B_{\mathscr{A}}$, and $X = Y$ if and only if $X = W$. A similar argument applies when $t > 1/2$. Hence it suffices to show that when $1 = \frac{1}{2}(X+Y)$, $X, Y \in B_{\mathscr{A}}$, then $X = Y = 1$.

Suppose $X, Y \in B_{\mathscr{A}}$ and $1 = \frac{1}{2}(X+Y)$. Let $A = \frac{1}{2}(X + X^*)$ and $B = \frac{1}{2}(Y + Y^*)$. Then $1 = \frac{1}{2}(A+B)$, and $A, B \in B_{\mathscr{A}}$ are self adjoint. Since $B = 21 - A$ and since $\sigma(A) \subset [-1,1]$, $\sigma(B) \subset [1,3]$ But since $B \in B_{\mathscr{A}}$, $\sigma(B) \subset [-1,1]$. Hence $\sigma(B) = \{1\}$, and hence $B = 1$, from which it follows that $A = 1$. Then X has the form $1 + iC$, where C is self adjoint. Therefore X is normal and its norm is its spectral radius. Hence if $C \neq 0$, $\|X\| > 1$, which is not the case. Hence $X = 1$, and in the same way, we conclude $Y = 1$.

Now let U by unitary, and suppose that for some $U = \frac{1}{2}(X+Y)$ for $X, Y \in B_{\mathscr{A}}$. Then $1 = \frac{1}{2}(U^*X + U^*Y)$, and $U^*X, U^*Y \in B_{\mathscr{A}}$. Hence $U^*X = U^*Y = 1$, and then $X = Y = U$. □

A theorem of Kadison [19] specifies the extreme points of the unit ball of any unital C^* algebra $\mathscr{A}$. There turn out to be closely related to unitaries: The extreme points are a special sort of *partial isometry.*

2.47 DEFINITION. An element $U \in \mathscr{A}$ is a *partial isometry* in case $P = U^*U$ is an idempotent; i.e., $P^2 = P$, and since P is self adjoint, this is the case exactly when $\sigma(P) \subset \{0,1\}$.

Since UU^* has the same spectrum as U^*U, U^* is a partial isometry if and only if U is a partial isometry. Of course, every unitary is a partial isometry, but not every partial isometry is unitary.

At this point we only prove Kadison's Theorem in the von Neumann algebra setting, in which its proof is facilitated by the use of the polar decomposition. However, the theorem is true as stated for all unital C^* algebras. It is the von Neumann algebra result that we need in the next sections.

2.48 THEOREM (Kadison's Extreme Point Theorem). *Let $\mathscr{A}$ be a unital C^* algebra. Then the extreme points of $B_{\mathscr{A}}$ are precisely the elements $U \in \mathscr{A}$ such that*

$$(1 - UU^*)\mathscr{A}(1 - U^*U) = 0 \ , \tag{2.29}$$

and all of these are partial isometries.

The following lemma will be used in the proof of Theorem 2.48. It provides a substitute for the polar decomposition provided by Lemma 2.39 for *invertible* elements of a C^* algebra.

2.49 LEMMA. *Let $\mathscr{A}$ be a unital C^* algebra, and let $A \in B_{\mathscr{A}}$. Define, as in (2.24), $|A| = (A^*A)^{1/2}$. Then there exits $U \in B_{\mathscr{A}}$ such that $A = U|A|^{1/2}$ and $A^*U = |A|^{3/2}$.*

Proof. For $n \in \mathbb{N}$ define $U_n := A(1/n + (A^*A))^{-1/2}(A^*A)^{1/4}$. For $\lambda \geq 0$ and $n \in \mathbb{N}$, define $f_n(\lambda) := (1/n + \lambda)^{-1/2}$, and then define $V_{m,n} = f_n(A^*A) - f_m(A^*A)$, and note that $U_n - U_m = AV_{m,n}(A^*A)^{1/4}$. Therefore, using Theorem 2.44

$$\|U_n - U_m\|^2 = \|(A^*A)^{1/4}V_{m,n}A^*AV_{m,n}(A^*A)^{1/4}\| = g_{m,n}(A^*A)$$

where $g_{m,n}(\lambda) = \lambda^{3/2}(f_n(\lambda) - f_m(\lambda))^2 = (\lambda^{3/4}f_n(\lambda) - \lambda^{3/4}f_m(\lambda))^2$. Note that on $[0,1]$, $f_n(\lambda)$ increases monotonically to the continuous function $f(\lambda) = \lambda^{1/4}$. Then by Dini's Theorem f_n converges uniformly to f on $[0,1]$. It follows that

$$\lim_{m,n\to\infty} \|U_n - U_m\|^2 = \lim_{m,n\to\infty} \left(\sup_{\lambda\in[0,1]} \left\{ (\lambda^{3/4}f_n(\lambda) - \lambda^{3/4}f_m(\lambda))^2 \right\} \right) = 0 \ .$$

Therefore $\{U_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with limit U.

To see that $A = U|A|^{1/2}$, use Theorem 2.44 again to conclude

$$\|A - U_n|A|^{1/2}\|^2 = \|A^*A + (A^*A)^2 f_n(A^*A)^2 - 2(A^*A)^{3/2} f_n(A^*A)\| \ .$$

By Dini's Theorem again, $\lim_{n\to\infty}(A^*A)^2 f_n(A^*A)^2 = \lim_{n\to\infty}(A^*A)^{3/2} f_n(A^*A) = A^*A$, with the convergence in norm, and hence $\lim_{n\to\infty} \|A - U_nA\| = 0$. Since $\lim_{n\to\infty} U_n = U$, this proves that $A = U|A|^{1/2}$. The proof that $A^*U = |A|^{3/2}$ is similar but simpler. □

Proof of Theorem 2.48. If U satisfies (2.29), then $0 = U^*((1-UU^*)U(1-U^*U)) = U^*U(1`-U^*U)^2$, and so $\sigma(U^*U) \subset \{0,1\}$, and hence U is a partial isometry.

We now show that (2.29) is necessary, starting with a proof of the consequent fact that every extreme point of $B_{\mathscr{A}}$ must be an isometry. For $X \in B_{\mathscr{A}}$, apply Lemma 2.49 to write $X = U|X|^{1/2}$ with $U \in B_{\mathscr{A}}$ and $X^*U = |X|^{3/2}$. Then

$$X = U|X|^{1/2} = \frac{1}{2}(U(2|X|^{1/2} - |X|) + U|X|) \ .$$

Since $|X|$ has spectrum in $[0,1]$, $U(2|X|^{1/2} - |X|), U|X| \in B_{\mathscr{A}}$. If X is extreme, $U(2|X|^{1/2} - |X|) = U|X|$, and multiplying on the left by X^* then yields $|X|^2 = |X|^{5/2}$. This is the case if and only if $|X|$, and hence $|X|^2 = X^*X$, has spectrum in $\{0,1\}$. That is, unless X is a partial isometry, X cannot be an extreme point.

Let U be a partial isometry in $\mathscr{A}$, and define the projections $P = U^*U$ and $Q = UU^*$. Simple computations show that $(U - UP)^*(U - UP) = 0$ and $(U - QU)^*(U - QU) = 0$. Hence $U = UP = QU$.

Let $A \in B_{\mathscr{A}}$, and define $B = Q^\perp AP^\perp$. Since $U = QUP$, $U + B = QUP + Q^\perp AP^\perp$. Then

$$\|U + B\|^2 = \|(U + B)^*(U + B)\| = \|PU^*QUP + P^\perp A^*Q^\perp AP^\perp\| = \max\{\|U^*U\|, \|A\|^2\} \leq 1 \ ,$$

where in the last line we have use the fact that $U^*U = PU^*QUP$ and $P^\perp A^* Q^\perp AP^\perp$ are commuting and self adjoint, and their product is zero. Thus, if $B \neq 0$, the decomposition $U = \frac{1}{2}(U+B) + \frac{1}{2}(U-B)$, the decomposition shows that U is not extreme. Thus, (2.29) is a necessary condition for U to be extreme.

Now suppose that U satisfies (2.29). As we have noted at the beginning, U is an isometry. As above, define $P = U^*U$ and $Q = UU^*$. Suppose $X, Y \in B_{\mathscr{A}}$ and $U = \frac{1}{2}(X+Y)$. Since $U = UP = QU$, we also have $U = \frac{1}{2}(XP + YP)$, and then

$$P = U^*U = \frac{1}{4}(PX^*XP + PX^*YP + PY^*XP + PY^*YP) \ . \tag{2.30}$$

Since P is the identity in the C^* algebra $P\mathscr{A}P$, P is extreme, and hence all of the terms on the right in (2.30) are equal to P. But then $P(X-Y)^*(X-Y)P = 0$, and so $XP = YP$. The same sort of argument shows that $QX = QY$, and then $X - Y = (1-Q)(X-Y)(1-P)$, and this is 0 by (2.29). Hence $X = Y$, and U is extreme. □

2.8 Homeomorphisms of C^* algebras

2.50 THEOREM. *Let $\mathscr{A}$ be a C^* algebra, and let $\mathscr{J}$ be a norm-closed ideal in $\mathscr{A}$. Then $\mathscr{J}$ is closed under the involution.*

Proof of Theorem 2.50. Let $A \in \mathscr{J}$. By part *(1)* of Theorem 2.32, there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of positive elements of $\mathscr{J}$, with $\|E_n\| \leq 1$ for all n, such that $\lim_{n\to\infty} \|AE_n - A\| = 0$. Since the involution is an isometry, $\|E_nA^* - A^*\| = \|AE_n - A\|$, and $E_nA^* \in \mathscr{J}$, $\lim_{n\to\infty} \|E_nA^* - A^*\| = 0$. Then since $\mathscr{J}$ is closed, $A^* \in \mathscr{J}$. □

Now let $\mathscr{A}$ be a C^* algebra, and let $\mathscr{J}$ be a closed ideal in $\mathscr{A}$. As usual, let $\{A\}$ denote equivalence class of A mod $\mathscr{J}$, and let $\|\{A\}\|\|$ denote the quotient norm of $\{A\}$; that is,

$$\|\{A\}\|\| = \inf\{\ \|A - B\|\ :\ B \in \mathscr{J}\ \} \ .$$

Then $\mathscr{A}/\mathscr{J}$ is a Banach algebra with the quotient norm. By Theorem 2.50, $A - B \in \mathscr{J} \iff A^* - B^* \in \mathscr{J}$, and therefore we may define an involution on $\mathscr{A}/\mathscr{J}$ by $\{A\}^* = \{A^*\}$. Evidently this involution is an isometry, and so for all $A \in \mathscr{A}$, $\|\{A\}^*\{A\}\| \leq \|\{A\}\|^2$. To show that $\mathscr{A}/\mathscr{J}$ is a C^* algebra with this involution, we need only show that for all $A \in \mathscr{A}$,

$$\|\{A\}\|^2 \leq \|\{A\}^*\{A\}\| \ . \tag{2.31}$$

It will be convenient to have a unit. If $\mathscr{A}$ is not unital, let $\mathscr{A}_1$ be the C^* algebra obtained by adjoining a unit as in Vowden's Theorem. Then $\mathscr{J}$, identified with its canonical embedding into $\mathscr{A}_1$, is still a closed ideal in $\mathscr{A}_1$, and for all $(\lambda, A), (\mu, B)$ in $\mathscr{A}_1$, $(\lambda, A) \sim (\mu, B)$ if and only if $(\lambda - \mu, A - B) \in \mathscr{J}$. Evidently this is the case if and only if $\lambda = \mu$ and $A \sim B \in \mathscr{A}$. In particular for all $A \in \mathscr{A}$, the equivalence class of $(0, A)$ in $\mathscr{A}_1$ is precisely the set of $(0, B)$ with $B \sim A$ in $\mathscr{A}_1$, and $\|\{(0,A)\}|| = \|\{A\}\|$. Hence it suffices to prove (2.31) in unital algebras.

2.51 LEMMA. *Let $\mathscr{A}$ be a unital C^* algebra, and let $\mathscr{J}$ be a closed ideal in $\mathscr{A}$. For all $A \in \mathscr{A}$, the quotient norm of $\{A\}$ is given by*

$$\|\{A\}\| = \inf\{\ \|A - AE\|\ :\ E \in \mathscr{J} \quad \text{and} \quad E \in \mathscr{A}^+ \cap B_{\mathscr{A}}\ \} \ . \tag{2.32}$$

Proof. Whenever $E \in \mathscr{J}$, $A \sim (A - AE)$ so that $\|\{A\}\|$ is no greater than the right hand side of (2.32). To prove the equality, pick $\epsilon > 0$ and $B \in \mathscr{J}$ so that $\|\{A\}\| \geq \|A - B\| - \epsilon$. By Theorem 2.32 applied to $\{B\}$, we can choose $E \in \mathscr{A}^+ \cap B_{\mathscr{A}}$ so $\|B - BE\| < \epsilon$. By (2.13), $\|1 - E\| \leq 1$, and so

$$\|A - B\| \geq \|A - B\|\|1 - E\| \geq \|(A - B)(1 - E)\| = \|(A - AE) - (B - BE)\| \geq \\ \|A - AE\| - \|B - BE\| \geq \|A - AE\| - \epsilon \ .$$

Hence $\|\{A\}\| \geq \|A - AE\| - 2\epsilon$, and since $\epsilon > 0$ is arbitrary, (2.32) is proved. □

Now to prove (2.31), pick $\epsilon > 0$ and $E = E^* \in \mathscr{J}$ with $E \in \mathscr{A}^+ \cap B_{\mathscr{A}}$ so that

$$\|A^*A(1 - E)\| \leq (1 + \epsilon)\|\{A^*A\}\| = (1 + \epsilon)\|\{A\}^*\{A\}\| \ .$$

Then, again using $\|1 - E\| \leq 1$,

$$\|\{A\}\|^2 \leq \|A(1 - E)\|^2 = \|(1 - E)A^*A(1 - E)\| \leq \\ \|(1 - E)\|\|A^*A(1 - E)\| \leq \|A^*A(1 - E)\| \leq (1 + \epsilon)\|\{A\}^*\{A\}\| \ . \quad (2.33)$$

Since $\epsilon > 0$ is arbitrary, (2.31) is proved. We have shown:

2.52 THEOREM. *Let $\mathscr{A}$ be a C^* algebra, and let $\mathscr{J}$ be a norm closed ideal in $\mathscr{A}$, then $\mathscr{J}$ is closed under the involution, and the definition $\{A\}^* = \{A^*\}$ defines an involution on $\mathscr{A}/\mathscr{J}$ so that, equipped with the quotient norm, $\mathscr{A}/\mathscr{J}$ is a C^* algebra.*

2.53 LEMMA. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^* algebras, and let $\pi : \mathscr{A} \to \mathscr{B}$ be a $*$-homomorphism. Then π is a contraction; i.e., $\|\pi(A)\| \leq \|A\|$ for all $A \in \mathscr{A}$. If moreover π is one-to-one, π is an isometry.*

Proof. For all $A \in \mathscr{A}$, by the Spectral Contraction Theorem, $\nu(\pi(A)^*\pi(A)) = \nu(\pi(A^*A)) \leq \nu(A^*A)$. Since for self adjoint elements of a C^* algebra, the norm is the spectral radius, $\|\pi(A)^*\pi(A)\| \leq \|A^*A\| = \|A^*\|\|A\| = \|A\|^2$. Thus, $\|\pi(A)\|^2 \leq \|A\|^2$, and this proves that π is a contraction.

Notice that if $\nu(\pi(A^*A)) = \nu(A^*A)$, the argument gives $\|\pi(A)\| = \|A\|$. Hence it remains to show that if π is one-to-one, π cannot decrease the spectral radius of any self adjoint element of $\mathscr{A}$.

Indeed, let $A = A^* \in \mathscr{A}$, and suppose that $\nu(\pi(A)) < \nu(A)$. Then there is a non-zero continuous bounded function f supported on $[-\nu(A), \nu(A)]$ that vanishes identically on $[-\nu(\pi(A)), \nu(\pi(A))]$. Since f may be approximated by polynomials, $\pi(f(A)) = f(\pi(A))$. However, since f vanishes identically on the spectrum of $\pi(A)$, $f(\pi(A)) = 0$. Thus, $f(A)$ is in the kernel of π, which is a contradiction. Hence, when π is one-to-one, it preserves the spectral radius of self adjoint elements. □

We summarize with the following theorem:

2.54 THEOREM (Homomorphisms of C^* algebras)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be C^* algebras, and let $\pi : \mathscr{A} \to \mathscr{B}$ be a $*$-homomorphism. Then π is a contraction, $\pi(\mathscr{A})$ is a C^*-subalgebra of $\mathscr{B}$, and π induces an isometric isomorphism of $\mathscr{A}/\ker(\pi)$ onto $\pi(\mathscr{A})$.*

3 Concrete C^* algebras

3.1 The role of various topologies on $\mathscr{B}(\mathscr{H})$

Let $\mathscr{H}$ be a Hilbert space, not necessarily separable. For $A \in \mathscr{B}(\mathscr{H})$, let A^* denote the Hermitian adjoint of A. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ that is closed under $A \mapsto A^*$ is called *$*$-subalgebra of* $\mathscr{B}(\mathscr{H})$. If furthermore $\mathscr{A}$ is closed in the operator norm, $\mathscr{A}$ is a *concrete C^*-algebra.* This chapter is devoted to concrete C^* algebras.

In addition to the norm topology, three weaker, non-metrizable, topologies are essential to the development of the theory. One of these in known as the *strong operator topology.* This is the natural topology for a full development of the functional calculus based on the Abstract Spectral Theorem. It will be used to extend this thoerem so that it may be applied to a much broader class of functions of operators. The second of these topologies in as the *σ-weak topology.* It turns out that $\mathscr{B}(\mathscr{H})$ is the dual Banach space to anther space of operators, $\mathscr{T}(\mathscr{H})$, the space of trace class operators on $\mathscr{H}$, as explained in this chapter. This means that $\mathscr{B}(\mathscr{H})$ may be endowed with the weak-$*$ topology that come along with its being a dual space. This is the source of important compactness results, and much, much more. These two topologies are not comparable with one another, but both are stronger than a third, known as the *weak operator topology.* Especially since it coincides with the σ-weak topology on bounded subsets of $\mathscr{B}(\mathscr{H})$, it provides a kind of bridge between the other two topologies.

The interplay between these three topologies is quite right and varied, as the result presented in this chapter will show. Although the three topologies are distinct (in the infinite dimensional case), a number of the results that follow will express a *coincidence of topologies result* for certain classes of subsets. For example, it turns out that a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ is closed in any one of these three topologies if and only if it is also closed in the other two – and then of course also in the stronger norm topology. It turns out that C^*-subalgebras of $\mathscr{B}(\mathscr{H})$ that are closed in, say, the weak operator topology are always unital, and this will bring us to the notion of a *von Neumann algebra*, which is a $*$ subalgebra of $\mathscr{B}(\mathscr{H})$ that is closed in the weak operator topology, and contains the identity in $\mathscr{B}(\mathscr{H})$.

The next section introduces some terminology and notation, and recalls some facts concerning the polarization identity and the parallelogram law in Hilbert space, though it is assumed that the reader is familiar with the basic theory of Hilbert space.

3.2 The Polarization Identity and the Parallelogram Law.

Let $\mathscr{H}$ be a Hilbert space, not necessarily separable, with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of bounded linear operators on $\mathscr{H}$. (We may write $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ and $\| \cdot \|_{\mathscr{H}}$ for specificity if there is more than one Hilbert space under consideration.) For $A \in \mathscr{B}(\mathscr{H})$, $\mathrm{ran}(A)$ denotes the range of A, and $\ker(A)$ denotes the null space of A. If $\mathscr{K}$ is a subspace of $\mathscr{H}$, $\mathscr{K}^\perp$ is the orthogonal complement of $\mathscr{K}$, which is a closed subspace of $\mathscr{H}$.

Rank-one operators will occur frequently in what follows, and it will be convenient to use Dirac notation for these: For $\eta, \xi \in \mathscr{H}$, $|\eta\rangle\langle\xi|$ denotes the operator acting on $\mathscr{H}$ by

$$|\eta\rangle\langle\xi|\zeta = \langle \xi, \zeta \rangle \eta \ . \tag{3.1}$$

3.1 LEMMA (Polarization identity)**.** *For all $A \in \mathscr{B}(\mathscr{H})$ and all ζ and ξ in $\mathscr{H}$,*

$$
\begin{aligned}
\langle \zeta, A\xi \rangle = & \frac{1}{4}\left[\langle (\zeta+\xi), A(\zeta+\xi)\rangle - \langle (\zeta-\xi), A(\zeta-\xi)\rangle\right] \\
& - \frac{i}{4}\left[\langle (\zeta+i\xi), A(\zeta+i\xi)\rangle - \langle (\zeta-i\xi), A(\zeta-i\xi)\rangle\right] \ .
\end{aligned}
$$

Proof. We compute

$$
\frac{1}{4}\left[\langle (\zeta+\xi), A(\zeta+\xi)\rangle - \langle (\zeta-\xi), A(\zeta-\xi)\rangle\right] = \frac{1}{2}\left[\langle \zeta, A\xi\rangle + \langle \xi, A\zeta\rangle\right]
$$

and

$$
\frac{1}{4}\left[\langle (\zeta+i\xi), A(\zeta+i\xi)\rangle - \langle (\zeta-i\xi), A(\zeta-i\xi)\rangle\right] = \frac{i}{2}\left[\langle \zeta, A\xi\rangle - \langle \xi, A\zeta\rangle\right] \ .
$$

□

3.2 COROLLARY. *Let $A \in \mathscr{B}(\mathscr{H})$ be such that $\langle \xi, A\xi\rangle \in \mathbb{R}$ for all $\xi \in \mathscr{H}$. Then $A = A^*$.*

Proof. Since each of the four inner products on the right in the polarization identity are real, $\langle \zeta, A\xi\rangle = \overline{\langle \xi, A\zeta\rangle}$. By the definition of the inner product, $\overline{\langle \xi, A\zeta\rangle} = \langle A\zeta, \xi\rangle$, and so $\langle \zeta, A\xi\rangle = \langle A\zeta, \xi\rangle = \langle \zeta, A^*\xi\rangle$. □

3.3 DEFINITION. An element of $\mathscr{B}(\mathscr{H})$ is *positive* in case $\langle \xi, A\xi\rangle \geq 0$ for all $\xi \in \mathscr{H}$, and is *strictly positive* in case $\langle \xi, A\xi\rangle > 0$ when $\xi \neq 0$. The positive elements of $\mathscr{B}(\mathscr{H})$ are denoted by $\mathscr{B}(\mathscr{H})^+$.

Since $\mathscr{B}(\mathscr{H})$ is a unital C^* algebra, we have another definition of what it means for $A \in \mathscr{B}(\mathscr{H})$ to be positive, namely that A is self adjoint and the spectrum of A lies in $[0, \infty)$. These two definitions agree; they identify the same set of operators as "positive". One way to see this, which will be useful for other purposes, turns on the Parallelogram Law:

$$
\langle \xi+\eta, A(\xi+\eta)\rangle + \langle \xi-\eta, A(\xi-\eta)\rangle = 2\langle \xi, A\xi\rangle + 2\langle \eta, A\eta\rangle \ , \tag{3.2}
$$

which is evidently valid for all $A \in \mathscr{B}(\mathscr{H})$ and all $\eta, \xi \in \mathscr{H}$. In case $A = 1$, it reduces to

$$
\|\xi+\eta\|^2 + \|\xi-\eta\|^2 = 2\|\xi\|^2 + 2\|\eta\|^2 \ , \tag{3.3}
$$

which is what is commonly called the Parallelogram Law, but the more general case is is useful here. The identity (3.3) expresses a *uniform convexity* property of the unit ball in $\mathscr{H}$. To see this, re-write it as follows:

$$
\left\|\frac{\xi+\eta}{2}\right\|^2 = \frac{\|\xi\|^2 + \|\eta\|^2}{2} - \left\|\frac{\xi-\eta}{2}\right\|^2 \ . \tag{3.4}
$$

In particular, if $\|\xi\| - \|\eta\| = 1$, then $\|\frac{1}{2}(\xi+\eta)\|^2 \leq 1 - \|\frac{1}{2}(\xi-\eta)\|^2$, so that the midpoint of the line segment joining ξ and η has a norm strictly less than 1 by an amount that depends only on $\|\xi - \eta\|$.

3.4 DEFINITION. A real valued function Ψ on $\mathscr{H}$ is λ *mid-point convex* for $\lambda \in \mathbb{R}$ in case for all ζ, ξ in $\mathscr{H}$,

$$
\tfrac{1}{2}\Psi(\zeta) + \tfrac{1}{2}\Psi(\xi) - \Psi\left(\tfrac{1}{2}(\zeta+\xi)\right) \geq \lambda\tfrac{1}{4}\|\zeta-\xi\|^2 \ , \tag{3.5}
$$

. The function Ψ is *mid-point convex* in case (3.5) is valid for $\lambda = 0$.

3.5 THEOREM. *Let Ψ be λ mid-point convex for some $\lambda > 0$, and lower semicontinuous. Let K be a non empty closed convex set in $\mathscr{H}$. Suppose that Ψ is bounded below on K. Then there exists a unique $\xi_0 \in K$ such that $\Psi(\xi_0) \leq \Psi(\xi)$ for all $\xi \in K$, and if $\{\xi_n\}_{n\in\mathbb{N}}$ is any sequence in $\mathscr{H}$ with*

$$\lim_{n\to\infty} \Psi(\xi_n) = \inf\{\Psi(\xi) \ : \ \xi \in K \ \} \ ,$$

then $\lim_{n\to\infty} \|\xi_n - \xi_0\| = 0$.

Proof. Let $\{\xi_n\}_{n\in\mathbb{N}}$ is any sequence in $\mathscr{H}$ with $\lim_{n\to\infty} \Psi(\xi_n) = c := \inf\{\Psi(\xi) \ : \ \xi \in K \ \}$. Fix $\epsilon > 0$. Then there exits $N \in \mathbb{N}$ such that for all $n \geq N$, $c + \epsilon > \Psi(\xi_n) \geq c$. Then by (3.5), for $n, m \geq N$, and the fact that $\frac{1}{2}(\xi_m + \xi_n) \in K$, $\epsilon \geq \frac{\lambda}{4}\|\xi_m - \xi_n\|^2$. Hence $\{\xi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and its limit, ξ_0, belongs to K since K is closed. Since Ψ is lower semicontinuous, $\Psi(\xi_0) \leq \lim_{n\to\infty} \Psi(\xi_n) = c$, and hence $\Psi(\xi_0) = c$. If $\Psi(\xi_1) = c$, $\xi_1 \in K$, then by (3.5) once more $\|\xi_0 - \xi_1\|^2 = 0$, which proves the uniqueness of ξ_0. □

Let $A \in \mathscr{B}(\mathscr{H})$ be positive in the sense of Definition 3.3. Fix $\epsilon > 0$ and $\zeta \in \mathscr{H}$. Define the function

$$\Phi_\zeta(\xi) = -2\Re(\langle \xi, \zeta\rangle) + \langle \xi, (A + \epsilon 1)\xi\rangle \ .$$

Using the identity (3.2) and the positivity of A, one readily checks that Φ_ζ is ϵ midpoint convex, and it is evidently continuous. By Theorem 3.5, there exists a unique $\xi_0 \in \mathscr{H}$ such that $\Phi_\zeta(\xi_0) < \Phi_\zeta(\xi)$ for all $\xi \neq \xi_0$. Differentiating, for all $\eta \in \mathscr{H}$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Phi_\zeta(\xi_0 + t\eta)|_{t=0} = -2\Re(\langle \eta, \zeta\rangle) + 2\Re(\langle \eta, (A + \epsilon 1)\xi_0\rangle) \ ,$$

where we have used the self adjointness of $(A + \epsilon 1)$ in the final term. Replacing η by $i\eta$, we conclude that $\langle \eta, \zeta\rangle = \langle \eta, (A + \epsilon 1)\xi_0\rangle$ for all η, and hence $(A + \epsilon 1)\xi_0 = \zeta$. If also $(A + \epsilon 1)\xi_1 = \zeta$, $0 = \langle \xi_1 - \xi_0, (A + \epsilon 1)(\xi_1 - \xi_0)\rangle \geq \epsilon\|\xi_1 - \xi_0\|^2$, and so $\xi_1 - \xi_0$. Finally,

$$\epsilon\|\xi_0\|^2 \leq \langle \xi_0, (A + \epsilon 1)\xi_0\rangle = \langle \xi_0, \zeta\rangle \leq \|\xi_0\|\|\zeta\| \ ,$$

and so $\|\xi_0\| \leq \epsilon^{-1}\|\zeta\|$. Thus $(A + \epsilon 1)$ is invertible, $(A + \epsilon 1)^{-1}\zeta = \xi_0$, and $\|(A + \epsilon 1)^{-1}\| \leq \epsilon^{-1}$.

The converse is clear: If A is self adjoint and has positive spectrum, then $A^{1/2}$ is well-defined, self adjoint and $A = A^{1/2}A^{1/2}$. Then $\langle \xi, A\xi\rangle = \|A^{1/2}\xi\|^2 \geq 0$. This shows that a self adjoint operator A is positive in the sense of Definition 3.3 if and only if A has non-negative spectrum. Further use of Theorem 3.5 will be made later.

3.6 REMARK. *The fact that for all $\epsilon > 0$ and all $A \in \mathscr{B}(\mathscr{H})$, the operator $\epsilon 1 + A$ is invertible and $\|(\epsilon 1 + A)^{-1}\| \leq \epsilon^{-1}$ is the Hilbert-space version of the Lax-Milgram Theorem. Since for any $A \in \mathscr{B}(\mathscr{H})$, and any $\xi \in \mathscr{H}$, $\langle \xi, A^*A\xi\rangle = \langle A\xi, A\xi\rangle \geq 0$, A^*A is positive (in any of the equivalent senses), and then by what we have just seem $1 + A^*A$ is invertible. Hence, in $\mathscr{B}(\mathscr{H})$, hypothesis (vi) of Gelfand and Neumark is valid, but this simple proof of it uses the vectors on which elements of $\mathscr{B}(\mathscr{H})$ operate, and not only the algebra itself.*

3.3 The strong and weak operator topologies

The strong and weak operator topologies were introduced by von Neumann [30] in 1930:

3.7 DEFINITION (Strong and weak operator topologies)**.** The *strong operator topology* on $\mathscr{B}(\mathscr{H})$ is the weakest topology such that for each $\xi \in \mathscr{H}$, the function $A \mapsto A\xi$ from $\mathscr{B}(\mathscr{H})$ to $\mathscr{H}$ is continuous with the usual norm topology on $\mathscr{H}$. The *weak operator topology* on $\mathscr{B}(\mathscr{H})$ is the weakest topology such that for each $\xi \in \mathscr{H}$, the function $A \mapsto \langle \xi, A\xi \rangle$ is continuous from $\mathscr{B}(\mathscr{H})$ to $\mathbb{C}$.

It follows from the definitions that a basic set of neighborhoods of 0 for the strong operator topology is given by the sets

$$U_{\epsilon,\xi_1,\dots,\xi_n} = \{ A \in \mathscr{B}(\mathscr{H}) \ : \ \|A\xi_j\| < \epsilon \quad \text{for} \quad j = 1,\dots,n \ \} \tag{3.6}$$

where $\epsilon > 0$ and $\xi_1,\dots,\xi_n \in \mathscr{H}$. Likewise, it follows that a basic set of neighborhoods of 0 for the weak operator topology is given by the sets

$$V_{\epsilon,\xi_1,\dots,\xi_n} = \{ A \in \mathscr{B}(\mathscr{H}) \ : \ |\langle \xi_j, A\xi_j \rangle| < \epsilon \quad \text{for} \quad j = 1,\dots,n \ \} \tag{3.7}$$

$\epsilon > 0$ and $\xi_1,\dots,\xi_n \in \mathscr{H}$. Both topologies are evidently Hausdorff. Since for all $A \in \mathscr{B}(\mathscr{H})$ and all $\xi \in \mathscr{H}$, $|\langle \xi, A\xi \rangle| = |\langle \xi, A^*\xi \rangle|$, $V_{\epsilon,\xi_1,\dots,\xi_n} = V^*_{\epsilon,\xi_1,\dots,\xi_n}$, and hence $A \mapsto A^*$ is continuous in the weak operator topology.

3.8 REMARK. *Since $U_{\epsilon,\xi_1,\dots,\xi_n} = U_{1,\epsilon^{-1}\xi_1,\dots,\epsilon^{-1}\xi_n}$ and $V_{\epsilon,\xi_1,\dots,\xi_n} = V_{1,\epsilon^{-1}\xi_1,\dots,\epsilon^{-1}\xi_n}$, one could dispense with the parameter ϵ. However, it will sometimes be useful to take $\{\xi_1,\dots,\xi_n\}$ to be orthonormal. For this and other reasons, the redundancy is useful.*

3.9 LEMMA. *The weak operator topology on $\mathscr{B}(\mathscr{H})$ is the weakest topology such that for each $\xi, \zeta \in \mathscr{H}$, the function $A \mapsto \langle \zeta, A\xi \rangle$ is continuous from $\mathscr{B}(\mathscr{H})$ to $\mathbb{C}$.*

Proof. This follows immediately from the polarization identity. □

Since for each $\xi \in \mathscr{H}$, $A \mapsto A\xi$ is continuous in the norm topology on $\mathscr{B}(\mathscr{H})$, the norm topology is at least as strong as the strong operator topology. Similarly, for all $\xi \in \mathscr{H}$, the function $A \mapsto \langle \xi, A\xi \rangle$ is continuous in the strong operator topology, and hence the strong operator topology is at least as strong as the weak operator topology.

The following proposition shows that the norm topology is *strictly stronger* than the strong operator topology, which is in turn *strictly stronger* than the weak operator topology.

3.10 PROPOSITION (Continuity of the norm and adjoint)**.** *Let $\mathscr{H}$ be an infinite dimensional Hilbert space. Then:*

(1) The function $A \mapsto \|A\|$ from $\mathscr{B}(\mathscr{H})$ to $\mathbb{R}_+$ is continuous in the norm topology, but is only lower semicontinuous in the strong and weak operator topologies.

(2) The function $A \mapsto A^*$ is continuous from $\mathscr{B}(\mathscr{H})$ to $\mathscr{B}(\mathscr{H})$ in the norm and the weak operator topologies, but not in the strong operator topology.

Proof. Let $\{\zeta_j\}$ be an orthonormal sequence in $\mathscr{H}$. For $n \in \mathbb{N}$, let P_n denote the orthogonal projection onto the span of $\{\zeta_1, \dots, \zeta_n\}$. For all $\xi \in \mathscr{H}$, $\lim_{n\to\infty} \|P_n^\perp \xi\| = 0$ by Bessel's inequality, so that $\lim_{n\to\infty} P_n^\perp = 0$ in the strong operator topology. Since for $n \neq m$, $\|P_n^\perp - P_m^\perp\| = 1$, the sequence $\{P_n\}$ is not even Cauchy in the norm topology. Hence the norm is discontinuous in the strong operator topology, and then also in the weak operator topology.

To see that the norm is lower semicontinuous in the weak operator topology, we must show that the sub-level sets $\{\ A \in \mathscr{B}(\mathscr{H})\ :\ \|A\| \leq t\ \}$ are weakly closed for each $t > 0$. Suppose that B is in the weak operator topology closure of $\{\ A \in \mathscr{B}(\mathscr{H})\ :\ \|A\| \leq t\ \}$. Then for each pair η, ξ of unit vectors in $\mathscr{H}$, and each $n \geq 2$ there is an $A_n \in \{\ A \in \mathscr{B}(\mathscr{H})\ :\ \|A\| \leq t\ \}$ such that $B - A_n \in \{X \in \mathscr{B}(\mathscr{H})\ :\ |\langle \eta, X\xi\rangle| < 1/n\}$. If η and ξ are such that $|\langle \eta, B\xi\rangle| \geq (1-1/n)\|B\|$, then

$$\|B\| \leq (1-1/n)^{-1}|\langle \eta, B\xi\rangle| \leq \frac{n}{n-1}\left(|\langle \eta, A_n\xi\rangle| + \frac{1}{n}\right) \leq \frac{n}{n-1}t + \frac{1}{n}\ .$$

Since $n \geq 2$ is arbitrary, $\|B\| \leq t$. This proves the closure in the weak operator topology, and hence also in the strong operator topology.

For the second part, the continuity of the involution is obvious in the norm topology and we have seen it is true in the weak operator topology. To see that the involution is not continuous for the strong operator topology when $\mathscr{H}$ is infinite dimensional, note that every such Hilbert space contains a copy of ℓ_2, the Hilbert space of all square summable functions from $\mathbb{N}$ to $\mathbb{C}$. We may suppose without loss of generality that $\mathscr{H} = \ell_2$. Define the shift operator $A \in \mathscr{B}(\mathscr{H})$ by

$$(A\zeta)_j = \begin{cases} \zeta_{j-1} & j \geq 2 \\ 0 & j = 1 \end{cases}$$

Evidently, for all ζ, $\|A\zeta\| = \|\zeta\|$. The adjoint is given by $(A^*\zeta)_j = \zeta_{j+1}$ for all $j \in \mathbb{N}$. Therefore, $\|A^*\zeta\|^2 = \sum_{j=2}^\infty |\zeta_j|^2 = \|\zeta\|^2 - |\zeta_1|^2$. It follows that for all $\zeta \in \mathscr{H}$,

$$\lim_{n\to\infty} \|(A^n)^*\zeta\| = 0 \quad \text{while} \quad \|A^n\zeta\| = \|\zeta\|\ .$$

Hence the sequence $\{(A^n)^*\}$ converges to zero in the strong operator topology, but the sequence $\{A^n\}$ does not. □

3.11 THEOREM (Continuous linear functionals for the strong operator topology)**.** *Let $\mathscr{H}$ be a Hilbert space, and let φ be a linear functional on $\mathscr{B}(\mathscr{H})$ that is continuous in the strong operator topology. Then there exists $n \in \mathbb{N}$ and two sets of vectors $\{\zeta_1, \dots, \zeta_n\}$ and $\{\xi_1, \dots, \xi_n\}$ such that for all $A \in \mathscr{B}(\mathscr{H})$,*

$$\varphi(A) = \sum_{j=1}^n \langle \zeta_j, A\xi_j\rangle\ . \tag{3.8}$$

Evidently, every such linear functional is weakly continuous, and hence every strongly continuous linear functional is weakly continuous. Consequently, a convex subset of $\mathscr{B}(\mathscr{H})$ is strongly closed if and only if it is weakly closed.

Proof. If φ is strongly continuous, then $\varphi^{-1}(\{\lambda\ :\ |\lambda| < 1\})$ contains a neighborhood of 0 in $\mathscr{B}(\mathscr{H})$. Thus, there exists an $\epsilon > 0$ and a set of n vectors $\xi_1, \dots, \xi_n$, which without loss of generality we may assume to be orthonormal, such that if $\|A\xi_j\| < \epsilon$ for $j = 1, \dots, n$, $|\varphi(A)| < 1$. Note that if $A\xi_j = 0$ for $j = 1, \ldots, n$, then $\|tA\xi_j\| < \epsilon$ for all $t > 0$, and all $j = 1, \ldots, n$. Consequently $t|\varphi(A)| < 1$. It follows that

$$A\xi_j = 0 \quad \text{for} \quad j = 1, \ldots, n \quad \Rightarrow \quad \varphi(A) = 0 \ . \tag{3.9}$$

For $A \in \mathscr{B}(\mathscr{H})$, define $\widehat{A}$ by $\widehat{A} = \sum_{j=1}^{n} |A\xi_j\rangle\langle\xi_j|$, using the Dirac notation from (3.1). Then $(A - \widehat{A})\xi_j = 0$ for each j, and by (3.9),

$$\varphi(A) = \varphi(\widehat{A}) = \sum_{j=1}^{n} \varphi\left(|A\xi_j\rangle\langle\xi_j|\right) \ . \tag{3.10}$$

For each fixed j, and any $\eta \in \mathscr{H}$, consider the rank-one operator $|\eta\rangle\langle\xi_j|$. Then $\eta \mapsto \varphi(|\eta\rangle\langle\xi_j|)$ is a bounded linear functional on $\mathscr{H}$, and therefore by the Riesz Representation Theorem, there is a vector $\zeta_j \in \mathscr{H}$ such that $\varphi(|\eta\rangle\langle\xi_j|) = \langle\zeta_j, \eta\rangle$ for all $\eta \in \mathscr{H}$. In particular, $\varphi(|A\xi_j\rangle\langle\xi_j|) = \langle\zeta_j, A\xi_j\rangle$ for each j. Combining this with (3.10) yields (3.8). The final statement is a standard application of the Hahn-Banach Theorem. □

3.12 LEMMA. *When $\mathscr{H}$ is infinite dimensional, the product map, $(A, B) \mapsto AB$ from $\mathscr{B}(\mathscr{H}) \times \mathscr{B}(\mathscr{H})$ to $\mathscr{B}(\mathscr{H})$ is not jointly continuous in the strong operator topology, through its restriction to bounded subsets of $\mathscr{B}(\mathscr{H}) \times \mathscr{B}(\mathscr{H})$ is.*

Proof. Were the product map continuous, it would be continuous at the origin, and then since for all $\xi \in \mathscr{H}$, $U := \{X \in \mathscr{B}(\mathscr{H}) \ : \ \|X\xi\| < 1\}$ is strongly open, there would exist strongly open neighborhoods V, W of 0 in $\mathscr{B}(\mathscr{H})$ such that for all $Y \in V$ and all $Z \in W$, $YZ \in U$. This is impossible when $\mathscr{H}$ is infinite dimensional: V would contain $\{Y \ : \ \|Y\eta_j\| < 1 \ , j = 1, \ldots, m\}$ for some set $\{\eta_1, \ldots, \eta_m\} \subset \mathscr{H}$. Choose a unit vector $\eta_0 \in \{\eta_1, \ldots, \eta_m\}^\perp$. Then for all $\zeta \in \mathscr{H}$, the operator $Y_0 := \zeta\langle\eta_0, \cdot\rangle \in V$ since $Y_0\eta_j = 0$ for each j. Since W is open in the norm topology, W contain the open ball of some radius $r > 0$, and hence there is some $Z_0 \in W$ such that $|\langle\eta_0, Z_0\xi\rangle| = c > 0$, and then $\|Y_0Z_0\xi\| = c\|\zeta\|$ which can be arbitrarily large since ζ is arbitrary.

Now let $\mathcal{S}$ be a bounded subset of $\mathscr{B}(\mathscr{H})$; suppose that $\|A\| \leq r$ for all $A \in \mathcal{S}$. Again, for any $\xi \in \mathscr{H}$, let U defined to be $U := \{X \in \mathscr{B}(\mathscr{H}) \ : \ \|X\xi\| < 1\}$. Let $A_0 \in \mathcal{S}$ and $B_0 \in \mathscr{B}(\mathscr{H})$. It suffices to show that there exist strongly open neighborhood V, W of 0 in $\mathscr{B}(\mathscr{H})$ such that for all $Y \in V \cap \mathcal{S}$ and all $Z \in W$, $(A_0 + Y)(B_0 + Z) \in U + A_0B_0$, since then taking intersections we get the general strong basic neighborhood of 0. For any $Z \in \mathscr{B}(\mathscr{H})$ and any $Y \in \mathcal{S}$,

$$\begin{aligned}
\|(A_0 + Y)(B_0 + Z)\xi - A_0B_0\xi\| &\leq \|YB_0\xi\| + \|A_0Z\xi\| + \|YZ\xi\| \\
&\leq \|YB_0\xi\| + \|A_0\|\|Z\xi\| + \|Y\|\|Z\xi\| \leq \|YB_0\xi\| + 2r\|Z\xi\|
\end{aligned}$$

We may take $V := \{Y \ : \ \|YB_0\xi\| < \frac{1}{3}\}$ and $W := \{Z \ : \ \|Z\xi\| < \frac{1}{3r}\}$. □

3.4 The Baire functional calculus

If $\mathscr{H}$ and $\mathscr{K}$ are two Hilbert spaces, a *unitary transformation* U from $\mathscr{H}$ to $\mathscr{K}$ is a linear bijection from $\mathscr{H}$ onto $\mathscr{K}$ such that for all $\xi \in \mathscr{H}$, $\|U\xi\|_{\mathscr{K}} = \|\xi\|_{\mathscr{H}}$. Evidently, the inverse of U is also unitary and is equal to U^*. In this case the map

$$A \mapsto UAU^*$$

is an isometric $*$-isomorphism of $\mathscr{B}(\mathscr{H})$ onto $\mathscr{B}(\mathscr{K})$.

Let $A \in \mathscr{B}(\mathscr{H})$ be self adjoint, and let η be a non-zero vector in $\mathscr{H}$. Define

$$\mathscr{H}_\eta = \overline{\mathrm{Span}(\{A^n\eta\}_{n\geq 0})}\ . \tag{3.11}$$

That is, $\mathscr{H}_\eta$ is the norm closure of the span of the polynomials in A, and indeed of the polynomials with rational coefficients - a countable set. Hence $\mathscr{H}_\eta$ is separable even if $\mathscr{H}$ is not.

Evidently, $\mathscr{H}_\eta$ is invariant under $\mathscr{C}(\{1,A\})$. Our goal is to describe the closure of $\mathscr{C}(\{1,A\})$ in the strong operator topology. Recall that the spectrum of A in $\mathscr{C}(\{1,A\})$ is the same as the spectrum of A in $\mathscr{B}(\mathscr{H})$, and in this section we simply write $\sigma(A)$ to denote this spectrum.

Let $(X,\mathcal{F},\mu)$ be a measure space, and consider the Hilbert space $\mathscr{K} := L^2(X,\mathcal{F},\mu)$. An operator $T \in \mathscr{B}(\mathscr{K})$ is a *multiplication operator* in case for some function $f \in L^\infty(X,\mathcal{F},\mu)$, $T\xi(x) = f(x)\xi(x)$ for all $\xi \in \mathscr{K}$; in this case $\|T\| = \|f\|_\infty$. The next theorem shows that every self adjoint operator is unitarily equivalent to a multiplication operator, and a very simple one at that.

3.13 THEOREM (The Spectral Theorem in $\mathscr{B}(\mathscr{H})$)**.** *Let $A \in \mathscr{B}(\mathscr{H})$ be self adjoint. Let η be any non-zero vector in $\mathscr{H}$. Let $\mathscr{H}_\eta$ be defined by* (3.11)*. Then there is a Borel measure μ_η of total mass $\|\eta\|^2$ such that the map $f \mapsto f(A)\eta$, $f \in \mathscr{C}(\sigma(A))$ satisfies*

$$\|f(A)\eta\|^2 = \int_{\sigma(A)} |f(\lambda)|^2 \mathrm{d}\mu_\eta(\lambda)\ . \tag{3.12}$$

Let $\mathscr{K}_\eta$ denote the Hilbert space $L^2(\sigma(A),\mathcal{B}(\sigma(A)),\mu_\eta)$. The isometry $f \mapsto f(A)\eta$ extends to a unitary transformation U_η mapping $\mathscr{K}_\eta$ onto $\mathscr{H}_\eta$. Define $\widehat{A}$ to be the operator on $\mathscr{K}_\eta$ given by $\widehat{A} = U_\eta^* A U_\eta$. Then for all $\psi \in \mathscr{K}_\eta$,

$$\widehat{A}\psi(\lambda) = \lambda\psi(\lambda)\ . \tag{3.13}$$

For a bounded Borel function f on $\sigma(A)$, define the operator $f(\widehat{A})$ on $\mathscr{K}_\eta$ by

$$f(\widehat{A})\psi(\lambda) = f(\lambda)\psi(\lambda) \tag{3.14}$$

for all $\phi \in \mathscr{K}_\eta$. Then when $f \in \mathscr{C}(\sigma(A))$, $U_\eta f(\widehat{A})U_\eta^* = f(A)$, where $f(A)$ is given by the Abstract Spectral Theorem.

Proof. Define a linear functional μ_η on $\mathscr{C}(\sigma(A))$ through

$$\mu_\eta(f) = \langle \eta, f(A)\eta\rangle\ . \tag{3.15}$$

Then μ is a positive linear functional with $\mu_\eta(1) = \|\eta\|^2$. By the Riesz-Markoff Theorem, there is a positive Borel measure on $\sigma(A)$ of total mass $\|\eta\|^2$, also denoted by μ_η, so that for all $f \in \mathscr{C}(\sigma(A))$,

$$\mu_\eta(f) = \int_{\sigma(A)} f\mathrm{d}\mu_\eta\ . \tag{3.16}$$

Combining (3.15) and (3.16), we conclude that for all $f \in \mathscr{C}(\sigma(A))$, $\langle \eta, f(A)\eta\rangle_{\mathscr{H}} = \displaystyle\int_{\sigma(A)} f\mathrm{d}\mu_\eta$

Define an operator $U_\eta : \mathscr{C}(\sigma(A)) \to \mathscr{H}_\eta$ by $U_\eta : f \mapsto f(A)\eta$ where $f(A)$ is defined using the Abstract Spectral Theorem. Then, using the $*$-homomorphism property of $f \mapsto f(A)$,

$$\langle U_\eta f, U_\eta f\rangle = \langle f(A)\eta, f(A)\eta\rangle = \langle \eta, |f|^2(A)\eta\rangle = \int_{\sigma(A)} |f(\lambda)|^2 \mathrm{d}\mu_\eta \ .$$

Therefore, U_η is an isometry from a dense set in $\mathscr{K}_\eta$ whose range, which includes $p(A)\eta$ for all polynomials p, is dense in $\mathscr{H}_\eta$. Hence U_η extends to a unitary operator from $\mathscr{K}_\eta$ onto $\mathscr{H}_\eta$.

For any polynomial p, define $q(\lambda) = \lambda p(\lambda)$, Then

$$U_\eta q = Ap(A)\eta = AU_\eta p \quad \text{and hence} \quad U_\eta^* A U_\eta p = q \ .$$

This proves (3.13) when ψ is a polynomial, and since polynomials are dense in $\mathscr{K}$, (3.13) holds for all $\psi \in \mathscr{K}$.

More generally, for $f \in \mathscr{C}(\sigma(A))$, and p any polynomial,

$$U_\eta f p = f(A)p(A)\eta = f(A)U_\eta p \quad \text{and hence} \quad U_\eta^* f(A) U_\eta p = fp = f(\widehat{A})p \ ,$$

using the definition (3.14) of the multiplication operator $f(\widehat{A})$. Again, since polynomials are dense in $\mathscr{K}$, this proves that $U_\eta f(\widehat{A})U_\eta^* = f(A)$ when $f \in \mathscr{C}(\sigma(A))$. □

Recall that the *Baire functions* on a topological space $(X, \mathcal{O})$ are the smallest class of (complex valued) functions that is closed under the operation of taking pointwise limits of sequences of functions in the class, and which contains all of the continuous functions. The *Baire class 1 functions* are the functions f on X such that for some sequence $\{f_n\}$ of continuous functions on X, $f(x) = \lim_{n\to\infty} f_n(x)$ for *all* $x \in X$. The *Baire class 2 functions* are the functions f on X such that for some sequence $\{f_n\}$ of Baire class 1 functions on X, $f(x) = \lim_{n\to\infty} f_n(x)$ for *all* $x \in X$, and so forth. The full class of Baire functions X is obtained by transfinite induction, but we shall not need this. For our purposes, Baire class 1 functions suffice.

Let A be a self adjoint operator in $\mathscr{B}(\mathscr{H})$. Note that the Baire class 1 functions form a $*$-algebra on $\sigma(A)$ with the usual operations. Let f be any uniformly bounded Baire class 1 function on $\sigma(A)$; say $|f(\lambda)| \le K$ for all $\lambda \in \sigma(A)$. Since f is Borel measurable, the operator on $f(\widehat{A})$ may be defined by (3.14).

Let $\{f_n\}$ be a sequence of continuous functions on $\sigma(A)$ converging pointwise to f. Without loss of generality, we may suppose that for all n, $|f_n(\lambda)| \le K$ for all $\lambda \in \sigma(A)$. Then

$$\lim_{n\to\infty} \int_{\sigma(A)} |f_n(\lambda) - f(\lambda)|^2 \mathrm{d}\mu_\eta = 0$$

by the Lebesgue Dominated Convergence Theorem, and so

$$0 = \lim_{n\to\infty} \|U_\eta f_n - U_\eta f\|_{\mathscr{H}} = \lim_{n\to\infty} \|f_n(A)\eta - U_\eta f\|_{\mathscr{H}} \ .$$

In particular, for all non-zero $\eta \in \mathscr{H}$,

$$\lim_{n\to\infty} f_n(A)\eta = U_\eta f \ , \tag{3.17}$$

the limit exists in the norm topology on $\mathscr{H}$, and the limit depends only on f, η and A, and not on the approximating sequence $\{f_n\}$.

3.14 DEFINITION. Let A be a self adjoint operator in $\mathscr{B}(\mathscr{H})$, and let f be a uniformly bounded Baire class 1 function on $\sigma(A)$. For all non-zero $\eta \in \mathscr{H}$, define

$$f(A)\eta := U_\eta f \ , \tag{3.18}$$

and for $\eta = 0$, set $f(A)\eta = 0$.

As the strong limit of a sequence of uniformly bounded linear operators on $\mathscr{H}$, $f(A) \in \mathscr{B}(\mathscr{H})$, and moreover, it belongs to the strong closure of $\mathscr{C}(\{1, A\})$.

3.15 THEOREM. *For all uniformly bounded Baire class 1 functions f on $\sigma(A)$, the map*

$$\eta \mapsto f(A)\eta$$

defined in (3.18) *is a bounded linear transformation on $\mathscr{H}$ that is in the strong operator closure of $\mathscr{C}(\{1, A\})$. The map $f \mapsto f(A)$ from the $*$-algebra of uniformly bounded Baire class 1 function on $\sigma(A)$ into $\mathscr{B}(\mathscr{H})$ is a norm-reducing $*$-homomorphism, and moreover it is order preserving: If f and g are real valued functions with $f(\lambda) \geq g(\lambda)$ for all $\lambda \in \sigma(A)$, then $f(A) - g(A) \geq 0$ in $\mathscr{B}(\mathscr{H})$.*

Proof. The first statement has already been proved. Let f, g be uniformly bounded Baire class 1 functions on $\sigma(A)$. Let $\{f_n\}$, $\{g_n\}$ be uniformly bounded sequence of continuous functions on $\sigma(A)$ converging pointwise to f and g respectively. Since $\lim_{n\to\infty} f_n g_n(\lambda) = fg(\lambda)$ for all λ, and since $f_n(A)g_n(A) = f_n g_n(A)$ for all n, $f(A)g(A) = fg(A)$. Next, let g and h be the real and imaginary parts, respectively, of a uniformly bounded Baire class 1 function f, so that $f(\lambda) = g(\lambda) + ih(\lambda)$ for all $\lambda \in \sigma(A)$. Let $\{h_n\}$, $\{g_n\}$ be uniformly bounded sequence of continuous functions on $\sigma(A)$ converging pointwise to h and g respectively. Then $f(A) = g(A) + ih(A) = \lim_{n\to\infty}(h_n(A) + ig_n(A))$, and $\overline{f}(A) = g(A) - ih(A) = \lim_{n\to\infty}(h_n(A) - ig_n(A))$. Evidently, $\overline{f}(A) = (f(A))^*$. The linearity of $f \mapsto f(A)$ is even simpler, and is left as an exercise. Hence $f \mapsto f(A)$ is a $*$-homomorphism from the space of Baire class 1 functions on $\sigma(A)$, equipped with the usual operations, to $\mathscr{B}(\mathscr{H})$.

Finally, if f and g are real Baire class 1 functions f on $\sigma(A)$, and $f - g \geq 0$ on $\sigma(A)$, define $h = \sqrt{f - g}$, and note that h is a real Baire class 1 functions f on $\sigma(A)$. By what we have just proved, $f(A) - g(A) = (h(A))^2 = (h(A))^* h(A) \geq 0$. □

The $*$-homomorphism provided by Theorem 3.15 need not be an isomorphism. The following example is useful elsewhere: Let $\lambda_0 \in \sigma(A)$ and consider the function 1_{λ_0} given by $1_{\lambda_0}(\lambda) = 1$ for $\lambda = \lambda_0$ and zero otherwise. Then for all λ, $\lambda_0 1_{\lambda_0}(\lambda) = \lambda 1_{\lambda_0}(\lambda)$. By the $*$-homomorphism property,

$$\lambda_0 1_{\lambda_0}(A) = A 1_{\lambda_0}(A) \ .$$

It follows that any non-zero vector in the range of $1_{\lambda_0}(A)$ is an eigenvector of A with eigenvalue λ_0. Conversely, if $A\xi = \lambda_0 \xi$, then for all continuous functions f, $f(A)\xi = f(\lambda_0)\xi$. Let $\{f_n\}$ be a sequence of continuous functions on $\mathbb{R}$ with values in $[0, 1]$ such that $\lim_{n\to\infty} f_n(\lambda) = 1_{\lambda_0}(\lambda)$ for all λ. Then $1_{\lambda_0}(A)\xi = \lim_{n\to\infty} f_n(A)\xi = \lim_{n\to\infty} f_n(\lambda_0)\xi = \xi$. Hence $1_{\lambda_0}(A)$ is the projector onto the eigenspace of A with eigenvalues λ_0, provided λ_0 is an eigenvalue of A, and is zero otherwise, as in the following example: Consider $\mathscr{H} = L^2([0, 1])$ with respect to Lebesgue measure. Let A be the multiplication operator $A\psi(t) = t\psi(t)$. Then it is easy to see that A is self adjoint and $\sigma(A) = [0, 1]$, but A has no eigenvectors at all. Hence for each $\lambda_0 \in [0, 1]$, $1_{\lambda_0}(A)$ is the zero operator. In this case, the $*$-homomorphism provided by Theorem 3.15 is evidently not an isomorphism.

Next, let E be an interval of the form (a,b), $[a,b]$, $(a,b]$ or $[a,b)$. Then 1_E is easily seen to be a Baire class 1 function on $\mathbb{R}$, and hence on $\sigma(A)$ for any self adjoint A. Hence $1_E(A)$ is well defined and belongs to the strong operator topology closure of $\mathscr{C}(\{1,A\})$. Moreover, by the $*$-isomorphism established above, $1_E(A) = (1_E(A))^*$ and $(1_E(A))^2 = 1_E(A)$. Hence $1_E(A)$ is an orthogonal projection. Moreover, if f is any Baire class 1 function with support in E, for all $\eta \in \mathscr{H}$,

$$1_E(A)(f(A)\eta) = (1_E f(A))\eta = f(A)\eta \ ,$$

while if the support of F lies in E^c,

$$1_E(A)(f(A)\eta) = (1_E f(A))\eta = 0 \ .$$

3.16 DEFINITION. Let $A \in \mathscr{B}(\mathscr{H})$ be self adjoint. The set of operators of the form $1_E(A)$, where $E \subset \mathbb{R}$ is such that 1_E is a Baire class 1 function, is the set of *spectral projections of A*.

In particular, let $E = (-\infty, 0) \cup (0, \infty)$. Then $1_E(A)$ is a spectral projection, and by what we have noted above, $1_E(A)\eta = 0$ if and only if $A\eta = 0$. That is, $1_E(A)$ is the orthogonal projection onto the orthogonal complement of $\ker(A)$, or what is the same, the orthogonal projection onto the closure of the range of A. Combining the result obtained in this section, we have:

3.17 LEMMA. *$A \in \mathscr{B}(\mathscr{H})$ be self adjoint. Then the orthogonal projection onto the closure of the range of A belongs to the strong operator topology closure of $\mathscr{C}(\{1,A\})$.*

Let P_1 and P_2 be two orthogonal projections in $\mathscr{B}(\mathscr{H})$ with ranges $\mathscr{K}_1$ and $\mathscr{K}_2$ respectively. Then $P_1 \vee P_2$ denotes the orthogonal projection onto $\overline{\mathscr{K}_1 + \mathscr{K}_2}$, and $P_1 \wedge P_2$ denotes the orthogonal projection onto $\mathscr{K}_1 \cap \mathscr{K}_2$.

3.18 THEOREM. *Let $\mathscr{A}$ be a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ that is closed in the strong operator topology. Then $\mathscr{A}$ is the norm closure of the span of the orthogonal projections contained in $\mathscr{A}$. Moreover, if P_1 and P_2 are two orthogonal projections in $\mathscr{A}$, then $P_1 \vee P_2$ and $P_1 \wedge P_2$ both belong to $\mathscr{A}$.*

Proof. Let $A \in \mathscr{A}$ be self adjoint. Fix $n \in \mathbb{N}$, and define $E_j = [j/n, (j+1)/n)$. Let $k \in \mathbb{N}$, $k \geq \|A\|$. Define $f_n(\lambda) = \sum_{j=-k}^{k} \frac{j}{n} 1_{E_j}(\lambda)$. Define $f(\lambda) = \lambda$. Then $\sup_{\lambda \in \sigma(A)}\{|f_n(\lambda) - f(\lambda)|\} \leq 1/n$, and $A = f(A)$. Hence $\|A - f_n(A)\| \leq 1/n$, and each $f_n(A)$ is a finite linear combination of the orthogonal projections $E_j(A)$, which belong to $\mathscr{A}$ because $\mathscr{A}$ is closed in the strong operator topology. Hence every self adjoint element of $\mathscr{A}$, is the norm closure of the span of the orthogonal projections contained in $\mathscr{A}$, and since $\mathscr{A}$ is a $*$ algebra, every element in it is a sum of two self adjoint elements. This proves the first part.

For the second part, note that the spectrum of $P_1 + P_2$ lies in $\{0,1,2\}$. Let f be any continuous real valued function with $f(0) = 0$, $f(1) = f(2) = 1$. Then $P_1 \vee P_2 = f(P_1 + P_2)$, and $f(P_1 + P_2) \in \mathscr{A}$. The proof for $P_1 \wedge P_2$ is similar. □

3.5 The polar decomposition

3.19 DEFINITION (Operator absolute value)**.** Let $\mathscr{H}$ be a Hilbert space and let $A \in \mathscr{B}(\mathscr{H})$. Then the operator absolute value of A is the operator $|A|$ defined by

$$|A| = \sqrt{A^*A} \ , \tag{3.19}$$

where the square root is taken using the Abstract Spectral Theorem.

3.20 REMARK. *One should not be misled by the notation: It is not in general true that* $|AB| = |A||B|$*, or that* $|A^*| = |A|$ *or even that* $|A + B| \leq |A| + |B|$*.*

Partial isometries in a C^* algebra have already been defined in Definition 2.47; recall that U is a partial isometry when $\sigma(U^*U) \subset \{0, 1\}$. When the C^* algebra is $\mathscr{B}(\mathscr{H})$, there is more to say:

Provided $U \neq 0$, $1 \in \sigma(U^*U)$, and the corresponding eigenspace of U^*U is a closed subspace of $\mathscr{H}$, called the *initial space* of U, $\ker(U^*U) = \ker(U)$ is the orthogonal complement of the initial space of U, and U^*U is the orthogonal projection onto the initial space of U. As we have remarked earlier. UU^* has the same spectrum as U^*U, and hence U^* is also a partial isometry. Its initial space is called the *final space* of U. If η belongs to the final space of U, $\eta = U(U^*\eta) \in \mathrm{ran}(U)$, so that the final space is contained in $\mathrm{ran}(U)$. If $\eta \in \mathrm{ran}(U)$, there exists $\xi \in \ker(U)^\perp$ such that $\eta = U\xi$. Then $\xi = U^*U\xi$ and so $\eta = UU^*(U\xi)$, and hence η belongs to the final space of U. Altogether, $\mathrm{ran}(U)$, which is closed, is the final space of U, and U is a unitary transformation from its initial space onto its final space.

3.21 LEMMA. *For all* $A \in \mathscr{B}(\mathscr{H})$*, there is a unique partial isometry* U *in* $\mathscr{B}(\mathscr{H})$ *such that* $A = U|A|$*, the initial space of* U *is* $\ker(A)^\perp$*, and the final space of* U *is* $\overline{\mathrm{ran}(A)}$*. Moreover,* U *belongs to the strong closure of* $\mathscr{C}(\{1, A, A^*\})$*.*

Proof. For each $t > 0$, define the operator $U_t := A(t1 + |A|)^{-1}$, noting that $1t + |A|$ is invertible. For $s, t > 0$, the Resolvent Identity (1.15) yields

$$U_t - U_s = (s - t)A[(t1 + |A|)^{-1}(s1 + |A|)^{-1}] \ .$$

Hence for any $\xi \in \mathscr{H}$, and $0 < s < t$,

$$\begin{aligned} \|(U_t - U_s)\xi\|^2 &= (s - t)^2\langle \xi, |A|^2(t1 + |A|)^{-2}(s1 + |A|)^{-2}\xi\rangle_{\mathscr{H}} \\ &= (t - s)^2 \int_{\sigma(|A|)} \frac{\lambda^2}{(t + \lambda)^2(s + \lambda)^2} \mathrm{d}\mu_\xi \\ &\leq \int_{\sigma(|A|)\backslash\{0\}} \frac{t^2}{(t + \lambda)^2} \mathrm{d}\mu_\xi \end{aligned}$$

Since $0 \leq t^2/(t + \lambda)^2 \leq 1$ for all $\lambda > 0$, and since $\lim_{t\to 0} t^2/(t + \lambda)^2 = 0$ for all $\lambda > 0$, the Lebesgue Dominated Convergence Theorem yields $\displaystyle\lim_{t\to 0}\left(\sup_{s<t}\{\|(U_t - U_s)\xi\|^2\}\right) = 0$. Thus, the strong limit $U = \lim_{t\to 0} U_t$ exists. Note that $U|A| = \lim_{t\to 0} U_t|A| = A \lim_{t\to 0} f_t(|A|)$ where $f_t(\lambda) = \lambda/(t + \lambda)$. Since $\lim_{t\to 0} f_t(\lambda) = 1_{(0,\infty)}(\lambda)$ for all $\lambda \geq 0$, it follows from Theorem 3.15 that $\lim_{t\to 0} f_t(|A|) = 1_{(0,\infty)}(|A|) = 1 - 1_{\{0\}}(|A|)$. Since $1_{\{0\}}(|A|)$ is the projector onto $\ker(|A|) = \ker(A)$,

$$U|A| = A \ . \tag{3.20}$$

Next note that $U^*U = \lim_{t\to\infty} f_t^2(|A|)$ with $f_t(\lambda) = \lambda/(t + \lambda)$ once more. It follows that

$$U^*U = 1_{(0,\infty)}(|A|) \tag{3.21}$$

which is the orthogonal projection onto $\ker(A)^\perp$. It follows from (3.20) that $\mathrm{ran}(U) = \overline{\mathrm{ran}(A)}$, and hence U is a partial isometry from $\ker(A)^\perp$ onto $\overline{\mathrm{ran}(A)}$. □

Taking the adjoint of (3.20), we obtain $A^* = |A|U^*$ and hence $AA^* = UA^*AU^*$. Squaring both sides and observing that $AU^*U = A$, which follows from (3.21), $(AA^*)^2 = U(A^*A)^2U^*$. An induction now yields $(AA^*)^n = U(A^*A)^nU^*$ for all n, and then taking a polynomial approximation to the square root, we conclude that

$$U|A|U^* = |A^*| \ , \tag{3.22}$$

and then since $A^* = |A|U^* = U^*U|A|U^*$,

$$A^* = U^*(U|A|U^*) \tag{3.23}$$

is the polar decomposition of A^*.

3.22 THEOREM. *Let $\mathscr{A}$ be a unital $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ that is closed in the strong operator topology. Then for all $A \in \mathscr{A}$, the components $|A|$ and U of the polar decomposition $A = U|A|$ both belong to $\mathscr{A}$. Moreover, for all $A \in \mathscr{A}$, the orthogonal projections onto* $\ker(A)^\perp$ *and* $\overline{\mathrm{ran}(A)}$ *both belong to $\mathscr{A}$.*

Proof. Since the strong operator topology is weaker than the norm topology, $\mathscr{A}$ is a C^* algebra, and hence $|A| = \sqrt{A^*A} \in \mathscr{A}$. By Lemma 3.21, $U \in \mathscr{A}$. Since $\mathscr{A}$ is a $*$-algebra, U^*U and UU^* both belong to $\mathscr{A}$, by Lemma 3.21, these are, respectively, the orthogonal projections onto $\ker(A)^\perp$ and $\overline{\mathrm{ran}(A)}$. □

3.6 Three theorems on strong closure

$\mathscr{B}(\mathscr{H})_{\text{s.a.}}$ is partially ordered by the relation defined by $A \geq B$ in case $A - B$ is positive.

3.23 DEFINITION. A set $\mathcal{S}$ of self adjoint operators in $\mathscr{B}(\mathscr{H})$ is an *upward directed set* in case for all $A, B \in \mathcal{S}$, there exists $C \in \mathcal{S}$ such that $C \geq A$ and $C \geq B$. Likewise, it is a *downward directed set* in case for all $A, B \in \mathcal{S}$, there exists $C \in \mathcal{S}$ such that $C \leq A$ and $C \leq B$.

The next theorem gives a condition for a set of self adjoint operators in $\mathscr{B}(\mathscr{H})$ to have a maximal element in its strong operator topology closure. Since $\mathcal{S}$ is downward directed if and only if $-\mathcal{S}$ is upward directed, there is a trivial restatement in terms of downward directed sets.

3.24 THEOREM (Vigier's Theorem)**.** *Let $\mathcal{S}$ be an upward directed set in $\mathscr{B}(\mathscr{H})_{\text{s.a.}}$ that is norm bounded in $\mathscr{B}(\mathscr{H})$. Then there exists a unique $A_{\max}$ in the strong operator topology closure of $\mathcal{S}$ such that $A_{\max} \geq A$ for all $A \in \mathcal{S}$.*

Proof. Define a function q on $\mathscr{H}$ by $q(\xi) = \sup\{\langle \xi, B\xi\rangle \ : B \in \mathcal{S}\}$. Then for any $\epsilon > 0$ and any finite set $\{\xi_1, \dots, \xi_n\}$ in $\mathscr{H}$, there are $\{A_1, \dots, A_n\}$ in $\mathcal{S}$ such that $q(\xi_j) \geq \langle \xi_j, A_j\xi_j\rangle \geq q(\xi_j) - \epsilon$. $\mathcal{S}$ is upward directed, and hence there exists $A \in \mathcal{S}$ with $q(\xi) \geq \langle \xi_j, A\xi_j\rangle \geq q(\xi_j) - \epsilon$ for each j. That is, on any finite set, q may be uniformly approximated by functions of the form $\xi \mapsto \langle \xi, A, \xi\rangle$.

Define a function b on $\mathscr{H} \times \mathscr{H}$ by polarization of q:

$$b(\eta, \xi) = \frac{1}{4}[q(\eta + \xi) - q(\eta - \xi)] - \frac{i}{4}[q(\eta + i\xi) - q(\eta - i\xi)] \ .$$

By what has been proved about q, b is a bounded sesquilinear form on $\mathscr{H}$, and hence there is an operator $B \in \mathscr{B}(\mathscr{H})$, whose norm is no more than the norm bound on $\mathcal{S}$, such that $b(\eta, \xi) = \langle \eta, B\xi\rangle$

for all $\eta, \xi \in \mathscr{H}$. For all $\xi \in \mathscr{H}$, $\langle \xi, B\xi \rangle = q(\xi) \geq \langle \xi, A\xi \rangle$ for all $A \in \mathcal{S}$. That is, $B - A \geq 0$ for all $A \in \mathcal{S}$.

Finally, let c denote the norm bound on $\mathcal{S}$. Given $\epsilon > 0$ and $\{\xi_1, \ldots, \xi_n\} \subset \mathscr{H}$, for each $j = 1, \ldots, n$, by the definition of B, we may choose $A_j \in \mathcal{S}$ such that

$$\langle \xi_j, A_j \xi_j \rangle \geq q(\xi_j) - \frac{\epsilon^2}{2c} = \langle \xi_j, B\xi_j \rangle - \frac{\epsilon^2}{2c} \ . \tag{3.24}$$

Since $\mathcal{S}$ is upward directed, there exists $A \in \mathcal{S}$ such that $A \geq A_j$, $j = 1, \ldots, n$. Then evidently (3.24) is valid with A_j replaced by A for each j. Then for each j, there is a unit vector η_j such that

$$\|(B - A)\xi_j\| = \langle \eta_j, (B - A)\xi_j \rangle = \langle (B - A)^{1/2}\eta_j, (B - A)^{1/2}\xi_j \rangle \leq \|B - A\| \langle \xi_j, (B - A)\xi_j \rangle \ .$$

By (3.24) with A in place of A_j, $\|(B - A)\xi_j\| \leq \epsilon$ for each j. Hence B belongs to the strong closure of $\mathcal{S}$. □

3.25 DEFINITION. If $\mathcal{S}$ is a bounded upward directed set of self adjoint operators in $\mathscr{B}(\mathscr{H})$, the unique operator B in the strong closure of $\mathcal{S}$ such that $B \geq A$ for all $A \in \mathcal{S}$, is called the *least upper bound* of $\mathcal{S}$, and is denoted $\bigvee_{A \in \mathcal{S}} A$. Likewise if $\mathcal{S}$ is a bounded downward directed set of self adjoint operators in $\mathscr{B}(\mathscr{H})$, the unique operator B in the strong closure of $\mathcal{S}$ such that $B \leq A$ for all $A \in \mathcal{S}$ is called the *greatest lower bound* of $\mathcal{S}$, and is denoted $\bigwedge_{A \in \mathcal{S}} A$.

Vigier's theorem has a number of consequences. Here is the first of these.

3.26 COROLLARY. *Let $\mathscr{A}$ be a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ that is closed in the strong operator topology. Then $\mathscr{A}$ is unital.*

Proof. Let $\mathcal{S}$ denote the set of all orthogonal projections contained in $\mathscr{A}$. Evidently $\|P\| \leq 1$ for all $P \in \mathcal{S}$, and by Theorem 3.18, for all $P, Q \in \mathcal{S}$, $P \vee Q \in \mathcal{S}$. Then by Theorem 3.24, there exists $P_{\max} = \text{l.u.b.}(\mathcal{S})$ in the closure of $\mathcal{S}$ in the strong operator topology, and hence $\in \mathscr{A}$, such that $P_{\max} \geq P$ for all $P \in \mathcal{S}$. $P_{\max}$ is evidently an orthogonal projection whose range includes the range of any other orthogonal projection P in $\mathscr{A}$. Then by Theorem 3.18 once more, $P_{\max}A = AP_{\max} = A$ for all $A \in \mathscr{A}$. □

Note that the multiplicative identity $P_{\max}$ provided by Corollary 3.26, need not be the identity in $\mathscr{B}(\mathscr{H})$. However, every operator A in $\mathscr{A}$ annihilates the range of $P_{\max}^{\perp}$, and hence nothing is lost if we restrict $\mathscr{A}$ to the range of $P_{\max}$, and on this subspace of $\mathscr{H}$, $P_{\max}$ is the identity operator.

3.27 DEFINITION. Let $\mathcal{S} \subset \mathscr{B}(\mathscr{H})$. The *commutant* $\mathcal{S}'$ of $\mathcal{S}$ is the subset of $\mathscr{B}(\mathscr{H})$ given by

$$\mathcal{S}' = \{ \ A \in \mathscr{B}(\mathscr{H}) \ : \ AB - BA = 0 \quad \text{for all} \quad B \in \mathcal{S} \ \} \ .$$

We write $[A, B] = AB - BA$ to denote the *commutator* of A and B.

3.28 LEMMA. *Let $\mathcal{S} \subset \mathscr{B}(\mathscr{H})$. The commutant $\mathcal{S}'$ of $\mathcal{S}$ has the following properties:*

(1) $\mathcal{S}'$ is closed in the weak operator topology on $\mathscr{B}(\mathscr{H})$.

(2) $\mathcal{S}'$ is a unital subalgebra of $\mathscr{B}(\mathscr{H})$.

(3) If $\mathcal{S}$ is closed under the involution, $\mathcal{S}'$ is a weakly closed unital $*$-subalgebra of $\mathscr{B}(\mathscr{H})$.

Proof. For $\zeta, \xi \in \mathscr{H}$, $B \in \mathcal{S}$, define a linear functional $\varphi_{\zeta,\xi,B}$ on $\mathscr{B}(\mathscr{H})$ by

$$\varphi_{\zeta,\xi,B}(A) = \langle \zeta, ([A,B])\xi\rangle_{\mathscr{H}} = \langle \zeta, A(B\xi)\rangle_{\mathscr{H}} - \langle (B^*\zeta), A\xi\rangle_{\mathscr{H}} \ .$$

Since $\varphi_{\zeta,\xi,B}$ is weakly continuous, $\varphi^{-1}_{\zeta,\xi,B}(\{0\})$ is weakly closed. Then since

$$\mathcal{S}' = \bigcap \{ \ \varphi^{-1}_{\zeta,\xi,B}(\{0\}) \ : \ \zeta, \xi \in \mathscr{H} \ , \ B \in \mathcal{S} \ \} \ ,$$

(1) is proved. *(2)* is evident, and then *(3)* follows from the fact that $[A,B]^* = [B^*, A^*]$ together with *(1)* and *(2)*. □

3.29 LEMMA. *Let $\mathscr{A}$ be a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$. A closed subspace $\mathscr{K}$ of $\mathscr{H}$ is invariant under $\mathscr{A}$ if and only if the orthogonal projection P of $\mathscr{H}$ onto $\mathscr{K}$ belongs to $\mathscr{A}'$.*

Proof. First, $\mathscr{K}$ is invariant under $\mathscr{A}$ if and only if $\mathscr{K}^\perp$ is invariant under $\mathscr{A}$. To see this, let $\zeta \in \mathscr{K}^\perp$ and $\xi \in \mathscr{K}$, and $A \in \mathscr{A}$. If $\mathscr{K}$ is invariant, $A^*\xi \in \mathscr{K}$, and hence

$$\langle A\zeta, \xi\rangle_{\mathscr{H}} = \langle \zeta, A^*\xi\rangle_{\mathscr{H}} = 0 \ .$$

Thus the invariance of $\mathscr{K}$ implies the invariance of $\mathscr{K}^\perp$, and then by symmetry, the reverse implication is valid as well.

Now let P be the orthogonal projection onto $\mathscr{K}$. When $\mathscr{K}$ is invariant, for all $A \in \mathscr{A}$,

$$0 = PA(1-P) = PA - PAP = PA - AP \ .$$

Therefore, $P \in \mathscr{A}'$. Conversely, if $P \in \mathscr{A}'$ and $\xi \in \mathscr{K}$, then for all $A \in \mathscr{A}$, $A\xi = AP\xi = PA\xi \in \mathscr{K}$, showing the invariance of $\mathscr{K}$. □

3.30 THEOREM (von Neumann Double Commutant Theorem)**.** *Let $\mathscr{A}$ be a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ that contains the identity. Then $\mathscr{A}''$ is the weak operator topology closure of $\mathscr{A}$.*

Proof. Since $\mathscr{A}$ is convex, the weak and strong operator topology closures of $\mathscr{A}$ coincide. Hence it suffices to show that for all $A \in \mathscr{A}''$, every strong neighborhood of A contains some $B \in \mathscr{A}$. That is, it suffices to show that for all $n \in \mathbb{N}$ and all $\{\eta_1, \dots, \eta_n\} \subset \mathscr{H}$, and all $\epsilon > 0$, there is some $B \in \mathscr{A}$ such that $\|(B-A)\eta_j\| < \epsilon$ for all $j = 1, \dots, n$.

Let $\widehat{\mathscr{H}} = \mathscr{H} \oplus \cdots \oplus \mathscr{H}$, the direct sum of n copies of $\mathscr{H}$. The elements of $\mathscr{B}(\widehat{\mathscr{H}})$ are $n \times n$ matrices $[B_{i,j}]$ with entries in $\mathscr{B}(\mathscr{H})$.

Let $\widehat{\mathscr{A}}$ be the algebra of all operators on $\widehat{\mathscr{H}}$ of the form $[A\delta_{i,j}]$ with $A \in \mathscr{A}$. Evidently, its commutator $\widehat{\mathscr{A}'}$ consists of all $[B_{i,j}]$ with each $B_{i,j} \in \mathscr{A}'$. Thus, for all $A \in \mathscr{A}''$, $[A\delta_{i,j}] \in \widehat{\mathscr{A}''}$.

Let $\eta = \eta_1 \oplus \cdots \oplus \eta_n$, and define $\mathscr{K} = \overline{\widehat{\mathscr{A}}\eta}$, which is a closed subspace of $\widehat{\mathscr{H}}$ that is invariant under $\widehat{\mathscr{A}}$. Let P be the orthogonal projection onto $\mathscr{K}$. Since $\mathscr{K}$ is invariant under $\widehat{\mathscr{A}}$, $P \in \widehat{\mathscr{A}'}$ by Lemma 3.29, and hence $PB = BP$ for all $B \in \widehat{\mathscr{A}''}$. That is, $\mathscr{K}$ is invariant under $\widehat{\mathscr{A}''}$. In particular, for all $A \in \mathscr{A}''$, $\mathscr{K}$ is invariant under $[A\delta_{i,j}]$.

Since $\mathscr{A}$ contains the identity, $\eta \in \mathscr{K}$, so that $A\eta_1 \oplus \cdots \oplus A\eta_n \in \mathscr{K}$. Therefore, by the definition of $\mathscr{K}$ as the closure of $\widehat{\mathscr{A}}\eta$, for all $\epsilon > 0$, there exists $B \in \mathscr{A}$ such that

$$\|B\eta_1 \oplus \cdots \oplus B\eta_n - A\eta_1 \oplus \cdots \oplus A\eta_n\|^2_{\mathscr{H}} \le \epsilon^2 \ .$$

□

Let $\mathscr{A}$ be a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$, and let $\mathscr{M}$ be its closure in the strong operator topology. If $\mathscr{A}$ is unital, then the von Neumann Double Commutant Theorem implies that $\mathscr{M}$ is also a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$, even though the product map is not strongly continuous. The next theorem will show that $\mathscr{M}$ is a $*$-subalgebra even when $\mathscr{A}$ is not unital.

$B_{\mathscr{A}}$, the unit ball in $\mathscr{A}$, is contained in $B_{\mathscr{M}}$, the unit ball in $\mathscr{M}$. By Proposition 3.10, the norm function is lower semicontinuous in the strong topology. Therefore, $B_{\mathscr{B}(\mathscr{H})}$ is strongly closed, and hence $B_{\mathscr{M}}$ is strongly closed. However, again on account of the lower semicontinuity, it is not immediately clear that $B_{\mathscr{A}}$ is strongly dense in $B_{\mathscr{M}}$. It is not obvious that there are no elements of $B_{\mathscr{M}}$ with a weak neighborhood U that only contains elements $A \in \mathscr{A}$ with $\|A\|$ strictly larger – or even much larger – than one. However, this, and more, is true.

3.31 THEOREM (Kaplansky's Density Theorem)**.** *Let $\mathscr{A}$ be a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$, and let $\mathscr{M}$ be its closure in the strong operator topology. Then $B_{\mathscr{A}}$ is strongly dense in $B_{\mathscr{M}}$, and the self-adjoint part of $B_{\mathscr{A}}$ is strongly dense in the self adjoint part of $B_{\mathscr{M}}$.*

Proof. Let $\mathscr{A}_{\rm s.a}$ denote the space of self adjoint elements in $\mathscr{A}$, and let $\mathscr{M}_{\rm s.a}$ denote the space of self adjoint elements in $\mathscr{M}$. As a first step, we show that the strong closure of $\mathscr{A}_{\rm s.a}$ is $\mathscr{M}_{\rm s.a}$.

$\mathscr{B}(\mathscr{H})_{\rm s.a.}$ is the null space of the real linear map $X \mapsto X - X^*$. Since the involution is weakly continuous, the convex set $\mathscr{B}(\mathscr{H})_{\rm s.a.}$ is weakly, and hence strongly closed. $\mathscr{M}_{\rm s.a.} = \mathscr{M} \cap \mathscr{B}(\mathscr{H})_{\rm s.a.}$ is thus strongly closed. Therefore, the strong closure of $\mathscr{A}_{\rm s.a}$, $\overline{\mathscr{A}_{\rm s.a}}$, satisfies $\overline{\mathscr{A}_{\rm s.a}} \subset \mathscr{M}_{\rm s.a.}$. It then remains to show that $\mathscr{A}_{\rm s.a}$ is weakly dense in $\mathscr{M}_{\rm s.a}$ since, being convex, its weak closure is the same as its strong closure.

Let $Y \in \mathscr{M}_{\rm s.a.}$. If $V_{\epsilon,\xi_1,\dots,\xi_n}$ is any weak basic neighborhood of 0, as in (3.7), $Y + V_{\epsilon,\xi_1,\dots,\xi_n}$ contains some $X \in \mathscr{A}$, i.e., there is an $X \in \mathscr{A}$ such that $|\langle \xi_j, (Y - X)\xi_j\rangle| < \epsilon$ for all $j = 1, \dots, n$. Then since Y is self adjoint, $|\langle \xi_j, (Y - X^*)\xi_j\rangle| < \epsilon$ for all $j = 1, \dots, n$, and then by convexity, with $Z = \frac{1}{2}(X + X^*)$, $|\langle \xi_j, (Y - Z)\xi_j\rangle| < \epsilon$ for all $j = 1, \dots, n$, and $Z \in \mathscr{A}_{\rm s.a.}$. Hence Y is in the weak closure of $\mathscr{A}_{\rm s.a.}$. This completes the proof that $\overline{\mathscr{A}_{\rm s.a}} = \mathscr{M}_{\rm s.a.}$.

Let $B_{\mathscr{A}_{\rm s.a}}$ denote the unit ball in $\mathscr{A}_{\rm s.a}$, and let $B_{\mathscr{M}_{\rm s.a}}$ denote the unit ball in $\mathscr{M}_{\rm s.a}$. We now show that the strong closure of $B_{\mathscr{A}_{\rm s.a}}$ is $B_{\mathscr{M}_{\rm s.a}}$, which is the second part of the theorem.

Without loss of generality, we may suppose that $\mathscr{A}$ is norm closed; i.e., that $\mathscr{A}$ is a C^* algebra. Consider the function $f : \mathbb{R} \to [-1, 1]$ given by

$$f(\lambda) = \frac{2\lambda}{1 + \lambda^2}\ .$$

It is easy to see that f a homeomorphism from $[-1, 1]$ onto $[-1, 1]$. Let g denote the inverse function on $[-1, 1]$. The explicit formula is readily computed, but what really matters is that since both $f(0) = 0$ and $g(0) = 0$, we may apply the Abstract Spectral Theorem to define $f(A)$ for all $A \in \mathscr{A}_{\rm s.a.}$, and $g(A)$ for all $A \in B_{\mathscr{A}} \cap \mathscr{A}_{\rm s.a.}$, even when $\mathscr{A}$ is not unital. Let $B \in B_{\mathscr{M}} \cap \mathscr{M}_{\rm s.a.}$, and define $\widehat{B} = g(B)$. Let $\widehat{A} \in \mathscr{A}_{\rm s.a.}$ and define $A = f(\widehat{A})$. Evidently $A \in \mathscr{A}_{\rm s.a.} \cap B_{\mathscr{A}}$. By what we have noted above, and by Corollary 2.23, $f(\widehat{B}) = f(g(B)) = fg(B) = B$. Our aim now is to show that, for any $\xi \in \mathscr{H}$, $\epsilon > 0$, if $\widehat{A}$ is chosen from an appropriate strong neighborhood of $\widehat{B}$, then $\|(A - B)\xi\| < \epsilon$.

Using $\widehat{A}\widehat{B}^2 - \widehat{A}^2\widehat{B} = \widehat{A}(\widehat{B} - \widehat{A})\widehat{B}$ in the last step, computation yields

$$\begin{aligned} A - B &= f(\widehat{A}) - f(\widehat{B}) \\ &= 2\frac{1}{1+\widehat{A}^2}\left(\widehat{A}(1+\widehat{B}^2) - (1+\widehat{A}^2)\widehat{B}\right)\frac{1}{1+\widehat{B}^2} \\ &= 2\frac{1}{1+\widehat{A}^2}\left(\widehat{A} - \widehat{B}\right)\frac{1}{1+\widehat{B}^2} + \frac{2\widehat{A}}{1+\widehat{A}^2}\left(\widehat{B} - \widehat{A}\right)\frac{\widehat{B}}{1+\widehat{B}^2} \end{aligned}$$

Note that $\left\|\frac{1}{1+\widehat{A}^2}\right\|, \left\|\frac{2\widehat{A}}{1+\widehat{A}^2}\right\| < 1$, and therefore if $\widehat{A}$ satisfies

$$\left\|(\widehat{A} - \widehat{B})\frac{1}{1+\widehat{B}^2}\xi\right\| < \frac{\epsilon}{3} \quad \text{and} \quad \left\|(\widehat{A} - \widehat{B})\frac{\widehat{B}}{1+\widehat{B}^2}\xi\right\| < \frac{\epsilon}{3},$$

then $\|(A - B)\xi\| < \epsilon$. This proves that for every strong neighborhood U of B, there is a strong neighborhood V of $\widehat{B}$ so that if $\widehat{A} \in \mathscr{A}_{\text{s.a.}}$ is chosen in V, then $A = f(\widehat{A}) \in U$. Since $A \in \mathscr{A}_{\text{s.a.}} \cap B_{\mathscr{A}}$, this shows that $\mathscr{A}_{\text{s.a.}} \cap B_{\mathscr{A}}$ is strongly dense in $\mathscr{M}_{\text{s.a.}} \cap B_{\mathscr{M}}$.

To pass to the general case, consider $\mathscr{B}(\mathscr{H}) \otimes M_2(\mathbb{C})$, consisting of 2×2 block matrices $\begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$ with entries in $\mathscr{B}(\mathscr{H})$; we identify $\mathscr{B}(\mathscr{H}) \otimes M_2(\mathbb{C})$ with $\mathscr{B}(\mathscr{H} \oplus \mathscr{H})$. A neighborhood basis at 0 for the strong topology in $\mathscr{B}(\mathscr{H}) \otimes M_2(\mathbb{C})$ is given by the sets $\widetilde{U}_{\epsilon,\xi_1,\cdots\xi_n}$ consisting of operators of the form $\begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$ in $\mathscr{B}(\mathscr{H}) \otimes M_2(\mathbb{C})$ such that $B_{i,j} \in U_{\epsilon,\xi_1,\ldots,\xi_n}$ with $U_{\epsilon,\xi_1,\ldots,\xi_n}$ specified in (3.6). It is then easy to see that $\mathscr{A} \otimes M_2(\mathbb{C})$ is dense in $\mathscr{M} \otimes M_2(\mathbb{C})$. For $B \in \mathscr{M}$, define

$$\widetilde{B} = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}. \tag{3.25}$$

Then $\|\widetilde{B}\| = \|B\|$, so that for $B \in B_{\mathscr{M}}$, by what we have just proved, for any neighborhood $\widetilde{U}_{\epsilon,\xi_1,\cdots\xi_n}$, there is $\widetilde{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \in \mathscr{A} \otimes M_2(\mathbb{C})$ with $A_{2,1} = A_{1,2}^*$ and $A_{1,1}, A_{2,2} \in \mathscr{A}_{\text{s.a.}}$ and $\|\widetilde{A}\| \leq 1$, such that $\widetilde{A} - \widetilde{B} \in \widetilde{U}_{\epsilon,\xi_1,\cdots\xi_n}$. Then $A_{1,2} - B \in U_{\epsilon,\xi_1,\cdots\xi_n}$, and since $\|\widetilde{A}\| \leq 1$, $\|A_{1,2}\| \leq 1$. □

We now come to an important class of operator algebras singled out by von Neumann.

3.32 DEFINITION. A von Neumann algebra is C^* subalgebra of $\mathscr{B}(\mathscr{H})$ that contains the identity on $\mathscr{B}(\mathscr{H})$ and is closed in the strong operator topology.

Replacing the strong operator topology by the weak operator topology would not make the definition any more inclusive since the same convex sets (and hence subspaces) of $\mathscr{B}(\mathscr{H})$ are closed for the two topologies. Corollary 3.26 shows that any weakly closed C^* subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ is unital, and the unit is an orthogonal projection in $\mathscr{B}(\mathscr{H})$. Restricting the operators in $\mathscr{A}$ to the range of this projection, nothing essential is lost, and then the identity in the algebra is the identity operator on the smaller Hilbert space.

3.7 Trace class operators and the σ-weak topology

3.33 DEFINITION (Trace class)**.** Let $\mathscr{H}$ be a Hilbert space. An operator $A \in \mathscr{B}(\mathscr{H})$ is *trace class* in case

$$\|A\|_1 := \sup \left\{ \sum_{j=1}^{n} |\langle \eta_j, A\xi_j \rangle| \ , \{\eta_1, \ldots, \eta_n\} \ , \ \{\xi_1, \ldots, \xi_n\} \text{ orthonormal} \ , \ n \in \mathbb{N} \right\} < \infty \ . \tag{3.26}$$

The set of trace class operators is denoted by $\mathscr{T}(\mathscr{H})$. The function $A \mapsto \|A\|_1$ on $\mathscr{T}(\mathscr{H})$ defined by (3.26) is called the *trace norm* on $\mathscr{T}(\mathscr{H})$; this terminology is justified below.

3.34 THEOREM. *For any Hilbert space $\mathscr{H}$, $\mathscr{T}(\mathscr{H})$ is a (non-closed) $*$-ideal in $\mathscr{B}(\mathscr{H})$, and the function $A \mapsto \|A\|_1$ is a norm on A under which the involution $*$ is isometric. $\mathscr{T}(\mathscr{H})$ is complete in the trace norm. Moreover, for all $A \in \mathscr{T}(\mathscr{H})$ and all $B \in \mathscr{B}(\mathscr{H})$,*

$$\|AB\|_1 \leq \|B\|\|A\|_1 \quad \text{and} \quad \|BA\|_1 \leq \|B\|\|A\|_1 \ . \tag{3.27}$$

and there is equality in both inequalities in (3.27) *whenever B is unitary.*

Proof. Since $|\langle \eta, (A+B)\xi \rangle| \leq |\langle \eta, A\xi \rangle| + |\langle \eta, B\xi \rangle|$, it is evident that $\mathscr{T}(\mathscr{H})$ is closed under addition, and moreover that $\|A+B\|_1 \leq \|A\|_1 + \|B\|_1$. It is also evident that $\mathscr{T}(\mathscr{H})$ is closed under scalar multiplication. For $A \in \mathscr{T}(\mathscr{H})$, $\|A\|_1 \geq \|A\| = \sup\{|\langle \eta, A\xi \rangle| \ : \|\eta\| = \|\xi\| = 1\}$. Hence $\|A\|_1 = 0$ if and only if $A = 0$. This shows that the trace norm is indeed a norm. Since $|\langle \xi, A^*\eta \rangle| = |\langle \eta, A\xi \rangle|$, $\|A^*\|_1 = \|A\|_1$ and hence $\mathscr{T}(\mathscr{H})$ is closed under the involution, and the involution is isometric in the trace norm.

Finally, let $A \in \mathscr{T}(\mathscr{H})$, and let U be unitary in $\mathscr{B}(\mathscr{H})$. Since $\{U\eta_1, \ldots, U\eta_n\}$ is orthonormal if and only if $\{\eta_1, \ldots, \eta_n\}$ is orthonormal, it follows that AU and UA are trace class, and that $\|AU\|_1 = \|UA\|_1 = \|A\|_1$.

Now let $\epsilon > 0$, and let $B \in \mathscr{B}(\mathscr{H})$ satisfy $\|B\| < 1 - \epsilon$. By Theorem 2.42, for $m > 2/\epsilon$, B has the form $B = \frac{1}{m}\sum_{j=1}^{m} U_j$, where each j is unitary. Then $\|BA\|_1 \leq \dfrac{1}{m}\displaystyle\sum_{j=1}^{m} \|U_j A\|_1 = \|A\|_1$. Since $\epsilon > 0$ is arbitrary, it follows that for all $\|B\|$ in the unit ball of $\mathscr{B}(\mathscr{H})$, $\|BA\|_1 \leq \|A\|_1$, and then that for all $B \in \mathscr{B}(\mathscr{H})$, the first inequality in (3.27) is valid. The same argument (or use of the isometry property of the involution) proves the second inequality. Finally, (3.27) shows that $\mathscr{T}(\mathscr{H})$ is an ideal in $\mathscr{B}(\mathscr{H})$.

To prove completeness, recall that the operator norm is dominated by the trace norm; i.e., $\|\cdot\| \leq \|\cdot\|_1$, Let $\{A_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{T}(\mathscr{H})$. Then it is a Cauchy sequence in $\mathscr{B}(\mathscr{H})$ and there exists $A \in \mathscr{B}(\mathscr{H})$ such that $\lim_{n\to\infty} \|A - A_n\| = 0$. Pick $\epsilon > 0$ and $N \in \mathbb{N}$ so that for $m, n > N$, $\|A_m - A_n\|_1 < \epsilon$. Then for any $p \in \mathbb{N}$, and any orthonormal sets $\{\eta_1, \ldots, \eta_p\}$ and $\{\xi_1, \ldots, \xi_p\}$, $\displaystyle\sum_{j=1}^{p} |\langle \eta_j, (A_m - A_n)\xi_j \rangle| < \epsilon$. Since $\lim_{m\to\infty}\langle \eta_j, (A_m - A_n)\xi_j \rangle = \langle \eta_j, (A - A_n)\xi_j \rangle$,

$$\sum_{j=1}^{p} |\langle \eta_j, (A - A_n)\xi_j \rangle| < \epsilon \ ,$$

and this implies that $\|A - A_n\|_1 \leq \epsilon$, showing at the same time that $A \in \mathscr{T}(\mathscr{H})$ and that $\lim_{n\to\infty} \|A - A_n\|_1 = 0$. □

3.35 LEMMA. *For all $A \in \mathscr{T}(\mathscr{H})$, and all $\epsilon > 0$, there is a finite rank operator A_ϵ such that $\|A - A_\epsilon\|_1 < \epsilon$. Moreover, $\ker(A)^\perp$ and $\overline{\operatorname{ran}(A)}$ are separable subspaces of $\mathscr{H}$, and hence there exists a separable subspace $\mathscr{K}$ of $\mathscr{H}$ such that $A|_{\mathscr{K}^\perp} = 0$ and $A\mathscr{K} \subset \mathscr{K}$.*

Proof. Let $A \in \mathscr{T}(\mathscr{H})$, and let $A = U|A|$ be its polar decomposition. Then $|A| = U^*A \in \mathscr{T}(\mathscr{H})$, and $\||A|\|_1 = \|A\|_1$. We claim that

$$\||A|\|_1 := \sup\left\{ \sum_{j=1}^n \langle \xi_j, |A|\xi_j\rangle \ , \{\xi_1, \dots, \xi_n\} \subset \ker|A|^\perp \ \text{ orthonormal} \ , \ n \in \mathbb{N} \right\} < \infty \ . \tag{3.28}$$

To see this, use the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality to conclude that for any unit vectors η and ξ,

$$|\langle \eta, |A|\xi\rangle| = |\langle |A|^{1/2}\eta, |A|^{1/2}\xi\rangle| \le \frac{1}{2}\langle \eta, |A|\eta\rangle + \frac{1}{2}\langle \xi, |A|\xi\rangle \ .$$

Hence for positive A, there is nothing to be gained by using two distinct orthonormal sets in (3.26). Next, suppose that $\{\xi_1, \dots, \xi_n\}$ and $\{\xi_1', \dots, \xi_n'\}$ are two orthonormal sets with the same span. Then for each i, $\xi_i' = \sum_{j=1}^n \langle \xi_j, \xi_i'\rangle \xi_j$, and

$$\sum_{i=1}^n \langle \xi_i', |A|\xi_i'\rangle = \sum_{i,j,k=1}^n \langle \xi_j, \xi_i'\rangle\langle \xi_i', \xi_k\rangle\langle \xi_k, |A|\xi_j\rangle = \sum_{j,k=1}^n \langle \xi_j, \xi_k\rangle\langle \xi_k, |A|\xi_j\rangle = \sum_{j=1}^n \langle \xi_j, |A|\xi_j\rangle \ . \tag{3.29}$$

Thus, for any orthonormal set $\{\xi_1, \dots, \xi_n\}$ with span $\mathscr{V}$, we may replace $\{\xi_1, \dots, \xi_n\}$ by another orthonormal basis $\{\xi_1', \dots, \xi_n'\}$ of $\mathscr{V}$ such that each ξ_j belongs to either $\mathscr{V} \cap \ker|A|$ or to $\mathscr{V} \cap \ker|A|^\perp$ without changing the sum $\sum_{j=1}^n \langle \xi, |A|\xi_j\rangle$. The vectors in $\mathscr{V} \cap \ker|A|$ make no contribution to this sum, and may be deleted. This proves the claim.

Fix $\epsilon > 0$, and let P_ϵ, be the spectral projection of $|A|$ for the interval $[\epsilon, \|A\|]$. If $\{\eta_1, \dots, \eta_m\}$ is an orthonormal set in the range of P_ϵ, then $\sum_{j=1}^m |\langle \eta_j, |A|\eta_j\rangle| \ge \epsilon m$, and hence $m \le \|A\|_1/\epsilon$.

Let $P = \lim_{\epsilon \to 0} P_\epsilon$ which exists in the strong operator topology, and is the orthogonal projection onto $\ker(|A|)^\perp = \ker(A)^\perp$. Since the range of each P_ϵ is finite dimensional, the range of P is a separable subspace of $\mathscr{H}$. Replacing A with A^*, we see that the closure of the range of A is also a separable subspace of A. Let $\mathscr{K}$ be the closed span of $\ker(A)^\perp \cup \overline{\operatorname{ran}(A)}$. Then $\mathscr{K}$ is separable, invariant under A, and $A|_{\mathscr{K}^\perp} = 0$.

Moreover, for all ξ, $\langle \xi, |A|\xi\rangle - \langle P_\epsilon\xi, |A|P_\epsilon\xi\rangle = \langle(\xi - P_\epsilon\xi), |A|\xi\rangle + \langle P_\epsilon\xi, |A|(\xi - P_\epsilon\xi)\rangle$, and then by the strong convergence of P_ϵ to P, for all $\{\xi_1, \dots, \xi_m\}$ orthonormal,

$$\lim_{\epsilon \to 0} \sum_{j=1}^m \langle \xi_j, P_\epsilon|A|P_\epsilon\xi_j\rangle = \sum_{j=1}^m \langle \xi_j, |A|\xi_j\rangle \ .$$

Thus for all $\delta > 0$, there exists $\epsilon > 0$ such that $\|P_\epsilon|A|P_\epsilon\| \ge \||A|\|_1 - \delta$. Since P_ϵ commutes with $|A|$, $|A| = P_\epsilon|A|P_\epsilon + P_\epsilon^\perp|A|P_\epsilon^\perp$. By the triangle inequality, $\||A|\|_1 \le \|P_\epsilon|A|P_\epsilon\|_1 + \|P_\epsilon^\perp|A|P_\epsilon^\perp\|_1$. But by (3.28), restricting to sets $\{\xi_1, \dots, \xi_n\}$ such that for each j, ξ_j is in the range of either P_ϵ or $P_\epsilon^\perp$, one sees that $\||A|\|_1 \ge \|P_\epsilon|A|P_\epsilon\|_1 + \|P_\epsilon^\perp|A|P_\epsilon^\perp\|_1$, and hence $\||A|\|_1 = \|P_\epsilon|A|P_\epsilon\|_1 + \|P_\epsilon^\perp|A|P_\epsilon^\perp\|_1$. Thus $\|A - UP_\epsilon|A|P_\epsilon\|_1 = \|UP_\epsilon^\perp|A|P_\epsilon^\perp\|_1 \le \delta$. Since $\delta > 0$ is arbitrary and $UP_\epsilon|A|P_\epsilon$ has finite rank, this proves the density of finite rank operators. $\square$

3.36 LEMMA. *Let $A \in \mathscr{T}(\mathscr{H})$ be positive, and let $\mathscr{K}$ be any separable subspace of $\mathscr{H}$ that is invariant under A and is such that $A|_{\mathscr{K}^\perp} = 0$. Then for any orthonormal basis $\{\eta_j\}_{j\in\mathbb{N}}$ of $\mathscr{K}$,*

$$\|A\|_1 = \sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle \ . \tag{3.30}$$

Proof. By (3.28), in computing $\|A\|_1$, we need only consider orthonormal sets $\{\xi_1, \dots, \xi_n\} \subset \ker|A|^\perp$ and hence in $\mathscr{K}$. Let $\{\eta_j\}_{j\in\mathbb{N}}$ and $\{\xi_k\}_{k\in\mathbb{N}}$ be two orthonormal bases for $\mathscr{K}$. As in (3.29),

$$\sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle = \sum_{j=1}^{\infty} \|A^{1/2}\eta_j\|^2 = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left|\langle \xi_k, A^{1/2}\eta_j\rangle\right|^2 \ .$$

Since infinite series of non-negative terms may be summed in any order, the right hand side is actually symmetric in $\{\eta_j\}_{j\in\mathbb{N}}$ and $\{\xi_k\}_{k\in\mathbb{N}}$. Therefore, by symmetry in the two orthonormal bases, $\displaystyle\sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle = \sum_{k=1}^{\infty} \langle \xi_k, A\xi_k\rangle$, showing that $\sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle$ depends only on A, and not the particular orthonormal basis $\{\eta_j\}_{j\in\mathbb{N}}$.

Now pick $\epsilon > 0$ and $\{\eta_1, \dots, \eta_n\}$ orthonormal in $\mathscr{K}$ such that $\sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle \geq \|A\|_1 - \epsilon$. Then

$$\|A\|_1 - \epsilon \leq \sum_{j=1}^{n} \langle \eta_j, A\eta_j\rangle \leq \sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle \leq \|A\|_1 \ ,$$

and since $\epsilon > 0$ is arbitrary, the proof is complete. □

For any $A \in \mathscr{T}(\mathscr{H})$, write $A = X + iY$, with X, Y self adjoint. Since $X = \frac{1}{2}(A + A^*)$, $\|X\|_1 \leq \|A\|_1$ and likewise $\|Y\|_1 \leq \|A\|_1$. Then $X_+ = \frac{1}{2}(|X| + X)$ and $X_- = \frac{1}{2}(|X| - X)$, so that $\|X_+\|_1, \|X_-\|_1 \leq \|A\|_1$, and likewise $\|Y_+\|_1, \|Y_-\|_1 \leq \|A\|_1$. By what we have just proved, for any orthonormal basis $\{\eta_j\}_{j\in\mathbb{N}}$,

$$\sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle = \sum_{j=1}^{\infty} \langle \eta_j, X_+\eta_j\rangle - \sum_{j=1}^{\infty} \langle \eta_j, X_-\eta_j\rangle + i\sum_{j=1}^{\infty} \langle \eta_j, Y_+\eta_j\rangle - i\sum_{j=1}^{\infty} \langle \eta_j, Y_-\eta_j\rangle$$

converges absolutely and is independent of the orthonormal basis.

3.37 DEFINITION (Trace)**.** For all $A \in \mathscr{T}(\mathscr{H})$, the *trace* of A, $\mathrm{Tr}[A]$, is defined by

$$\mathrm{Tr}[A] = \sum_{j=1}^{\infty} \langle \eta_j, A\eta_j\rangle$$

where $\{\eta_j\}$ is any orthonormal basis of any separable subspace $\mathscr{K}$ invariant under A with $A|_{\mathscr{K}^\perp} = 0$.

3.38 THEOREM (Properties of the trace)**.** *The functional $A \mapsto \mathrm{Tr}[A]$ is linear on $\mathscr{T}(\mathscr{H})$ and $\mathrm{Tr}[A^*] = \mathrm{Tr}[A]^*$ for all $A \in \mathscr{T}(\mathscr{H})$. Moreover:*

(i) For all $A \in \mathscr{T}(\mathscr{H})$ and all $B \in \mathscr{B}(\mathscr{H})$

$$\mathrm{Tr}[AB] = \mathrm{Tr}[BA] \tag{3.31}$$

(ii) For all $A \in \mathscr{T}(\mathscr{H})$ and all $B \in \mathscr{B}(\mathscr{H})$

$$|\mathrm{Tr}[AB]| \leq \|A\|_1 \|B\| \tag{3.32}$$

(iii) For all $\zeta, \xi \in \mathscr{H}$, and all $B \in \mathscr{B}(\mathscr{H})$, $\mathrm{Tr}\big[|\zeta\rangle\langle\xi|A\big] = \langle \xi, A\zeta\rangle_{\mathscr{H}}$.

Proof. The linearity is evident, and since $\langle \eta, A\eta\rangle_{\mathscr{H}} = \overline{\langle \eta, A^*\eta\rangle_{\mathscr{H}}}$, $\mathrm{Tr}[A^*] = \overline{\mathrm{Tr}[A]}$. In view of the linearity and the fact that every $B \in \mathscr{B}(\mathscr{H})$ is a linear combination of unitaries, it suffices to prove (3.31) when B is unitary. In this case, $AB = B^*(BA)B$ and $\{B\eta_j\}_{j\in\mathbb{N}}$ is an orthonormal basis when $\{\eta_j\}_{j\in\mathbb{N}}$ is, and hence

$$\mathrm{Tr}[AB] = \sum_{j=1}^{\infty}\langle \eta_j, AB\eta_j\rangle = \sum_{j=1}^{\infty}\langle B\eta_j, BAB\eta_j\rangle = \mathrm{Tr}[BA]\ .$$

For part *(ii)*, note that for all $A \in \mathscr{T}(\mathscr{H})$, $|\mathrm{Tr}[A]| \le \|A\|_1$, and hence for all $A \in \mathscr{T}(\mathscr{H})$ and all $B \in \mathscr{B}(\mathscr{H})$, $|\mathrm{Tr}[AB]| \le \|AB\|_1 \le \|A\|_1\|B\|$, using (3.27) in the last step.

Part *(iii)* is a simple computation using any orthonormal basis $\{\eta_j\}_{j\in\mathbb{N}}$ in which $\eta_1 = \|\eta\|^{-1}\eta$ when $\eta \neq 0$, and is trivial otherwise. $\square$

3.39 THEOREM. *Every $B \in \mathscr{B}(\mathscr{H})$ defines a linear functional ψ_B on $\mathscr{T}(\mathscr{H})$ by*

$$\psi_B(X) = \mathrm{Tr}[BX] \quad \text{for all} \quad X \in \mathscr{T}(\mathscr{H})\ . \tag{3.33}$$

The mapping $B \mapsto \psi_B$ is an isometric isomorphism of $\mathscr{B}(\mathscr{H})$ onto $\mathscr{T}(\mathscr{H})^*$.

Proof. For any $\zeta, \zeta' \in \mathscr{H}$, consider the rank-one operator $|\zeta\rangle\langle\zeta'|$. Then

$$\||\zeta\rangle\langle\zeta'|\| = \||\zeta\rangle\langle\zeta'|\|_1 = \|\zeta\|\|\zeta'\|\ . \tag{3.34}$$

Let $\psi \in \mathscr{T}(\mathscr{H})^*$. Define a sesquilinear form q_ψ on $\mathscr{H} \times \mathscr{H}$ by $q_\psi(\zeta', \zeta) = \psi(|\zeta\rangle\langle\zeta')$ for all $\zeta, \zeta' \in \mathscr{H}$. By (3.34),

$$|\psi(|\zeta\rangle\langle\zeta'|)| \le \|\psi\|\||\zeta\rangle\langle\zeta'|\|_1 = \|\psi\|\|\zeta\|\|\zeta'\|\ .$$

By the Riesz Lemma, there exists $B_\psi \in \mathscr{B}(\mathscr{H})$ with $\|B_\psi\| \le \|\psi\|$ such that for all $\zeta, \zeta' \in \mathscr{H}$, $q_\psi(\zeta', \zeta) = \langle \zeta', B_\psi\zeta\rangle$. Then for $\zeta, \zeta' \in \mathscr{H}$, $\psi(|\zeta\rangle\langle\zeta'|) = \langle\zeta', B_\psi\zeta\rangle = \mathrm{Tr}[B_\psi|\zeta\rangle\langle\zeta'|]$. and hence by linearity, $\psi(X) = \mathrm{Tr}[B_\psi X]$ for all finite rank X. Since finite rank operators are dense in $\mathscr{T}(\mathscr{H})$ in the trace norm, this is valid for all $X \in \mathscr{T}(\mathscr{H})$. By (3.32), $\|\psi\| \le \|B_\psi\|$ This shows that $\psi \mapsto B_\psi$ is a linear isometry of $\mathscr{T}(\mathscr{H})^*$ into $\mathscr{B}(\mathscr{H})$. By (3.32), for all $B \in \mathscr{B}(\mathscr{H})$, $X \mapsto \psi_B(X) := \mathrm{Tr}[BX]$ defines an element of $\mathscr{T}(\mathscr{H})^*$, and evidently $B_{\psi_B} = B$ so that $\psi \mapsto B_\psi$ is surjective. $\square$

3.40 DEFINITION. The σ-weak operator topology on $\mathscr{B}(\mathscr{H})$ is the weak-$*$ topology on $\mathscr{B}(\mathscr{H})$, which by Theorem 3.39 is the weakest topology making all of the functions $B \mapsto \mathrm{Tr}[AB]$, $A \in \mathscr{T}(\mathscr{H})$, continuous.

By Theorem 3.11, the σ-weak operator topology is stronger than the weak operator topology, which is the weakest topology making all of the functions $B \mapsto \mathrm{Tr}[AB]$, A finite rank, continuous. However:

3.41 LEMMA. *The weak operator topology and the σ-weak topology induce the same relative topology on the unit ball $B_{\mathscr{B}(\mathscr{H})}$, or, more generally, on bounded subsets of $\mathscr{B}(\mathscr{H})$.*

Proof. We need only show that each relative σ-weak neighborhood contains a relative weak neighborhood. Let $\mathcal{S} \subset \mathscr{B}(\mathscr{H})$ satisfy $\|A\| \le L$ for all $A \in \mathcal{S}$. The general σ-weak neighborhood of $A_0 \in \mathcal{S}$ has the form

$$V_{T_1,\dots,T_n,\epsilon} = \{A \in \mathcal{S}\ :\ |\mathrm{Tr}[T_j(A - A_0)]| < 1\ ,\ j = 1, \dots, n\ \}\ ,$$

where $n \in \mathbb{N}$ and $\{T_1, \ldots, T_n\} \subset \mathscr{T}(\mathscr{H})$, and $\epsilon > 0$. Replacing each T_j by $\epsilon^{-1}T_j$, we may suppose $\epsilon = 1$. By Lemma 3.35, we may approximate each T_j in $\mathscr{T}(\mathscr{H})$ by $\widetilde{T}_j$ such that $\widetilde{T}_j$ is finite rank, and $\|T_j - \widetilde{T}_j\| < \frac{1}{4L}$. Then $V_{T_1,\ldots,T_n,1}$ contains

$$W_{\widetilde{T}_1,\ldots,\widetilde{T}_n,\frac{1}{2}} = \{A \in \mathcal{S} \ : \ |\mathrm{Tr}[\widetilde{T}_j(A - A_0)]| < \frac{1}{2} \ j = 1, \ldots, n \ \} \ ,$$

and this is a neighborhood of A_0 in the relative weak operator topology. □

Now let $\mathscr{M}$ be a von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$. Since $\mathscr{M}$ is weakly closed in $\mathscr{B}(\mathscr{H})$, it is σ-weakly closed. Define $\mathscr{M}^\perp$ to be the *annihilator* of $\mathscr{M}$ in the dual pairing between $\mathscr{T}(\mathscr{H})$ and $\mathscr{B}(\mathscr{H})$. That is,

$$\mathscr{M}^\perp = \{A \in \mathscr{T}(\mathscr{H}) \ : \ \mathrm{Tr}[AB] = 0 \text{ for all } B \in \mathscr{M} \ \} \ .$$

Evidently, $\mathscr{M}^\perp$ is norm closed in $\mathscr{T}(\mathscr{H})$. We define $\mathscr{M}_*$ to be the Banach space $\mathscr{T}(\mathscr{H})/\mathscr{M}^\perp$ consisting of equivalence classes in $\mathscr{T}(\mathscr{H})$ under the equivalence relation $A \sim C$ if and only if $A - C \in \mathscr{M}^\perp$, and we equip $\mathscr{M}_*$ with the quotient norm. Then, by a standard result in the theory of Banach spaces, the dual of $\mathscr{M}_*$ is isometrically isomorphic to $\mathscr{M}$. While the result is standard, in the interest of completeness, we recall the argument leading to it, adapted to the present context.

First note that $\mathscr{M}$ is the annihilator of $\mathscr{M}^\perp$ in the dual pairing between $\mathscr{T}(\mathscr{H})$ and $\mathscr{B}(\mathscr{H})$. That is:

$$\mathscr{M} = \{B \in \mathscr{B}(\mathscr{H}) \ : \ \mathrm{Tr}[AB] = 0 \text{ for all } A \in \mathscr{M}^\perp \ \} =: \mathscr{M}^{\perp\perp} \ . \tag{3.35}$$

This is a consequence of the Hahn-Banach Theorem: Clearly $\mathscr{M} \subset \mathscr{M}^{\perp\perp}$. If the containment were proper, there would exist $X \in \mathscr{M}^{\perp\perp}$ not belonging to the σ-weakly closed subspace $\mathscr{M}$, and then by the Hahn-Banach Theorem, there would exist a σ-weakly continuous linear functional ϕ on $\mathscr{B}(\mathscr{H})$ with $\phi(\mathscr{M}) = 0$ and $\phi(X) = 1$. Since ϕ is σ-weakly continuous, it has the form $B \mapsto \mathrm{Tr}[AB]$ for some $A \in \mathscr{T}(\mathscr{H})$, and then since $\phi(\mathscr{M}) = 0$, $A \in \mathscr{M}^\perp$. But since $X \in \mathscr{M}^{\perp\perp}$, $\phi(X) = \mathrm{Tr}[AX] = 0$, which is incompatible with $\phi(X) = 1$.

3.42 LEMMA. *Let $Q : \mathscr{T}(\mathscr{H}) \to \mathscr{T}(\mathscr{H})/\mathscr{M}^\perp = \mathscr{M}_*$ be the quotient map. Then for each $\psi \in (\mathscr{M}_*)^*$, $\psi \circ Q$ vanishes on $\mathscr{M}^\perp$, and therefore belongs to $\mathscr{M}^{\perp\perp} = \mathscr{M}$. The map $\psi \mapsto \psi \circ Q$ is an isometric isomorphism of $(\mathscr{M}_*)^*$ onto $\mathscr{M}$.*

Proof. The quotient map is, by the definition of the quotient norm, a contraction. Consequently, $\|\psi \circ Q\|_{(\mathscr{T}(\mathscr{H}))^*} \leq \|\psi\|_{(\mathscr{M}_*)^*}$. Also, since Q vanishes on $\mathscr{M}^\perp$, $\psi \circ Q$ is an element of $(\mathscr{T}(\mathscr{H}))^*$ that vanishes on $\mathscr{M}^\perp$, and hence $\psi \circ Q \in \mathscr{M}^{\perp\perp} = \mathscr{M}$.

We now construct the inverse map: Suppose that $\phi \in \mathscr{M} = (\mathscr{M}^\perp)^\perp$. Define the linear functional $\widehat{\phi}$ on $\mathscr{M}_*$ by $\widehat{\phi}(\{A\}) = \phi(A)$, which is well-defined since ϕ vanishes on $\mathscr{M}^\perp$. Moreover. for all $C \sim A$, $|\widehat{\phi}(\{A\})| = |\phi(C)| \leq \|\phi\|\|C\|$. Taking the infimum over $C \sim A$, $|\widehat{\phi}(\{A\})| \leq \|\phi\|\|\{A\}\|$. That is, $\|\widehat{\phi}\|_{(\mathscr{M}_*)^*} \leq \|\phi\|$, By construction, for all $A \in \mathscr{T}(\mathscr{H})$, $\widehat{\phi}\circ Q(A) = \widehat{\phi}(\{A\}) = \phi(A)$; That is, $\widehat{\phi}\circ Q = \phi$.

Therefore, the map $\psi \mapsto \psi \circ Q$ is a linear map from $(\mathscr{M}_*)^*$ onto $\mathscr{M} = \mathscr{M}^{\perp\perp}$, and its inverse on $\mathscr{M}$ is the map $\phi \mapsto \widehat{\phi}$ defined in the previous paragraph. Since both maps are contractions, they are both isometries. □

We summarize what we have proved in the preceding paragraphs:

3.43 THEOREM. *Every von Neumann algebra $\mathscr{M}$ is isomorphic as a Banach space to the dual of a Banach space. More precisely, $\mathscr{M}$ is isomorphic to the dual of $\mathscr{M}_*$ where $\mathscr{M}_*$ is the quotient space $\mathscr{T}(\mathscr{H})/\mathscr{M}^\perp$ and $\mathscr{M}^\perp$ is the subspace of $\mathscr{T}(\mathscr{H})$ annihilated by $\mathscr{M}$ in the dual pairing of $\mathscr{T}(\mathscr{H})$ and $\mathscr{B}(\mathscr{H})$.*

While any strongly closed subspace of $\mathscr{B}(\mathscr{H})$ is weakly closed since the set of weakly continuous linear functionals coincides with the set of strongly continuous linear functionals, the set of σ-weakly continuously linear functionals is strictly larger than the class of weakly continuous linear functionals, and hence it is not true that every σ-weakly closed subspace of $\mathscr{B}(\mathscr{H})$ is weakly closed. However, when the subspaces also happen to be $*$-subalgebras of $\mathscr{B}(\mathscr{H})$, matters are different.

3.44 THEOREM. *Let $\mathscr{M}$ be a σ-weakly closed $*$-subalgebra of $\mathscr{B}(\mathscr{H})$. Then $\mathscr{M}$ is weakly closed.*

Proof. Let $\overline{\mathscr{M}}$ denote the weak closure of $\mathscr{M}$ in $\mathscr{B}(\mathscr{H})$, noting that this is also the strong closure. Let $A \in \overline{\mathscr{M}}$, and suppose that $\|A\| \leq 1$. Let U be a σ-weak neighborhood of A. By Lemma 3.41, there is a weak neighborhood V of A such that $V \cap B_{\mathscr{B}(\mathscr{H})} \subset U \cap B_{\mathscr{B}(\mathscr{H})}$. Since V is also a strong neighborhood, and since A belongs to the strong closure of $\mathscr{M}$, by the Kaplansky Density Theorem, $V \cap B_{\mathscr{B}(\mathscr{H})}$ contains points of $\mathscr{M}$, and hence $U \cap B_{\mathscr{B}(\mathscr{H})}$ contains points of $\mathscr{M}$. Hence $A \in \mathscr{M}$. $\square$

The following lemma will prove useful later.

3.45 LEMMA. *Let $\mathscr{J}$ be a σ-weakly closed left ideal in a von Neumann algebra $\mathscr{M}$. Then $\mathscr{J}$ is weakly closed, and there exists a unique projection $P \in \mathscr{J}$ such that $A = AP$ for all $A \in \mathscr{J}$. If $\mathscr{J}$ is a two-sided ideal, then P belongs to the center $\mathscr{M} \cap \mathscr{M}'$.*

Proof. Let $\mathscr{N} := \mathscr{J} \cap \mathscr{J}^*$. Then $\mathscr{N}$ is a σ-weakly closed $*$-subalgebra of $\mathscr{M}$, and by Theorem 3.44, $\mathscr{N}$ is also weakly closed. Let P be the maximal projection in $\mathscr{N}$, and define $\widetilde{\mathscr{J}} := \{A \in \mathscr{M} \ : \ A = AP\}$. Since $P \in \mathscr{J}$, which is a left ideal, $\widetilde{\mathscr{J}} \subset \mathscr{J}$. Since P is the identity in $\mathscr{N}$, $PB = BP = B$, for all $B \in \mathscr{N}$, and hence $\mathscr{N} \subset \widetilde{\mathscr{J}}$. If $A \in \mathscr{J}$, then $A^*A \in \mathscr{J}$, and being self adjoint, $A^*A \in \mathscr{N}$. Consequently, $|A| \in \mathscr{N}$. Let $A = U|A|$ be the polar decomposition of A. Then $AP = U|A|P = U|A| = A$. Thus, $\mathscr{J} \subset \widetilde{\mathscr{J}}$. Altogether, $\mathscr{J} = \widetilde{\mathscr{J}}$ and $\widetilde{\mathscr{J}}$ is strongly, and therefore weakly, closed by the *separate* strong continuity of multiplication.

Now let Q be any projection in $\mathscr{J}$ such that $A = AQ$ for all $A \in \mathscr{J}$. Since $P \in \mathscr{J}$, $P = PQ$, and hence $P \leq Q$. The same reasoning yields $Q \leq P$. This proves the uniqueness.

Finally, if $\mathscr{J}$ is a two sided ideal, for all unitaries $V \in \mathscr{M}$, $V^*PV \in \mathscr{N}$, and hence $V^*PV = PV^*PV = PV^*V = P$. Thus P commutes with every unitary in $\mathscr{M}$, and hence $P \in \mathscr{M}'$. $\square$

4 C^* algebras as operator algebras

4.1 Representations of C^* algebras

4.1 DEFINITION. A *representation* of a C^*-algebra $\mathscr{A}$ is a $*$-homomorphism π from $\mathscr{A}$ into $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. For any subspace $\mathscr{K}$ of $\mathscr{H}$, we define

$$\pi(\mathscr{A})\mathscr{K} = \{ \ \pi(A)\eta \ : \ A \in \mathscr{A} \ , \ \eta \in \mathscr{K} \ \} \ .$$

A subspace $\mathscr{K}$ of $\mathscr{H}$ is *invariant under* π in case $\pi(\mathscr{A})\mathscr{K} \subset \mathscr{K}$. The representation π is *irreducible* in case no non-trivial subspace $\mathscr{K}$ of $\mathscr{H}$ is invariant under π. The representation π is *non-degenerate* in case $\overline{\pi(\mathscr{A})\mathscr{H}} = \mathscr{H}$. Let π_1 and π_2 be two representations of $\mathscr{A}$ on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Then π_1 and π_2 are *equivalent representations* of $\mathscr{A}$ in case there exists a unitary transformation from U from $\mathscr{H}_1$ onto $\mathscr{H}_2$ such that for all $A \in \mathscr{A}$,

$$\pi_2(A)U = U\pi_1(A) \ .$$

4.2 LEMMA. *Let $\mathscr{A}$ be a C^* algebra, and let π be a non-zero representation of it as an algebra of operators on some Hilbert space $\mathscr{H}$. Then a closed subspace $\mathscr{K}$ of $\mathscr{H}$ is invariant under $\pi(\mathscr{A})$ if and only if the orthogonal projection of $\mathscr{H}$ onto $\mathscr{K}$ belongs to $(\pi(\mathscr{A}))'$.*

Proof. This is an immediate consequence of Lemma 3.29. □

Lemma 4.2 permits us to make the following definition:

4.3 DEFINITION. For a representation π of a $\mathbb{C}^*$ algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$, and a non-zero projector $P \in (\pi(\mathscr{A}))'$, π_P is the subrepresntation obtained by restricting π to $\mathrm{ran}(P)$.

4.4 THEOREM. *Let $\mathscr{A}$ be a C^* algebra, and let π be a non-zero representation of it as an algebra of operators on some Hilbert space $\mathscr{H}$. Then π is irreducible if and only if $(\pi(\mathscr{A}))'$ consists of scalar multiples of the identity.*

Proof. If $(\pi(\mathscr{A}))'$ consists of scalar multiples of the identity, then $(\pi(\mathscr{A}))'$ contains no non-trivial orthogonal projections, and hence by Lemma 4.2, π is irreducible. On the other hand, if $(\pi(\mathscr{A}))'$ contains some operator that is not a multiple of the identity, then it contains a self adjoint operator A that is not a multiple of the identity. Any such $A \in (\pi(\mathscr{A}))'$ has a non-trivial spectral projection that is also in $(\pi(\mathscr{A}))'$ since $(\pi(\mathscr{A}))'$ is a strongly closed $*$-algebra containing A. □

4.2 States on a C^* algebra

4.5 DEFINITION. Let $\mathscr{A}$ be a C^* algebra. A linear functional φ on $\mathscr{A}$ regarded as a Banach space, is *positive* in case $\varphi(A) \geq 0$ for all $A \in \mathscr{A}^+$. A positive linear functional φ is *faithful* in case

$$\varphi(A^*A) = 0 \quad \Rightarrow \quad A = 0 \ . \tag{4.1}$$

Every self-adjoint $A \in \mathscr{A}$ is the difference of two elements of $\mathscr{A}^+$: For $f(t) := \max\{0, t\}$, define $A_+ = f(A)$ and $A_- = f(A) - A$. Then $A = A_+ - A_-$, $A_+, A_- \in \mathscr{A}^+$. Moreover, with this choice of A_+ and A_-, $\|A_+\|, \|A_1\| \leq \|A\|$ and $A_+A_- = A_-A_+ = 0$. In this case we say that A_+ is the *positive part* of A, and A_- is the *negative part of* A. Consequently, for a positive linear functional φ, $\varphi(A) = \varphi(A_+) - \varphi(A_-)$, and hence φ is real on the self-adjoint elements of $\mathscr{A}$.

Every $A \in \mathscr{A}$ has a canonical decomposition $A = X + iY$, $X, Y \in \mathscr{A}_{\text{s.a.}}$ and hence a canonical decomposition

$$A = X_+ - X_- + i(Y_+ - Y_-) \tag{4.2}$$

where $X_+, X_-, Y_+, Y_- \in \mathscr{A}^+$, and $\|X_+\|, \|X_-\|, \|Y_+\|, \|Y_-\| \leq \|A\|$.

4.6 LEMMA. *Let $\mathscr{A}$ be a C^* algebra. Every positive linear functional is bounded*

Proof. As noted above, each $A \in B_{\mathscr{A}}$ has decomposition $A = X_+ - X_- + i(Y_+ - Y_-)$ with X_+, X_-, Y_+, Y_- all in $\mathscr{A}^+ \cap B_{\mathscr{A}}$. Therefore, for any positive linear functional φ,

$$\sup_{\|A\|\leq 1} \{|\varphi(A)|\} \leq 4 \sup_{A \in \mathscr{A}^+ \cap B_{\mathscr{A}}} \{|\varphi(A)|\} .$$

Suppose φ is a positive linear functional, and that $\varphi \in \mathscr{A}^*$ is not bounded, By the remarks just above, there is a sequence $\{X_j\}_{j\in\mathbb{N}}$ in $\mathscr{A}^+$ with $\|X_j\| = 1$ and $|\varphi(X_j)| \geq 4^j$ for all j. Define

$$X := \sum_{j=1}^{\infty} 2^{-j} X_j .$$

Since $\mathscr{A}^+$ is closed, $X \in \mathscr{A}^+$, Moreover for all $n > m$, $\sum_{j=m+1}^{n} 2^{-j} X_j \in \mathscr{A}^+$, and taking $n \to \infty$, and using the closure of $\mathscr{A}^+$ once more, for all $m \in \mathbb{N}$, $X = \sum_{j=1}^{m} 2^{-j} X_j \in \mathscr{A}^+$. Therefore,

$$\varphi(X) \geq \sum_{j=1}^{m} 2^{-j} \varphi(X_j) \geq m ,$$

and this evidently yields a contradiction for m large enough. □

Lemma 4.6 says that for every C^* algebra $\mathscr{A}$, every positive linear functional on $\mathscr{A}$ belongs to $\mathscr{A}^*$, the Banach space dual to $\mathscr{A}$. We write $\mathscr{A}^*_+$ to denote the set of positive linear functionals.

Lemma 4.6 can be sharpened considerably. For this, and many other purposes, it is useful to associate a sesquilinear form with each φ of $\mathscr{A}^*_+$. For all such φ on a $\mathbb{C}^*$ algebra A, the map

$$(A, B) \mapsto \varphi(A^*B) =: \langle A, B\rangle_\varphi . \tag{4.3}$$

defines a (possibly degenerate) inner product on $\mathscr{A}$; this inner product is non-degenerate if and only if φ is faithful. In any case, the fact that $\langle A, A\rangle_\varphi \geq 0$ for all $A \in \mathscr{A}$ yields the Cauchy-Schwarz inequality:

$$|\langle A, B\rangle_\varphi| \leq \langle A, A\rangle_\varphi^{1/2} \langle B, B\rangle_\varphi^{1/2} . \tag{4.4}$$

4.7 LEMMA. *Let $\mathscr{A}$ be a C^* algebra, and let $\varphi \in \mathscr{A}^*_+$. Let $\langle\cdot,\cdot\rangle_\varphi$ be the sesquilinear form on $\mathscr{A}$ defined in* (4.3). *Define*

$$\mathscr{J} := \{A \in \mathscr{A} \ : \ \varphi(A^*A) = 0\} . \tag{4.5}$$

Then $\mathscr{J}$ is a closed left ideal in $\mathscr{A}$ and

$$\mathscr{J} := \bigcap_{B\in\mathscr{A}} \{A \in \mathscr{A} \ : \ \varphi(BA) = 0\} = \bigcap_{B\in\mathscr{A}} \{A \in \mathscr{A} \ : \ \langle B, A\rangle_\varphi = 0\} . \tag{4.6}$$

Proof. Since $(BA)^*(BA) \leq \|B\|^2 A^*A$, $BA \in \mathscr{J}$ whenever $A \in \mathscr{J}$. If $A, B \in \mathscr{J}$, $A + B \in \mathscr{J}$ by the Cauchy-Schwarz inequality. It now follows that $\mathscr{J}$ a left ideal.

For $A \in \mathscr{J}$ and $B \in \mathscr{A}$, $|\varphi(BA)|^2 \leq \varphi(B^*B)\varphi(A^*A) = 0$, again by the Cauchy-Schwarz inequality, and hence $\mathscr{J} \subset \bigcap_{B\in\mathscr{A}}\{A \in \mathscr{A} \ : \ \varphi(BA) = 0\}$. Conversely, if $\varphi(BA) = 0$ for all $B \in \mathscr{A}$, taking $B = A^*$, yields $A \in \mathscr{J}$. This proves the first identity in (4.6), which displays $\mathscr{J}$ as an intersection of closed sets. The second follows easily since $\mathscr{A}$ is closed under the involution. □

4.8 LEMMA (Positivity and norm bounds)**.** *Let $\mathscr{A}$ be a C^* algebra. For all $\varphi \in \mathscr{A}_+^*$,*

$$\|\varphi\| = \sup_{A \in \mathscr{A}^+ \cap B_{\mathscr{A}}} \{|\varphi(A)|\} \ . \tag{4.7}$$

If furthermore $\mathscr{A}$ is unital, then $\varphi \in \mathscr{A}^*$ is positive if and only if $\|\varphi\| = \varphi(1)$.

Proof. Let $\epsilon > 0$, and pick $A \in B_{\mathscr{A}}$ such that $|\varphi(A)| + \epsilon \geq \|\varphi\|$. By Theorem 2.32, there exists $E \in \mathscr{A}^+ \cap B_{\mathscr{A}}$ such that $\|AE - A\| \leq \epsilon$. Then $|\varphi(AE)| \geq |\varphi(A)| - \|\varphi\|\epsilon$, and by the Cauchy-Schwarz inequality,

$$|\varphi(AE)| \leq \left(\varphi(A^*A)(\varphi(E^2)\right)^{1/2} \leq \sup_{X \in \mathscr{A}^+ \cap B_{\mathscr{A}}} \{|\varphi(X)|\}$$

since both A^*A and E^2 belong to $\mathscr{A}^+ \cap B_{\mathscr{A}}$. Since $\epsilon > 0$ is arbitrary, this proves (4.7).

For the second part, suppose that $\mathscr{A}$ is unital. If $\varphi \in \mathscr{A}_+^*$, $\|\varphi\| \geq \varphi(1) \geq \varphi(A)$ for all $A \in \mathscr{A}^+ \cap B_{\mathscr{A}}$, and hence $\sup_{A \in \mathscr{A}^+ \cap B_{\mathscr{A}}} \{|\varphi(A)|\} = \varphi(1)$, and then by (4.7), $\|\varphi\| = \varphi(1)$.

The proof of the converse uses an argument of Phelps [35]: Suppose that $\varphi \in \mathscr{A}^*$ and $\varphi(1) = \|\varphi\|$. If $\varphi = 0$, it is positive. If $\varphi \neq 0$, we may divide by $\|\varphi\|$ and thus may suppose that $\|\varphi\| = \varphi(1) = 1$.

We claim that for all $\varphi \in \mathscr{A}^*$ such that $\varphi(1) = \|\varphi\| = 1$, $\varphi(A)$ belongs to the convex hull of $\sigma_{\mathscr{A}}(A)$ for all $A \geq 0$ in $\mathscr{A}$. To see this, suppose that the closed disc of radius r centered on λ contains $\sigma_{\mathscr{A}}(A)$. Then $\lambda 1 - A$ is normal, and its spectrum is continued in $\{\lambda - t \ : \ t \in \sigma_{\mathscr{A}}(A)\}$, and hence the spectral radius of $\lambda 1 - A$ is at most r. Since $\lambda 1 - A$ is normal, $\|\lambda 1 - A\| \leq r$. Therefore,

$$|\lambda - \varphi(A)| = |\varphi(\lambda 1 - A)| \leq \|\lambda 1 - A\| \leq r \ .$$

Thus $\varphi(A)$ is contained in the closed disc of radius r centered on λ which contains $\sigma_{\mathscr{A}}(A)$. The intersection over all such discs is the convex hull of $\sigma_{\mathscr{A}}(A)$. □

4.9 DEFINITION (State and quasi-state)**.** Let $\mathscr{A}$ be a C^* algebra. A *state* on $\mathscr{A}$ is an element φ of $\mathscr{A}_+^*$ with $\|\varphi\| = 1$. A *quasi-state* on $\mathscr{A}$ is an element φ of $\mathscr{A}_+^*$ with $\|\varphi\| \leq 1$. We write $S_{\mathscr{A}}$ to denote the set of states on $\mathscr{A}$, and $Q_{\mathscr{A}}$ to denote the set of quasi-states. We equip both $S_{\mathscr{A}}$ and $Q_{\mathscr{A}}$ with the relative weak-$*$ topology.

Since $Q_{\mathscr{A}} = \cap_{A \in \mathscr{A}^+} \{\varphi \in \mathscr{A}^* \ : \ \|\varphi\| \leq 1 \quad \text{and} \quad \varphi(A) \geq 0\}$, $Q_{\mathscr{A}}$ is a weak-$*$ closed subset of the unit ball in $\mathscr{A}^*$, and hence, by the Banach-Alaoglu Theorem, $Q_{\mathscr{A}}$ is weak-$*$ compact.

By Lemma 4.8, when $\mathscr{A}$ is unital, when $\varphi \in \mathscr{A}_+^*$, $\|\varphi\| = \varphi(1)$, and hence in this case,

$$S_{\mathscr{A}} = B_{\mathscr{A}^*} \cap \{\varphi \in \mathscr{A}^* \ : \ \varphi(1) = 1\}$$

which displays $S_{\mathscr{A}}$ weak-$*$ closed subset of the unit ball of $\mathscr{A}^*$. Hence in the unital case, the state space is compact.

4.10 LEMMA. *Suppose that $\mathscr{A}$ is a non-unital C^* algebra. Let $\mathscr{A}_1$ be the C^* algebra obtained by adjoining a unit in the canonical manner. Let $\varphi \in Q_{\mathscr{A}}$, and define $\widetilde{\varphi}$ on $\mathscr{A}_1$ by*

$$\widetilde{\varphi}((\lambda, A)) = \|\varphi\|\lambda + \varphi(A) \ . \tag{4.8}$$

Then the map $\varphi \mapsto \widetilde{\varphi}$ is isometric from $Q_{\mathscr{A}}$ into $Q_{\mathscr{A}_1}$, and also form $S_{\mathscr{A}}$ into $S_{\mathscr{A}_1}$. Let ψ_0 denote the state on $\mathscr{A}_1$ given by $\psi_0((\lambda, A)) = \lambda$. Then every $\psi \in S_{\mathscr{A}_1}$ is a convex combination of ψ_0 and a state of the form $\widetilde{\varphi}$, $\varphi \in S_{\mathscr{A}}$.

Proof. Let $(\lambda, A) \in \mathscr{A}_1^+$. Then $(\lambda, A) = (\mu, B)^*(\mu, B)$, so that $\lambda = |\mu|^2 \geq 0$ and $\sigma_{\mathscr{A}}(A) \subset [-\lambda, \infty)$. Hence

$$\widetilde{\varphi}((\lambda, A)) = \|\varphi\|\lambda + \varphi(A_+) - \varphi(A_-) \geq \|\varphi\|\lambda - \varphi(A_-) \ .$$

Since $\sigma_{\mathscr{A}}(A_-) \subset [0, \lambda]$, $\|A_-\| \leq \ \lambda$. Therefore $\varphi(A_-) \leq \|\varphi\|\|A_-\| \leq \|\varphi\|\lambda$. Thus, $\widetilde{\varphi}((\lambda, A)) \geq 0$, showing that $\widetilde{\varphi}$ is positive. By the second part of Lemma 4.8, $\|\widetilde{\varphi}\| = \widetilde{\varphi}(1) = \|\varphi\|$, so that $\widetilde{\varphi} \in Q_{\mathscr{A}_1}$, and if $\varphi \in S_{\mathscr{A}}$, then $\widetilde{\varphi} \in S_{\mathscr{A}}$. Hence $\varphi \mapsto \widetilde{\varphi}$ maps $Q_{\mathscr{A}}$ into $Q_{\mathscr{A}_1}$, and also of $S_{\mathscr{A}}$ into $S_{\mathscr{A}_1}$.

Let $\psi \in S_{\mathscr{A}_1}$. Define a positive linear functional $\psi|_{\mathscr{A}}$ on $\mathscr{A}$ by $\psi|_{\mathscr{A}}(A) = \psi((0, A))$. Evidently $\|\psi|_{\mathscr{A}}\| \leq \|\psi\| = 1$, so that $\psi|_{\mathscr{A}} \in Q_{\mathscr{A}}$. Let ψ_0 denote the state on $\mathscr{A}_1$ given by $\psi_0((\lambda, A)) = \lambda$. For $\psi \in S_{\mathscr{A}_1}$, $\psi \neq \psi_0$, $\|\psi|_{\mathscr{A}}\| \in (0, 1]$. Then with $c := \|\psi|_{\mathscr{A}}\|$, $c^{-1}\psi|_{\mathscr{A}} \in S_{\mathscr{A}}$, and

$$\psi((\lambda, A)) = \lambda + \psi((0, A)) = \lambda + \psi|_{\mathscr{A}}(A) = (1 - c)\lambda + c\left(\lambda + c^{-1}\psi|_{\mathscr{A}}(A)\right) \ .$$

That is,

$$\psi = c\psi_0 + (1 - c)\widetilde{c^{-1}\psi|_{\mathscr{A}}} \ ,$$

which shows that every state $\psi \in S_{\mathscr{A}_1}$ is a convex combination of ψ_0 and a state of the form $\widetilde{\varphi}$, $\varphi \in S_{\mathscr{A}}$. $\square$

4.11 LEMMA. *Let $\mathscr{A}$ be a C^* algebra. For all non-zero self adjoint $A \in \mathscr{A}$, there exists a state φ such that $|\varphi(A)| = \|A\|$.*

Proof. Suppose that $\mathscr{A}$ is unital. Consider the C^* algebra $C(\{1, A\})$ generated by 1 and A. This is a commutative C^* algebra, and by Corollary 1.34, there exists a character φ_0 of $C(\{1, A\})$ such that $|\varphi_0(A)| = \nu(A) = \|A\|$. Since φ_0 is a character $\varphi(1) = 1$. Then by Lemma 4.8, $\varphi_0 \in \mathscr{A}_+^*$, and so φ is a state on $C(\{1, A\})$.

By the Hahn-Banach Theorem, there is a norm preserving extension φ of φ_0 (as a linear functional) to $\mathscr{A}$. Then $\varphi(1) = \varphi_0(1) = 1$, and hence by Lemma 4.8, φ is a state, and since φ extends φ_0, $|\varphi(A)| = \|A\|$.

When $\mathscr{A}$ is not unital, consider $\mathscr{A}_1$, the C^* algebra $\mathscr{A}_1$ obtained by adjoining a unit in the canonical manner. Then $(0, A)$ is self adjoint in $\mathscr{A}_1$, and by what we have just proved, there is a state ψ on $\mathscr{A}_1$ such that $|\psi((0, A))| = \|(0, A)\| = \|A\|$. Define φ to be the quasi-state $\varphi = \psi|_{\mathscr{A}}$; i.e., for all $B \in \mathscr{A}$, $\varphi(B) = \psi((0, B))$. Since $|\varphi(A)| = \|A\|$, $\|\varphi\| = 1$, and φ is a state on $\mathscr{A}$. $\square$

For a unital C^* algebra $\mathscr{A}$, $S_{\mathscr{A}}$ is compact in the weak-$*$ topology, and it is evidently convex. The Krein-Milman Theorem says that every non-empty convex set in $\mathscr{A}^*$ that is compact in the weak-$*$ topology is the convex hull of its extreme points. Hence there exist extreme points in $S_{\mathscr{A}}$.

4.12 DEFINITION (Pure state)**.** A *pure state* of a unital C^* algebra $\mathscr{A}$ is an extreme point of $S_{\mathscr{A}}$.

4.13 THEOREM. *Let $\mathscr{A}$ be a unital C^* algebra. For all $A \in \mathscr{A}^+$, there exists a pure state φ such that $\varphi(A) = \|A\|$.*

Proof. By Lemma 4.11, the set $\mathcal{S}$ of states φ such that $\varphi(A) = \|A\|$ is non-empty, and evidently it is convex and closed in the weak-$*$ topology. By the Krein-Milman Theorem, $\mathcal{S}$ has at least one extreme point ψ. We now show that ψ is extreme in $S_{\mathscr{A}}$ as well as in $\mathcal{S}$.

Suppose that $\psi_1, \psi_2 \in S_{\mathscr{A}}$ and that $\psi = t\psi_1 + (1-t)\psi_2$ for some $t \in (0,1)$. Evaluating both sides at A,

$$\|A\| = \psi(A) = t\psi_1(A) + (1-t)\psi_2(A) \le t\|A\| + (1-t)\|A\| = \|A\| \ .$$

Hence $\psi_1, \psi_2 \in \mathcal{S}$, and so $\psi_1 = \psi_2 = \psi$. □

4.14 DEFINITION. Let π be a representation of a C^* algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$. A vector $\eta \in \mathscr{H}$ is cyclic for π in case $A \mapsto \pi(A)\eta$ has dense range, and is a *separating* vector for π in case $A \mapsto \pi(A)\eta$ is injective. If a cyclic vector exists, then π is a *cyclic representation.*

By Thoerem 2.54 when π is a representation of a C^* algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$, then $\pi(A)$ is a C^* subalgebra of $\mathscr{B}(\mathscr{H})$. And of course if $\mathscr{A}$ is any C^* subalgebra of $\mathscr{B}(\mathscr{H})$, then the inclusion map is a representation of $\mathscr{A}$ on $\mathscr{H}$. Thus, given a C^* subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$, we say that a vector $\eta \in \mathscr{H}$ is cyclic for $\mathscr{A}$ in case $\mathscr{A}\eta$ is dense in $\mathscr{H}$, and that η is separating for $\mathscr{A}$ in case$A\eta \neq 0$ for any non-zero $A \in \mathscr{A}$.

4.15 LEMMA. *Let $\mathscr{A}$ be a unital C^* subalgebra of $\mathscr{B}(\mathscr{H})$, $\mathscr{H}$ a Hilbert space. Let $\eta \in \mathscr{H}$. Then η is cyclic for $\mathscr{A}$ if and only if η is separating for $\mathscr{A}'$.*

Proof. Suppose that η is cyclic for $\mathscr{A}$ Let $B \in \mathscr{A}'$. If $B\eta = 0$, then for all $A_1, A_2 \in \mathscr{A}$,

$$0 = \langle A_1^* A_2 \eta, B\eta \rangle = \langle A_2\eta, BA_1\eta \rangle \ ,$$

and since $\mathscr{A}\eta$ is dense in $\mathscr{H}$, this shows that $B = 0$. Hence η is separating for $\mathscr{A}'$.

Conversely, suppose that η is not cyclic for $\mathscr{A}$. Then, $\mathscr{K}$, the closure of $\mathscr{A}\eta$ is a proper, non zero subspace of $\mathscr{H}$. Let P be the orthogonal projection from $\mathscr{H}$ onto $\mathscr{K}$. Then $P, P^\perp \in \mathscr{A}'$ and since $\mathscr{A}$ is unital, $\eta \in \mathscr{K}$, and hence $P^\perp\eta = 0$. Hence η is not separating for $\mathscr{A}'$. □

If $\mathscr{A}$ is a von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$, then there is a greater symmetry since $\mathscr{A} = \mathscr{A}''$, and then η is cyclic for $\mathscr{A}'$ if and only if it is separating for $\mathscr{A}$.

4.16 EXAMPLE. Let $\mathscr{H}$ be the space $M_n(\mathbb{C})$ of $n \times n$ matrices equipped with the Hilbert-Schmidt inner product $\langle X, Y \rangle = \mathrm{Tr}[X^*Y]$. Let $\mathscr{A}$ be $M_n(\mathbb{C})$ with the usual involutions and operator norm. Define a representation $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ by $\pi(A) = L_A$, the operator of left multiplication by A. That is, $\pi(A)X = AX$. Evidently, $\ker(\pi) = \{o\}$, and hence π is an isomorphism. We may identify $\pi(A)$ with its image in $\mathscr{B}(\mathscr{H})$, which is the subalgebra generated by the left multiplication operators L_A. Let X be an invertible element of $M_n(\mathbb{C})$, regarded as a vector in $\mathscr{H}$. If $L_AX = 0$, then $A = (L_AX)X^{-1} = 0$. Thus X is separating. Moreover, for any $Y \in \mathscr{H}$, if $A = YX^{-1}$, then $L_AX = Y$, and hence X is separating. Later in this chapter we develop an infinite dimensional version of this construction.

For any representation π of $\mathscr{A}$ on $\mathscr{H}$, and any *unit vector* $\eta \in \mathscr{H}$, the functional $\varphi_\eta \in \mathscr{A}^*$ defined by

$$\varphi_\eta(A) = \langle \eta, \pi(A)\eta \rangle \tag{4.9}$$

is a state. Evidently, $\varphi_\eta(A^*A) = \langle \eta, \pi(A^*A)\eta \rangle = \|\pi(A)\eta\|^2_{\mathscr{H}}$, and hence η is separating for π if and only if φ_η is faithful. The next theorem links purity, cyclicity and irreducibility.

4.17 THEOREM. *Let π be a representation of a C^* algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$, and let η be a cyclic unit vector for π. Then with φ_η denoting the state defined in (4.9). Then π is irreducible iff and only if φ_η is pure.*

The following lemma of Dye [11, Lemma 2.2] will prove useful now and many times later:

4.18 LEMMA. *Let π be a representation of a unital C^* algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$, and let η be a cyclic unit vector for π. Then with φ_η denoting the state defined in (4.9), suppose that $\psi \in S_{\mathscr{A}}$, and that for some $r \in (0,\infty)$,*

$$\psi(A) \le r\varphi_\eta(A) \quad \text{for all} \quad A \in \mathscr{A}^+ \ . \tag{4.10}$$

Then there is a positive operator $X \in (\pi(\mathscr{A}))'$ such that $\|X\| \le r$ and for all $A, B \in \mathscr{A}$,

$$\psi(A^*B) = \langle \eta, \pi(A), X\pi(B)\eta\rangle \ . \tag{4.11}$$

Proof. Define a positive sesquilinear form q on $\pi(\mathscr{A})\eta$ by $q(\pi(A)\eta, \pi(B)\eta) = \psi(A^*B)$. Note that $q(\zeta,\xi)$ depends only on the vectors $\zeta,\xi \in \pi(\mathscr{A})\eta$ since if $\zeta = \pi(A)\eta = \pi(C)\eta$, then by (4.10), $\psi((B-C)^*(B-C)) = 0$, and then by the Cauchy-Schwarz inequality, $|\psi(A^*B) - \psi(C^*B)| = |\psi((A-C)^*B)| = 0$. Similar reasoning applies to the other argument.

We have, for $\zeta = \pi(A)\eta$,

$$|q(\zeta,\zeta)| = |q(\pi(A)\eta, \pi(A)\eta)| \le r|\langle \pi(A)\eta, \pi(A)\eta\rangle| \le r\|\pi(A)\|^2_{\mathscr{H}} = \|\zeta\|^2_{\mathscr{H}} \ .$$

Since η is cyclic, q is densely defined on $\mathscr{H}$ and extends to a positive sesquilinear form on all of $\mathscr{H}$, still denoted by q. By the Cauchy-Schwarz inequality, $|q(\zeta,\xi)| \le r\|\zeta\|_{\mathscr{H}}\|\xi\|_{\mathscr{H}}$ for all $\zeta,\xi \in \mathscr{H}$. By Reisz's Lemma, there exists a self adjoint operator $X \in \mathscr{B}(\mathscr{H})$ with $\|X\| \le r$ such that $q(\zeta,\xi) = \langle \zeta, X\xi\rangle$ for all $\zeta,\xi \in \mathscr{H}$. Since $q(\zeta,\zeta) \ge 0$ for ζ in the dense set $\pi(\mathscr{A})\eta$, X is positive.

Finally, note that for all $A, B, C \in \mathscr{A}$, $A^*(BC) = (B^*A)^*C$, and hence

$$\psi(A^*(BC)) = \psi((B^*A)^*C) \ .$$

This simple but fundamental identity means that $q(\pi(A)\eta, \pi(B)\pi(C)\eta) = q(\pi(B^*)\pi(A)\eta, \pi(C)\eta)$ which is the same as $\langle \pi(A)\eta, X\pi(B)\pi(C)\eta\rangle = \langle \pi(A)\eta, \pi(B)X\pi(C)\eta\rangle$. Thus for all ζ,ξ in a dense subset of $\mathscr{H}$, $\langle \zeta, X\pi(B)\xi\rangle = \langle \zeta, \pi(B)X\xi\rangle$, showing that X commutes with $\pi(B)$ for all $B \in \mathscr{A}$. □

Proof of Theorem 4.17. Suppose that π is irreducible. Let ψ_1, ψ_2 be two states such that $\varphi_\eta = t\psi_1 + (1-t)\psi_2$ for some $t \in (0,1)$. By Lemma 4.18, applied to ψ_1, which satisfies $\psi_1 \le t^{-1}\varphi_\eta$, there is a positive $X \in (\pi(\mathscr{A}))'$

$$\psi_1(A^*B) = \langle \eta, \pi(A), X\pi(B)\eta\rangle \quad \text{for all} \quad A, B \in \mathscr{A} \ . \tag{4.12}$$

Since π is irreducible, X must be a scalar multiple of the identity. Since ψ_1 is a state, taking $A = B = 1$ in (4.12), $1 = \psi(1) = \langle \eta, X\eta\rangle$, which shows that $X = 1$. Then taking $A = 1$ in (4.12) shows that $\psi(B) = \varphi_\eta(B)$ so that $\psi_1 = \varphi_\eta$. By symmetry, $\psi_2 = \varphi_\eta$ as well, and this proves φ_η is extreme.

For the converse, suppose that π is not irreducible. Then there exists a projection $P \in (\pi(\mathscr{A})')$ such that neither P nor $P^\perp$ is zero. Suppose that $P\eta = 0$. Then for all $A \in \mathscr{A}$, $P(\pi(A)\eta) =$

$\pi(A)P\eta = 0$ and this would mean that P vanishes on a dense subspace, which is not the case. Hence $\|P\eta\|_{\mathscr{H}} > 0$, and the same reasoning shows that $\|P^\perp\eta\|_{\mathscr{H}} > 0$. Define $\eta_1 = \|P\eta\|_{\mathscr{H}}^{-1}P\eta$ and $\eta_2 = \|P^\perp\eta\|_{\mathscr{H}}^{-1}P^\perp\eta$. For all $A \in \mathscr{A}$,

$$\langle \eta_1, \pi(A)\eta_2\rangle = \langle P\eta_1, \pi(A)P^\perp\eta_2\rangle = \langle \eta_1, PP^\perp\pi(A)\eta_2\rangle = 0 \ .$$

Define $t \in (0,1)$ by $t = \|P\eta\|_{\mathscr{H}}^2$ so that $\|P^\perp\eta\|_{\mathscr{H}}^2 = 1-t$. By the orthogonality proved just above, for all $A \in \mathscr{A}$,

$$\begin{aligned}\varphi_\eta(A) &= \langle[\sqrt{t}\eta_1 + \sqrt{1-t}\eta_2], \pi(A)[\sqrt{t}\eta_1 + \sqrt{1-t}\eta_2]\rangle \\ &= t\langle \eta_1, \pi(A)\eta_1\rangle + (1-t)\langle \eta_2, \pi(A)\eta_2\rangle \ ,\end{aligned}$$

and this displays φ_η as a convex combination of states. If φ_η is extreme, it must be the case that for all $A \in \mathscr{A}$, $\varphi_\eta(A) = \langle \eta_1, \pi(A)\eta_1\rangle$, or equivalently, that for all $A \in \mathscr{A}$,

$$t\langle \eta, \pi(A)\eta\rangle = \langle \eta, P\pi(A)P\eta\rangle = \langle \eta, \pi(A)P\eta\rangle \ .$$

That is, $\langle \pi(A^*)\eta, (P\eta - t\eta)\rangle = 0$. Replacing A^* by A^*B, we have $\langle \pi(B)\eta, (P\pi(A)\eta - t\pi(A)\eta)\rangle = 0$. Since η is cyclic, this means that $P = t1$ which is impossible since $t \in (0,1)$, and $\sigma(P) \subset \{0,1\}$. □

4.3 Grothendieck's Decomposition Theorem

4.19 DEFINITION. Let $\mathscr{A}$ be a C^* algebra. A linear functional $\phi \in \mathscr{A}^*$ is *Hermitian* in case $\phi(A^*) = \overline{\phi(A)}$ for all $A \in \mathscr{A}$. Since any $A \in \mathscr{A}$ can be written as $A = X + iY$, $X, Y \in \mathscr{A}_{\text{s.a.}}$, and then $\phi(A^*) = \phi(X) - i\phi(Y)$, ϕ is Hermitian if and only if $\phi : \mathscr{A}_{\text{s.a.}} \to \mathbb{R}$.

The real vector space of all Hermitian elements of $\mathscr{A}^*$ is denoted $(\mathscr{A}^*)_{\text{s.a.}}$. Let $(\mathscr{A}_{\text{s.a.}})^*$ denote the real Banach space dual to the real Banach space $\mathscr{A}_{\text{s.a.}}$. Note that the elements of $(\mathscr{A}^*)_{\text{s.a.}}$ are bounded complex linear functionals on $\mathscr{A}$, while the elements of $(\mathscr{A}_{\text{s.a.}})^*$ are bounded real linear functionals on $\mathscr{A}_{\text{s.a.}}$.

4.20 LEMMA. *For $\phi \in (\mathscr{A}^*)_{\text{s.a.}}$, define ϕ_R to be the restriction of ϕ to $\mathscr{A}_{\text{s.a}}$. For $\varphi \in (\mathscr{A}_{\text{s.a.}})^*$, define $\phi_C : \mathscr{A} \to \mathbb{C}$ by*

$$\varphi_C(X + iY) = \varphi(X) + i\varphi(Y) \tag{4.13}$$

for $X, Y \in \mathscr{A}_{\text{s.a.}}$. Then $\varphi_C \in (\mathscr{A}^*)_{\text{s.a.}}$, and the map $\varphi \mapsto \varphi_C$ is an isometric isomorphism of $(\mathscr{A}_{\text{s.a.}})^*$ onto $(\mathscr{A}^*)_{\text{s.a.}}$. In particular, if ϕ is Hermitian,

$$\|\phi\| = \sup\{\phi(A) \ , \ A \in \mathscr{A}_{\text{s.a.}} \|A\| \le 1 \ \} \ . \tag{4.14}$$

Proof. If $\phi \in \mathscr{A}^*$ is Hermitian, then the restriction ϕ_R of ϕ to the real Banach space $\mathscr{A}_{\text{s.a.}}$ is a bounded real linear functional on $\mathscr{A}_{\text{s.a.}}$, and since ϕ is determined by its action on $\mathscr{A}_{\text{s.a.}}$, $\phi \mapsto \phi_R$ is one-to-one.

For any $\varphi \in (\mathscr{A}_{\text{s.a.}})^*$, define an extension φ_C to $\mathscr{A}$ by (4.13). Then $\varphi_C(i(X + iY)) = i\varphi(X) - \varphi(Y) = i\varphi_C(X + iY)$, and from here one readily sees that φ_C is complex linear functional on $\mathscr{A}$, and moreover, φ_C is Hermitian, and the restriction of φ_C to $\mathscr{A}_{\text{s.a.}}$ is simply φ itself.

Furthermore, for $A = X + iY$, $X, Y \in \mathscr{A}_{\text{s.a.}}$, if $A \in B_{\mathscr{A}}$, then $X, Y \in B_{\mathscr{A}_{\text{s.a.}}}$. It follows immediately that for all $A \in B_{\mathscr{A}}$, $|\varphi_C(A)| \le 2\|\varphi\|$, so that so that $\varphi_C \in \mathscr{A}^*$. In fact, $\|\varphi_C\| = \|\varphi\|$.

To see this, fix $\epsilon > 0$, and choose $A \in B_{\mathscr{A}}$ such that $\|\varphi_C\| - \epsilon < |\varphi_C(A)|$. Choose $\theta \in [0, 2\pi)$ so that $e^{i\theta}\varphi(A) > 0$. Replace A by $e^{i\theta}A$, and write $A = X + iY$, $X, Y \in \mathscr{A}_{\text{s.a.}}$. Then $\varphi_C(A) = \varphi(X) + i\varphi(Y)$ is real, so that $\varphi(Y) = 0$ and, since $\|X\| \leq \|A\| \leq 1$,

$$\|\varphi_C\| - \epsilon \leq \varphi_C(A) = \varphi(X) \leq \|\varphi\|\|X\| \leq \|\varphi\| \ .$$

Hence $\|\varphi_C\| \leq \|\varphi\|$, and since φ is the restriction of φ_C to $\mathscr{A}_{\text{s.a.}}$, the opposite inequality is trivially true. □

4.21 LEMMA. *The unit ball in* $(\mathscr{A}^*)_{\text{s.a.}}$ *is the convex hull of* $Q_{\mathscr{A}}$ *and* $-Q_{\mathscr{A}}$*.*

Proof. Let K be the convex hull of $Q_{\mathscr{A}}$ and $-Q_{\mathscr{A}}$. Let K_R be the set of restrictions of elements of K to $\mathscr{A}_{\text{s.a.}}$, By the isometry property established in Lemma 4.20, $K = B_{\mathscr{A}^*}$ if and only if K_R is the unit ball in $(\mathscr{A}_{\text{s.a.}})^*$. K is the image of $[0,1] \times Q_{\mathscr{A}} \times Q_{\mathscr{A}}$ under the map $(t, \varphi, \psi) \mapsto (1-t)\varphi - t\psi$, which is continuous with the natural product topology on the domain. As the continuous image of a compact set, K is compact. The same reasoning shows that K_R is also weak $*$-compact and convex in $(\mathscr{A}_{\text{s.a.}})^*$.

Since the unit ball in $(\mathscr{A}_{\text{s.a.}})^*$ evidently contains K_R, we need only show every ψ in the unit ball of $(\mathscr{A}_{\text{s.a.}})^*$ belongs to K_R. Suppose on the contrary that $\psi \in (\mathscr{A}_{\text{s.a.}})^*$, $\|\psi\| \leq 1$ and $\psi \notin K_R$. Then by the Hahn-Banach Separation Theorem, there is an $A \in \mathscr{A}_{\text{s.a.}}$ and a $t \in \mathbb{R}$ such that $\psi(A) > t$, but $\varphi(A) \leq t$ for all $\varphi \in K_R$. Since $-\varphi \in K_R$ whenever $\varphi \in K_R$, $|\varphi(A)| \leq t$. By Theorem 4.11, $\|A\| \leq t$. But then $\psi(A) > t$ is impossible for ψ in the unit ball of $(\mathscr{A}_{\text{s.a.}})^*$. □

4.22 LEMMA. *Let* $\mathscr{A}$ *be a* C^* *algebra. For every two positive linear functionals* φ *and* ψ *on* $\mathscr{A}$*, the following are equivalent:*

(1) $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|$

(2) For all $\epsilon > 0$*, and all* $A \in \mathscr{A}^+$*, there exists* $X, Y \in \mathscr{A}^+$ *with* $X + Y \in B_{\mathscr{A}}$ *and* $X - Y \in B_{\mathscr{A}}$ *such that*

$$\varphi(X) < \epsilon \ , \quad \psi(Y) < \epsilon \ , \quad \varphi(Y) + \epsilon > \|\varphi\| \quad \text{and} \quad \psi(X) + \epsilon > \|\psi\| \ . \tag{4.15}$$

and

$$\|A(X+Y) - A\| = \|(X+Y)A - A\| < \epsilon \ . \tag{4.16}$$

Furthermore, when $\mathscr{A}$ *is unital, one may take* X *and* Y *to satisfy* $X + Y = 1$*, so that* (4.16) *is trivially true for all* A*.*

Proof. Suppose that *(2)* is valid for some $A \in \mathscr{A}^+$, e.g., $A = 0$. Pick $\epsilon > 0$, and suppose that X and Y satisfy the conditions in *(2)*. Then $X - Y \in B_{\mathscr{A}}$ and $(\varphi - \psi)(Y - X) \geq \|\varphi\| + \|\psi\| - 4\epsilon$. Hence $\|\varphi - \psi\| \geq \|\varphi\| + \|\psi\|$, and the opposite inequality is trivial. This prove that *(2)* implies *(1)*.

Suppose that *(1)* is valid. Pick $B, C \in \mathscr{A}^+ \cap B_{\mathscr{A}}$ such that $\varphi(B) + \epsilon > \|\varphi\|$ and $\psi(C) + \epsilon > \|\psi\|$. Since $\varphi - \psi$ is Hermitian, by (4.14), for all $\epsilon > 0$ there exists D in the unit ball of $\mathscr{A}_{\text{s.a.}}$ such that

$$\varphi(D) - \psi(D) + \epsilon \geq \|\varphi - \psi\| \ .$$

Suppose $\mathscr{A}$ is not unital. By Theorem 2.32 applied to $\{A, B, C, D\}$, there exits $E \in \mathscr{A}^+ \cap B_{\mathscr{A}}$ such that $\|ZE - Z\| < \epsilon$ and $\|EZ - Z\| < \epsilon$ when Z is any element of $\{A, B, C, D\}$. Then $EZE - Z = E(ZE - Z) + EZ - Z$ so that $\|EZE - Z\| < 2\epsilon$ when Z is any element of $\{A, B, C, D\}$.

Let $\mathscr{A}_1$ be the C^* algebra obtained by adjoining an identity to $\mathscr{A}$ in the canonical manner. Then for self adjoint Z in $B_{\mathscr{A}}$, the spectrum of $1 \pm Z$ lies in $[0,2]$ and hence it follows that $E(1 \pm Z)E = E^2 \pm EZE$ have spectrum in $[0,2]$. Thus, $E^2 \geq \pm EZE$ and $\frac{1}{2}(E^2 \pm EZE)$ both belong to $\mathscr{A}^* \cap B_{\mathscr{A}}$.

If $\mathscr{A}$ is unital, things are much simpler: We may simply take $E = 1$. In either case, by *(1)*,

$$\varphi(EDE) - \psi(EDE) + (1 + 2\|\varphi\| + 2\|\psi\|)\epsilon \geq \|\varphi\| + \|\psi\| \geq \varphi(E^2) + \psi(E^2) \ . \tag{4.17}$$

Define

$$X := \frac{1}{2}(E^2 - EDE) \quad \text{and} \quad Y := \frac{1}{2}(E^2 + EDE) \ .$$

Then as noted above, $X, Y \in \mathscr{A}^+$, and $X + Y = E^2 \in B_{\mathscr{A}}$ while $X - Y = -EDE \in B_{\mathscr{A}}$. Hence, with these definitions, (4.17) becomes

$$\psi(Y) + \varphi(X) \leq \frac{1}{2}(1 + 2\|\varphi\| + 2\|\psi\|)\epsilon \ . \tag{4.18}$$

Also, $\varphi(X + Y) = \varphi(E^2) \geq \varphi(EBE) \geq \|\varphi\| - (1 + 2\|\varphi\|)\epsilon$. Then by (4.18),

$$\varphi(Y) \geq \|\varphi\| - (2 + 3\|\varphi\| + \|\psi\|)\epsilon \ ,$$

In the same way, we obtain a similar lower bound for $\psi(X)$. Finally, since $X + Y = E^2$ and $E^2A - A = E(EA - E) + (EA - A)$, $\|E^2A - A\| \leq 2\epsilon$ Then since $\epsilon > 0$ is arbitrary, this proves that *(1)* implies *(2)*. Note that when $\mathscr{A}$ is unital $E^2 = 1$, and hence $X + Y = 1$. □

4.23 DEFINITION. Let $\mathscr{A}$ be a C^* algebra. Two positive linear functionals on $\mathscr{A}$ are *mutually singular* in case $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|$, in which case we write $\varphi \perp \psi$.

4.24 THEOREM (Grothendieck's Decomposition Theorem)**.** *Let $\mathscr{A}$ be a C^* algebra and let $\phi \in (\mathscr{A}^*)_{\text{s.a.}}$. Then ϕ has a unique decomposition $\phi = \varphi - \psi$ where φ and ψ are mutually singular positive linear functionals on $\mathscr{A}$.*

Proof. We may suppose without loss of generality that $\|\phi\| = 1$. By Lemma 4.21, there are $\phi_1, \phi_2 \in Q_{\mathscr{A}}$ and $t \in [0,1]$ such that $\phi = (1-t)\phi_1 - t\phi_2$. Define $\varphi = (1-t)\phi_1$ and $\psi = t\phi_2$. Then

$$1 = \|\phi\| = \|\varphi - \psi\| \leq \|\varphi\| + \|\psi\| = (1-t)\|\phi_1\| + t\|\phi_2\| \leq 1 \ .$$

Hence equality must hold throughout, and hence φ and ψ are mutually singular. This proves the existence of such a decomposition.

Now let $\phi = \varphi - \psi$ be one such decomposition, and suppose that $\phi = \widetilde{\varphi} - \widetilde{\psi}$ is another. Pick $\epsilon > 0$ and $A \in \mathscr{A}^+$, and pick X, Y satisfying the conditions of *(2)* of Lemma 4.22 for the first decomposition $\phi = \varphi - \psi$. As we have observed above, $\phi(Y - X) \geq \|\phi\| - 4\epsilon$. Hence $\widetilde{\varphi}(Y) + \widetilde{\psi}(X) - \widetilde{\varphi}(X) - \widetilde{\psi}(Y) \geq \|\widetilde{\varphi}\| + \|\widetilde{\psi}\| - 4\epsilon$. It follows that $\widetilde{\varphi}(X), \widetilde{\psi}(Y) \leq 4\epsilon$.

Next,

$$\begin{aligned} \|\varphi\| - \epsilon < \varphi(Y) &= \phi(Y) + \psi(Y) \\ &= \widetilde{\varphi}(Y) - \widetilde{\psi}(Y) + \psi(Y) \leq \widetilde{\varphi}(Y) + \epsilon \end{aligned}$$

and hence $\widetilde{\varphi}(Y) \geq \|\varphi\| - 2\epsilon$. It follows that $\|\widetilde{\varphi}\| \geq \|\varphi\|$. By symmetry, $\|\widetilde{\varphi}\| = \|\varphi\|$, and hence $\widetilde{\varphi}(Y) \geq \|\widetilde{\varphi}\| - 2\epsilon$. A similar argument applies to X and $\widetilde{\psi}$. Thus, replacing ϵ by 4ϵ, we have that

(4.16) is of satisfied with the *same* pair X, Y for both decompositions, and (4.16) is independent of the decomposition. Hence for each $A \in \mathscr{A}^+$, and each $\epsilon > 0$, there is a single pair X, Y satisfying the conditions in *(2)* of Lemma 4.22. Using this, we show the two decompositions coincide.

By the Cauchy-Schwarz inequality, $|\varphi(AX)| \leq \left(\varphi(X^{1/2}AX^{1/2})\varphi(X)\right)^{1/2} \leq \epsilon$, and likewise for $\widetilde{\varphi}(AX)$, $\psi(AY)$ and $\widetilde{\psi}(AY)$. Hence

$$|\varphi(AX)|\ , |\widetilde{\varphi}(AX)|\ , |\psi(AY)|\ ,\ |\widetilde{\psi}(AY)| < \epsilon\ . \tag{4.19}$$

Now for $\epsilon > 0$, let X, Y satisfy the conditions *(2)* of Lemma 4.22 for both decompositions. Then since $\|A(X+Y) - A\| < \epsilon$, and since $\varphi - \widetilde{\varphi} = \psi - \widetilde{\psi}$

$$\begin{aligned}
|\varphi(A) - \widetilde{\varphi}(A))| &\leq |\varphi(A(X+Y)) - \widetilde{\varphi}(A(X+Y))| + 2\|\varphi\|\epsilon \\
&\leq |\varphi(AX)) - \widetilde{\varphi}(AX)| + |\varphi(AY) - \widetilde{\varphi}(AY)| + 2\|\varphi\|\epsilon \\
&= |\varphi(AX) - \widetilde{\varphi}(AX)| + |\psi(AY) - \widetilde{\psi}(AY)| + 2\|\varphi\|\epsilon \\
&\leq |\varphi(AX)| + |\widetilde{\varphi}(AX)| + |\psi(AY)| + |\widetilde{\psi}(AY)| + 2\|\varphi\|\epsilon \\
&\leq (4 + 2\|\varphi\|)\epsilon\ ,
\end{aligned}$$

where (4.19) was used in the final step. Since $\epsilon > 0$ is arbitrary, and since Hermitian linear functional are determined by their values on $\mathscr{A}^+$, $\widetilde{\varphi} = \varphi$. □

For $\varphi \in \mathscr{A}^*$, define φ^* by

$$\varphi^*(A) = \overline{\varphi(A^*)} \quad \text{for all} \quad A \in \mathscr{A}\ . \tag{4.20}$$

Note that φ is Hermitian if and only if $\varphi^* = \varphi$. In any case, $\dfrac{1}{2}(\varphi + \varphi^*)$ and $\dfrac{1}{2i}(\varphi - \varphi^*)$ are Hermitian, and $\varphi = \dfrac{1}{2}(\varphi + \varphi^*) + i\dfrac{1}{2}(\varphi + \varphi^*)$, so that every $\varphi \in \mathscr{A}^*$, is a linear combination of two elements of $(\mathscr{A}^*)_{\text{s.a.}}$.

4.4 Normal functionals

For any Banach space X, a linear functional in X^{**} is continuous with respect to the weak-$*$ topology on X^* if and only if it is in the image of the canonical embedding of X into X^{**}. Thus, if $\mathscr{M}$ is a von Neumann algebra, and $\mathscr{M}_*$ is its predual, as described in Theorem 3.43, the elements of $\mathscr{M}^*$ that are continuous with respect to the weak-$*$ topology are precisely the elements of $\mathscr{M}^*$ that are in the image of $\mathscr{M}_*$ under the canonical embedding. Since the σ-weak topology is just another name for the weak-$*$ topology in this context, this is the same as saying that the elements of $\mathscr{M}^*$ that are continuous with respect to the σ-weak topology are precisely the elements of $\mathscr{M}^*$ that are in the image of $\mathscr{M}_*$ under the canonical embedding.

4.25 DEFINITION. Let $\mathscr{M}$ be a von Neumann algebra, and let $\mathscr{M}_*$ be its predual, as described in Theorem 3.43. An element of $\mathscr{M}^*$ is *normal* if and only if it is continuous with respect to the σ-weak topology on $\mathscr{M}$, or, what is the same by the remarks above, if and only if it is in the image of $\mathscr{M}_*$ under the canonical embedding of $\mathscr{M}_*$ into $\mathscr{M}^*$. We therefore use $\mathscr{M}_*$ to denote the set of normal functionals on $\mathscr{M}$.

4.26 LEMMA. *Let $\mathscr{M}$ be a von Neumann algebra. Then $\mathscr{M}_*$ is a norm closed subspace of $\mathscr{M}^*$. Furthermore, let ϕ be a normal functional on a von Neumann algebra $\mathscr{M}$. Then:*

(i) ϕ^* is also normal, and hence the Hermitian and skew-Hermitian parts of ϕ are normal.

(ii) For all $B \in \mathscr{M}$, define $B\phi$ to be the linear functional $B\phi(X) = \phi(BX)$ for all $X \in \mathscr{M}$, and define ϕB be the linear functional $\phi B(X) = \phi(XB)$ for all $X \in \mathscr{M}$. Then $B\phi$ and ϕB are also normal.

(iii) Let ϕ be a Hermitian normal functional on $\mathscr{M}$, and let $\phi = \phi_+ - \phi_-$ be its Grothendieck decomposition. then ϕ_+ and ϕ_- are normal.

Proof. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{M}_*$ converging in norm to $\phi \in \mathscr{M}^*$. Since the canonical embedding is isometric, $\{\phi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence sequence in $\mathscr{M}_*$, and $\mathscr{M}_*$ is complete. Hence $\phi \in \mathscr{M}_*$.

Let ϕ be normal; i.e, $\phi \in \mathscr{M}_*$. By Theorem 3.43, there is $A \in \mathscr{T}(\mathscr{H})$ such that $\phi(X) = \mathrm{Tr}[AX]$ for all $X \in \mathscr{M}$, and conversely, any linear functional of this form is normal.

To prove *(i)*, note that $\phi^*(X) = \overline{\mathrm{Tr}[AX^*]} = \mathrm{Tr}[XA^*] = \mathrm{Tr}[A^*X]$. Since $A^* \in \mathscr{T}(\mathscr{B})$, $\phi^* \in \mathscr{M}_*$, and then of course $\frac{1}{2}(\phi + \ \phi^*)$ and $\frac{1}{2i}(\phi - \phi^*)$ belong to $\mathscr{M}_*$.

To prove *(ii)*, note that $B\phi^*(X) = \mathrm{Tr}[BAX]$ and $BA \in \mathscr{T}(\mathscr{H})$. The same reasoning applies to ϕB.

To prove *(iii)*, pick $\epsilon > 0$, and then by the final part of Lemma 4.22, since $\mathscr{M}$ is unital, there exist $Y \geq 0$ with $Y \in B_{\mathscr{M}}$ such that $\phi_+(1-Y), \phi_-(Y) < \epsilon$. Then for any $A \in \mathscr{M}$,

$$\begin{aligned} |\phi_+(A) - Y\phi(A)| &= |\phi_+((1-Y)A) - \phi_-(YA)| \\ &\leq (\phi_+(1-Y)\phi_+(A^*(1-Y)A))^{1/2} + (\phi_-(Y)\phi_+(A^*YA))^{1/2} \\ &\leq \epsilon^{1/2}(\|\phi_+\| + \|\phi_-\|)\|A\| = \epsilon^{1/2}\|\phi\|\|A\| \ . \end{aligned}$$

since $Y\phi$ is normal by part *(ii)*, and since A and $\epsilon > 0$ are arbitrary, ϕ_+ is in the norm closure of $\mathscr{M}_*$, which is $\mathscr{M}_*$ itself. □

4.27 THEOREM (Sakai's Polar Factorization Theorem)**.** *Let $\mathscr{M}$ be a von Neumann algebra, and let $\phi \in \mathscr{M}_*$. Then there exists a unique positive normal functional $|\phi|$ such that $\||\phi|\| = \|\phi\|$ and such that*

$$|\phi(X)|^2 \leq \|\phi\||\phi|(X^*X) \tag{4.21}$$

for all $X \in \mathscr{M}$. Furthermore, there is a partial isometry $U \in \mathscr{M}$ such that $\phi = U|\phi|$ and $|\phi| = U^*\phi$.

Proof. Let $\|\phi\| = 1$. The set $\{X \in B_{\mathscr{M}} \ : \ \phi(X) = 1\}$ is a non-empty, since norming functionals always exist in a dual Banach space, and it is a convex, σ-weakly compact subset of $B_{\mathscr{M}}$. By the Krein-Milman Theorem, this set contains an extreme point U^*, which is also evidently an extreme point of $B_{\mathscr{M}}$. By Theorem 2.48, U^* is a partial isometry such that with $P := U^*U$ and $Q := UU^*$, $P^\perp \mathscr{M} Q^\perp = 0$.

Now define two linear functionals ψ_ℓ and ψ_r by $\psi_\ell := U^*\phi$ and $\psi_r := \phi U^*$. Evidently, $\|\psi_\ell\|, \|\psi_r\| \leq \|\phi\| = 1$, and both are normal by Corollary 4.26. Since

$$\|\psi_\ell\| \geq \psi_\ell(1) = \phi(U^*) = 1 \geq \|\psi_\ell\| \ ,$$

ψ_ℓ is positive and $\|\psi_\ell\| = \|\phi\| = 1$. Likewise, ψ_r is positive and $\|\psi_r\| = \|\phi\| = 1$.

As shown in the proof of Theorem 2.48, $U = UP = QU$. Taking adjoints, $U^* = PU^*$ and $U^* = U^*Q$. Then $\psi_\ell(Q) = \phi(U^*Q) = \phi(U^*) = 1$, and hence $\psi_\ell(Q^\perp) = 0$. The same sort of argument shows that $\psi_r(P^\perp) = 0$. We now claim that $P^\perp\phi = 0$. To see this, let $X \in \mathscr{M}$, and use the fact that $P^\perp \mathscr{M} Q^\perp = 0$ as follows:

$$\begin{aligned} \phi(P^\perp X) &= \phi(P^\perp XQ) + \phi(P^\perp XQ^\perp) = \phi(P^\perp XQ) \\ &= \phi(P^\perp XUU^*) = \psi_r(P^\perp XU) \le \left(\psi_r(P^\perp)\right)^{1/2} (\psi_r(U^*X^*XU))^{1/2} = 0 \ . \end{aligned}$$

Since X is arbitrary, $P^\perp\phi = 0$, and $\phi = P\phi$. But then for all X, $U\psi_\ell(X) = \psi_\ell(UX) = \phi(PX) = \phi(X)$, and hence $U\psi_\ell = \phi$. (The same sort of argument shows that $\phi = \phi Q$, and that $\psi_r U = \phi$,)

By the Cauchy-Schwarz inequality, for all $X \in \mathscr{A}$,

$$|\phi(X)|^2 = |\psi_\ell(UX)|^2 \le \psi_\ell(U^*U)\psi_\ell(X^*X) \le \|\psi_\ell\|\psi_\ell(X^*X) \ .$$

Likewise, $|\phi(X)|^2 = |\psi_r(XU)|^2 \le \|\psi_r\|\psi_r(XX^*)$. We now show that there is exactly one positive functional such ψ with $\|\psi\| = 1$ such that for all X,

$$|\phi(X)|^2 \le \|\psi\|\psi(X^*X) \ . \tag{4.22}$$

This will prove that (4.22) characterizes ψ_ℓ. (The reasoning will show that $|\phi(X)|^2 \le \|\psi\|\psi(XX^*)$ characterizes ψ_r.)

Suppose ψ is any positive functional with $\|\psi\| = 1$ that satisfies (4.22). For X self adjoint,

$$(\psi_\ell(X))^2 = |\phi(U^*X)|^2 \le \psi(XUU^*X) \le \psi(X^2) \ .$$

Replacing X by $1 + \epsilon X$, this yields $(\psi_\ell(1 + \epsilon X))^2 \le \psi((1 + \epsilon X)^2)$, and since $\psi_\ell(1) = \psi(1) = 1$, and since $\epsilon > 0$ is arbitrary, this yields. $\psi_\ell(X) \le \psi(X)$ for all self adjoint X. Replacing X with $-X$, we get that $\psi_\ell = \psi$. Now define $|\phi| = \psi_\ell$. □

Let ψ be a positive normal functional on a von Neumann algebra $\mathscr{M}$ on the Hilbert space $\mathscr{H}$. Let $A \in \mathscr{T}(\mathscr{H})$ be such that $\phi(X) = \mathrm{Tr}[AX]$ for all $X \in \mathscr{M}$. Let $\{A\}$ denote the equivalence class of such elements of $\mathscr{T}(\mathscr{H})$. It is easy to see that $\{A\} \cap \mathscr{M}_{\text{a.s.}} \neq \emptyset$: For all $X \in \mathscr{M}_{\text{s.a.}}$, $\phi(X) = \mathrm{Tr}[AX] = \mathrm{Tr}[(AX)^*] = \mathrm{Tr}[XA^*] = \mathrm{Tr}[A^*X]$. Since ϕ is determined by its action on $\mathscr{M}_{\text{s.a.}}$, $\frac{1}{2}(A + A^*) \in \{A\}$. It is a deeper fact that there is a positive element $B \in \{A\}$.

4.28 THEOREM (Dixmier's Extension Theorem)**.** *Let ϕ be a positive normal functional on a von Neumann algebra $\mathscr{M}$ on the Hilbert space $\mathscr{H}$. Then there exists a positive $B \in \mathscr{T}(\mathscr{H})$ with $\mathrm{Tr}[BX] = \phi(X)$ for all $X \in \mathscr{M}$. In particular, taking $X = 1$, $\|B\|_1 = \|\phi\|$, and hence ϕ has a norm-preserving normal extension to $\mathscr{B}(\mathscr{H})$ that is also positive.*

Proof. Let $A \in \mathscr{T}(\mathscr{H})_{\text{s.a.}}$ be such that $\phi(X) = \mathrm{Tr}[AX]$ for all $X \in \mathscr{M}$. Then A has the spectral resolution $A = \displaystyle\sum_{\mathrm{J}=1}^{\infty} \lambda_j |\zeta_j\rangle\langle\zeta_j|$. where $\{\zeta_j\}_{j\in\mathscr{N}}$ is orthonormal, $|\lambda_{j+1}| \ge |\lambda_j|$, and $\sum_{j=1}^\infty |\lambda_j| = \|A\|_1$.

Let $\widehat{\mathscr{H}} = \sum_{j=1}^\infty \mathscr{H}_j$ where each $\mathscr{H}_j$ is a copy of $\mathscr{H}$. For $(\eta_j)_{j\in\mathbb{N}} \in \widehat{\mathscr{H}}$ and $X \in \mathscr{M}$, define $\pi(X)(\eta_j)_{j\in\mathbb{N}} = (X\eta_j)_{j\in\mathbb{N}}$. Using the spectral resolution of A, define

$$\widehat{\xi}_0 := \|A\|_1^{-1/2}(|\lambda_j|^{1/2}\zeta_j)_{j\in\mathbb{N}} \ \in \widehat{\mathscr{H}} \ .$$

Let $\widehat{\mathscr{K}}$ be the closure of $\pi(\mathscr{M})\widehat{\xi}_0$; $\widehat{\mathscr{K}}$ is invariant under π, and $\widehat{\xi}_0$ is a cyclic unit vector for π.

Observe that for all $X \in \mathscr{M}^+$,

$$\langle \widehat{\xi}_0, \pi(X)\widehat{\xi}_0 \rangle = \frac{1}{\|A\|_1} \sum_{j=1}^{\infty} |\lambda_j| \langle \zeta_j, X\zeta_j \rangle \geq \frac{1}{\|A\|_1} \sum_{j=1}^{\infty} \lambda_j \langle \zeta_j, X\zeta_j \rangle = \frac{1}{\|A\|_1} \phi(X) \ .$$

By Lemma 4.18, there exists a positive operator $\widehat{B}$ in $(\pi(\mathscr{M}))'$ such that

$$\phi(X) = \langle \widehat{\xi}_0, \widehat{B}\pi(X)\widehat{\xi}_0 \rangle = \langle \widehat{B}^{1/2}\widehat{\xi}_0, \pi(X)\widehat{B}^{1/2}\widehat{\xi}_0 \rangle \ .$$

Taking $X = 1$, we see that $\|\widehat{B}^{1/2}\widehat{\xi}_0\|^2 = \|\phi\|$. Define $\{\eta_j\}_{j\in\mathbb{N}}$ and $B \in \mathscr{T}(\mathscr{H})$ by

$$(\eta_j)_{j\in\mathbb{N}} := \widehat{B}^{1/2}\widehat{\xi}_0 \quad \text{and} \quad B := \sum_{j=1}^{\infty} |\eta_j\rangle\langle\eta_j| \ ,$$

it being evident from $\|\widehat{B}^{1/2}\widehat{\xi}_0\|^2 = \sum_{j=1}^{\infty} \|\eta_j\|^2 = \|\phi\|$ that the series defining B converges in trace norm. With these definitions, $\phi(X) = \mathrm{Tr}[BX]$ for all X, and $B \in \mathscr{T}(\mathscr{H})^+$. □

4.29 COROLLARY (Sakai)**.** *Let $\mathscr{M}$ be a von Neumann algebra on $\mathscr{H}$. Every normal functional ϕ on $\mathscr{M}$ has a normal extension $\widehat{\phi}$ to $\mathscr{B}(\mathscr{H})$ such $\|\widehat{\phi}\| = \|\phi\|$.*

Proof. This is an immediate consequence of Dixmier's Extension Theorem and Sakai's Polar Decomposition Theorem. □

4.5 The support projection of a normal state

The main results in this section are due to Dye [11] who worked in a somewhat less general setting. The extension to the present setting is due to Dixmier [10].

4.30 THEOREM. *Let $\mathscr{M}$ be a von Neumann algebra, and let φ be a normal state on $\mathscr{M}$. The set of projections $Q \in \mathscr{M}$ such that $\varphi(Q) = 0$ contains a maximal element Q_φ, and for all $A \in \mathscr{M}$,*

$$\varphi(AQ_\varphi) = \varphi(Q_\varphi A) = 0 \ , \tag{4.23}$$

and

$$\varphi(A^*A) = 0 \quad \text{if and only if} \quad AQ_\varphi = A \ . \tag{4.24}$$

Proof. Let $\mathscr{J} := \{A \in \mathscr{M} \ \ \varphi(A^*A) = 0\}$. By Lemma 4.7, $\mathscr{J}$ is a left ideal, and by (4.6), $\mathscr{J} = \bigcap_{B\in\mathscr{M}}\{\ker(B\varphi)\}$. By Lemma 4.26, each $B\varphi$ is normal, and hence each $\ker(B\varphi)$ is σ-weakly closed. By Lemma 3.45, $\mathscr{J}$ is weakly closed and there is a unique projector Q_φ in $\mathscr{J}$ such that $AQ_\varphi = A$ for all $A \in \mathscr{J}$.

If P is any projector such that $\varphi(P) = 0$, then $P \in \mathscr{J}$ and hence $PQ_\varphi = P$, from which it follows that $P \leq Q_\varphi$. Hence Q_φ has the asserted maximality property. Then (4.23) follows from the Cauchy-Schwarz inequality: $|\varphi(AQ_\varphi)|^2 \leq \varphi(A^*A)\varphi(Q_\varphi) = 0$, and likewise for $\varphi(Q_\varphi A)$.

By the first part of the proof, if $\varphi(A^*A) = 0$, $AQ_\varphi = A$. On the other hand, since $Q_\varphi \in \mathscr{J}$, which is an ideal, if $AQ_\varphi = A$, then $A \in \mathscr{J}$. □

4.31 DEFINITION (Support projection of a normal state)**.** Let φ be a normal state on a von Neumann algebra $\mathscr{M}$. The *support* projection P_φ is the orthogonal complement of the projection Q_φ that is the maximal projection in $\mathscr{M}$ on which φ vanishes.

4.32 LEMMA. *Let φ and ψ be normal states on a von Neumann algebra $\mathscr{M}$, and let P_φ and P_ψ, respectively, be their support projections. Then φ and ψ are mutually singular (in the sense of Definition 4.23) if and only if their support projections satisfy $P_\varphi \leq P_\psi^\perp$ and $P_\psi \leq P_\varphi^\perp$*

Proof. Suppose that $P_\varphi \leq P_\psi^\perp$ and $P_\psi \leq P_\varphi^\perp$. We shall apply Lemma 4.22. Define $Y := P_\varphi$ and $X := P_\varphi^\perp$. Clearly X, Y, $X+Y = 1$ and $X-Y$ all belong to $B_{\mathscr{A}} \cap \mathscr{M}^+$. Moreover, $\varphi(Y) = 1 = \|\varphi\|$ and $\varphi(X) = 0$. Since $P_\psi \leq P_\varphi^\perp = X$, $\psi(X) \geq \psi(P_\psi) = 1 = \|\psi\|$. Since $Y := P_\varphi \leq P_\psi^\perp$, $\psi(Y) = 0$. Thus, by Lemma 4.22, φ and ψ are mutually singular.

Conversely, suppose that φ and ψ are mutually singular. Since norming linear functionals always exist, there is an $A \in B_{\mathscr{M}}$, self adjoint since $\varphi - \psi$ is Hermitian, such that

$$\begin{aligned} \|\varphi\| + \|\psi\| = \|\varphi - \psi\| &= \varphi(A) - \psi(A) = \varphi(A_+) + \psi(A_-) - \varphi(A_-) - \psi(A_+) \\ &\leq \|\varphi\| + \|\psi\| - \varphi(A_-) - \psi(A_+) \ . \end{aligned}$$

Hence $\|\varphi\| = \varphi(A_+) = \varphi(P_\varphi A_+ P_\varphi)$, $\|\psi\| = \psi(A_-) = \psi(P_\psi A_- P + \psi)$, $\varphi(A_-) = 0$ and $\psi(A_+) = 0$. Clearly, $P_\varphi - P_\varphi A_+ P_\varphi \geq 0$, and if $P_\varphi \neq P_\varphi A_+ P_\varphi$, there is some non-zero spectral projection Q of the difference and some $r > 0$ such that $P_\varphi - P_\varphi A_+ P_\varphi \geq rQ$. Then since $\|\varphi\| = \varphi(1) = \varphi(P_\varphi)$, $0 = \varphi(P_\varphi - P_\varphi A_+ P_\varphi) \geq r\varphi(Q)$. But since $Q \leq P_\varphi$, $\varphi(Q) > 0$, and this is impossible. Hence $P_\varphi = P_\varphi A_+ P_\varphi$. Since $\varphi(A_-) = 0$, $P_\varphi(A_-)P_\varphi = 0$ for similar reasons. Hence $P_\varphi = P_\varphi A_+ P_\varphi - P_\varphi A_- P_\varphi = P_\varphi A P_\varphi$. Likewise, $P_\psi = P_\psi A_- P_\psi = -P_\psi A P_\psi$.

Let η be a unit vector in $\mathrm{ran}(P_\varphi)$. Then $1 = \langle \eta, P_\varphi \eta\rangle = \langle \eta, A_+ \eta\rangle$, and by the conditions for equality in the Cauchy-Schwarz inequality, $A\eta = \eta$. Therefore, every vector in $\mathrm{ran}(P_\varphi)$ is an eigenvector of A with eigenvalue 1, while every vector in $\mathrm{ran}(P_\psi)$ is an eigenvector of A with eigenvalue -1. It follows that $\mathrm{ran}(P_\varphi)$ and $\mathrm{ran}(P_\psi)$ are mutually orthogonal. This proves that $P_\varphi \leq P_\psi^\perp$ and $P_\psi \leq P_\varphi^\perp$. □

The next lemma describes the range of the support projection of a normal state.

4.33 LEMMA. *Let φ be a normal state on a von Neumann algebra $\mathscr{M}$, and let P_φ be its support projection. Let $A \in \mathscr{T}(\mathscr{H})$ be such that $A \geq 0$, $\|A\|_1$ and $\varphi(X) = \mathrm{Tr}[AX]$ for all $X \in \mathscr{M}$. Such an A exists by Theorem 4.28. Let $A = \sum_{j\in J}^\infty \lambda_j |\eta_j\rangle\langle\eta_j|$ be a spectral resolution of A with $\lambda_j > 0$ for all $j \in J$. Then the subspace $\mathscr{V}_\varphi$ defined by*

$$\mathscr{V}_\varphi := \mathrm{Span}(\{B\eta_j \ , \ B \in \mathscr{M}' , \ j \in J\}) \tag{4.25}$$

is dense in the range of P_φ. That is, P_φ is the orthogonal projection onto the closure of $\mathscr{V}_\varphi$.

Proof. Since $\varphi(X) = \varphi(P_\varphi X P_\varphi)$ for all X, it follows that $\varphi(X) = \mathrm{Tr}[(P_\varphi A P_\varphi)X]$ for all $X \in \mathscr{M}$. Thus, $\|P_\varphi A P_\varphi\|_1 = \|A\|_1$, and this means that $\mathrm{Tr}[A] = \mathrm{Tr}[P_\varphi A P_\varphi] = \mathrm{Tr}[AP_\varphi]$. Since

$$(A^{1/2} - A^{1/2}P_\varphi)^*(A^{1/2} - A^{1/2}P_\varphi) = A + P_\varphi A P_\varphi - AP_\varphi - P_\varphi A$$

is a positive trace class operator such that

$$\begin{aligned} \|A^{1/2} - A^{1/2}P_\varphi\|^2 &= \|(A^{1/2} - A^{1/2}P_\varphi)^*(A^{1/2} - A^{1/2}P_\varphi)\| \\ &\leq \mathrm{Tr}[(A^{1/2} - A^{1/2}P_\varphi)^*(A^{1/2} - A^{1/2}P_\varphi)] = 0 \ , \end{aligned}$$

$A^{1/2} = A^{1/2}P_\varphi$, and hence $A = P_\varphi A P_\varphi$. Thus the eigenvectors η_j figuring in the spectral resolution $A = \sum_{j\in J}^{\infty} \lambda_j |\eta_j\rangle\langle\eta_j|$ are all in $\mathrm{ran}(P_\varphi)$, and moreover, for all $B \in \mathscr{M}'$ and all $j \in J$,

$$B\eta_j = BP_\varphi\eta_j = P_\varphi B\eta_j \ ,$$

and hence $B\eta_j \in \mathrm{ran}(P_\varphi)$. This proves that $\mathscr{V}_\varphi \subset \mathrm{ran}(P_\varphi)$. Let Q be the orthogonal projection onto the closure of $\mathscr{V}_\varphi$. Then since $\mathscr{V}_\varphi \subset \mathrm{ran}(P_\varphi)$, $Q \leq P_\varphi$, and since each $\eta_j \in \mathscr{V}_\varphi$, $QAQ = A$. Hence $\varphi(Q^\perp) = 0$. This means that $P_\varphi \leq Q$. Altogether, $P_\varphi = Q$. □

Let $\mathscr{M}$ be a von Neumann algebra, and φ a normal state on $\mathscr{M}$. There is a natural topology on $\mathscr{M}$ induced by the seminorm $A \mapsto \varphi(A^*A)^{1/2} = \langle A, A\rangle_\varphi$ associated to φ: A neighborhood base at the origin consists of the sets

$$\mathcal{U}_{\varphi,\epsilon} := \{ A \in \mathscr{M} \ : \ \langle A, A\rangle_\varphi \leq \epsilon^2 \ \} \ .$$

This topology is evidently the weakest topology making all of the maps $A \mapsto \langle A, A\rangle_\varphi$ continuous, and it will not be Haussdorf unless φ is faithful. However, a set S of normal states is said to be *faithful* in case for each $A \neq 0$, there is some $\varphi \in S$ such that $\langle A, A\rangle_\varphi > 0$, and then the weakest topology making all of the maps $A \mapsto \langle A, A\rangle_\varphi$, $\varphi \in S$, continuous, is Haussdorf.

4.34 THEOREM. *Let φ be a normal form, and let P_φ be its support functional. Let $\mathscr{M}_\varphi$ be the subspace of $\mathscr{M}$ consisting of operators of the form TP_φ, $T \in \mathscr{M}$. Then $\mathscr{M}_\varphi$ is closed in the strong operator topology. The sesquilinear form $\langle\cdot,\cdot\rangle_\varphi$ is non-degenerate on $\mathscr{M}_\varphi$, and induces an Hilbertian metric on $\mathscr{M}_\varphi$. On $B_{\mathscr{M}} \cap \mathscr{M}_\varphi$, the topology of this Hermitian metric coincides with the strong operator topology. In particular, if $\{T_n\}$ is any Cauchy sequence in $B_{\mathscr{M}} \cap \mathscr{M}_\varphi$ for the Hilbertian metric, then there exists $T \in B_{\mathscr{M}} \cap \mathscr{M}_\varphi$ such that $\lim_{n\to\infty}\langle T_n - T, T_n - T\rangle_\varphi = 0$.*

Proof. $\mathscr{M}_\varphi$ is the null-space of the strongly continuous linear map $T \mapsto TP_\varphi^\perp$, and hence $\mathscr{M}_\varphi$ is strongly dosed. The non-degeneracy of the inner product follows from the definition of the support projection.

Let $T, T_0 \in B_{\mathscr{M}} \cap \mathscr{M}_\varphi$, and $\epsilon > 0$. Let $A \in \mathscr{T}(\mathscr{H})$ be positive with $\|A\|_1 = 1$ and $\mathrm{Tr}[AX] = \varphi(X)$ for all $X \in \mathscr{M}$. Let $A = \sum_{j\in J} \lambda_j |\eta_j\rangle\langle\eta_j|$ be its spectral resolution as in Lemma 4.33. For any $n \in \mathbb{N}$, let $A_n = \sum_{j\leq n} \lambda_j |\eta_j\rangle\langle\eta_j|$, and then for some finite n, $\|A - A_n\|_1 < \epsilon/6$, and hence

$$\varphi((T-T_0)^*(T-T_0) \leq \mathrm{Tr}[A_n(T-T_0)^*(T-T_0)] + 2\epsilon/3 \leq \sum_{j=1}^{n} \|(T-T_0)\eta_j\|^2 + 2\epsilon/3 \ .$$

Therefore, $\{T \in B_{\mathscr{M}} \cap \mathscr{M}_\varphi \ : \ \langle T - T_0, T - T_0\rangle_\varphi < \epsilon\}$ contains the strongly open set

$$\left\{ T \in B_{\mathscr{M}} \cap \mathscr{M}_\varphi \ : \ \|(T-T_0)\eta_j\|^2 < \frac{\epsilon}{3n} \ , \ j = 1, \ldots, n \right\} \ .$$

Conversely, the strong operator topology on *bounded* sets is the weakest topology making all of the maps $T \mapsto T\xi$ continuous for ξ in any fixed *dense* subset of $\mathscr{H}$. By Lemma 4.33 $\mathscr{V}_\varphi$, as deifned in (4.25). is dense in $\mathrm{ran}P_\varphi$. Since operators in $\mathscr{M}_\varphi$ annihilate all vectors in the range of $P_\varphi^\perp$, it suffices to show that each set of the form $\{T \in B_{\mathscr{M}} \cap \mathscr{M}_\varphi \ : \ \|(T-T_0)\xi\| < \epsilon \ \}$, where ξ is any vector in $\mathscr{V}_\varphi$, contains a neighborhood of T_0 for the Hilbertian metric.

If $\xi \in \mathscr{V}_\varphi$, we may write $\xi = \sum_{j=1}^m B_j\eta_j$. Then

$$\begin{aligned}\|(T-T_0)\xi\| \le \sum_{j=1}^m \|(T-T_0)B_j\eta_j\| &= \sum_{j=1}^m \|B_j(T-T_0)\eta_j\| \le \sum_{j=1}^m \|B_j\|\|(T-T_0)\eta_j\| \\ &\le \sum_{j=1}^m \frac{\|B_j\|}{\sqrt{\lambda_j}}\|(T-T_0)\sqrt{\lambda_j}\eta_j\| \\ &\le \left(\sum_{j=1}^m \frac{\|B_j\|}{\sqrt{\lambda_j}}\right)\langle T-T_0, T-T_0\rangle_\varphi^{1/2} .\end{aligned}$$

This proves that $\{T \in B_{\mathscr{M}} \cap \mathscr{M}_\varphi \ :\ |(T-T_0)\xi\| \le \epsilon\}$ contains a neighborhood of T_0 in the Hilbertian metric topology, and completes the proof the the two topologies coincide on $B_{\mathscr{M}} \cap \mathscr{M}_\varphi$.

Now consider any Cauchy sequence $\{T_n\}$ in $B_{\mathscr{M}} \cap \mathscr{M}_\varphi$ for the Hilbertian metric. By what we have just shown, for each $\zeta \in \mathscr{H}$, $\{T_n\zeta\}$ is a Cauchy sequence on $\mathscr{H}$, and has a limit that we denote by $T\zeta$. It is easy to see that $\zeta \mapsto T\zeta$ is a bounded linear operator such that T_n converges to T in the strong operator topology. Therefore, $T \in \mathscr{M}$ and $\|T\| \le \sup\{\|T_n\|\} \le 1$. Finally, it is clear that that $TP_\varphi^\perp = 0$, and hence $T \in B_{\mathscr{M}} \cap \mathscr{M}_\varphi$. By what we have shown above, the strong convergence of T_n to T means that T_n converges to T in the Hilbertian metric. □

4.6 Order and normality

Let $\mathcal{S}$ be a bounded upward directed set of operators in a von Neumann algebra $\mathscr{M}$. Then its least upper bound, $\bigvee_{A\in\mathcal{S}} A$, exists and is in the strong closure of $\mathcal{S}$. It is therefore also in the weak closure, and since the weak and the σ-weak topologies coincide on bounded sets, $\bigvee_{A\in\mathcal{S}} A$ belongs to the σ-weak closure of $\mathcal{S}$.

It is then evident that if ϕ is any positive normal functional on $\mathscr{M}$,

$$\phi\left(\bigvee_{A\in\mathcal{S}} A\right) = \bigvee_{A\in\mathcal{S}} \phi(A) . \tag{4.26}$$

In fact the property (4.26) is characteristic of normal positive functionals:

4.35 THEOREM. *A positive functional ϕ on a von Neumann algebra $\mathscr{M}$ is normal if and only if* (4.26) *is valid for every bounded upward directed set $\mathcal{S}$ in $\mathscr{M}$.*

Proof. It remains to show that for any positive ϕ, if (4.26) is valid for every bounded upward directed set $\mathcal{S}$ in $\mathscr{M}$, then ϕ is normal. We make two temporary definitions to be used in the proof: A positive linear functional ϕ is *order-continuous* in case (4.26) is valid for every bounded upward directed set $\mathcal{S}$ in $\mathscr{M}$, and we say that an element $B \in \mathscr{M}$ is *regular* for the order continuous positive functional ϕ in case ϕB is normal.

Step 1: Let $\mathcal{S}$ be any bounded upward directed set of regular elements of $\mathscr{M}$. Let $B = \bigvee_{A\in\mathcal{S}} A$ which belongs to $\mathscr{M}$ by Vigier's Theorem. Then B is regular.

To prove this, it suffices to show that ϕB is the norm limit of a sequence in $\{\phi A_n\}$, $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ since $\mathscr{M}_*$ is norm closed. Let c denote the least upper bound on the norms of elements of $\mathcal{S}$. By the order continuity of ϕ, for each $n \in \mathbb{N}$, there exists $A_n \in \mathcal{S}$ such that $\phi(B - A_n) < 1/n^2$. Then for any $X \in B_{\mathscr{M}}$,

$$\begin{aligned} |(\phi B - \phi A_n)(X)| &= |\phi(X(B - A_n)| = |\phi(X(B - A_n)^{1/2}(B - A_n)^{1/2}| \\ &= \phi(B - A_n)^{1/2}\phi(X(B - A_n)X^*)^{1/2} \leq \frac{1}{n}(2c)^{1/2}\|\phi\|^{1/2} \ . \end{aligned}$$

Step 2: Let $M \in \mathscr{M}$ be non-zero and positive. Then there exists a non-zero $B \in \mathscr{M}$ and a $\xi \in \mathscr{H}$ such that $0 \leq B \leq M$, and such that

$$0 \leq X \leq cB \quad \text{for some } c > 0 \quad \Rightarrow \quad \phi(X) \leq \langle \xi, X\xi \rangle \ . \tag{4.27}$$

To prove this, choose a vector $\xi \in \mathscr{H}$ such that $\langle \xi, M\xi \rangle > \phi(M)$, and define

$$\mathcal{R} := \{ A \in \mathscr{M} \ : \ 0 \leq A \leq M \ , \langle \xi, A\xi \rangle \leq \phi(A) \ \} \ .$$

The linear functional $X \mapsto \langle \xi, A\xi \rangle$ is evidently normal, and hence order continuous, as is ϕ. Hence if $\mathcal{S}$ is any totally ordered set in $\mathcal{R}$, it has a least upper bound that is in $\mathscr{M}$ by Vigier's Theorem, and which belongs to $\mathcal{R}$ by order continuity. Then by Zorn's Lemma, $\mathcal{R}$ contains a maximal element C. Define $B = M - C$. Then $0 \leq B \leq M$ and $\langle \xi, B\xi \rangle = \langle \xi, M\xi \rangle - \langle \xi, C\xi \rangle > \phi(M - C) \geq 0$. Hence $B \neq 0$.

Suppose for some $0 \leq X \leq B$, $\phi(X) \geq \langle \xi, X\xi \rangle$. Then

$$\phi(C + X) = \phi(C) + \phi(X) \geq \langle \xi, C\xi \rangle + \langle \xi, X\xi \rangle = \langle \xi, (C + X)\xi \rangle \ ,$$

and $0 \leq C + X \leq C + B = M$. For $X \neq 0$, this contradicts the maximality of C. Hence for all $0 \leq X \leq B$, $\phi(X) \leq \langle \xi, X\xi \rangle$. Then (4.27) follows by homogeneity in X.

Step 3: For B satisfying (4.27), ϕB is normal. Consequently, for every non-zero positive $M \in \mathscr{M}$, there exists a non-zero regular $B \in \mathscr{M}$ such that $0 \leq B \leq M$.

To see this, note that for all $A \in \mathscr{M}$, by the Cauchy-Schwarz inequality, $|\phi B(A)|^2 = |\phi(AB)|^2 \leq \phi(1)\phi(B^*A^*AB) \leq \|\phi\|\phi(B^*A^*AB)$. Then since $B^*A^*AB \leq \|A\|^2 B^2 \leq \|A\|^2\|B\|B$, (4.27) implies that $\phi(B^*A^*AB) \leq \langle \xi, B^*A^*AB\xi \rangle = \|AB\xi\|^2$. Altogether, $|\phi B(A)| \leq \|\phi\|^{1/2}\|AB\xi\|$ and ϕB is continuous in the strong operator topology. But the strong and weak operator topologies have the same set of continuous linear functionals, and the σ-weak topology is stronger than the weak operator topology. Hence ϕB is normal.

Step 4: By Zorn's Lemma and the result of the first step, we may choose a maximal regular A such that $0 \leq A \leq 1$. Let $M = 1 - A$. If $M > 0$, by what we proved in the second and third steps, there is a non-zero regular B such that $0 \leq B \leq M$. Then $A + B$ is regular, contradicting the maximality of A. Hence $M = 1$, 1 is regular, and ϕ is normal. □

4.36 DEFINITION (Normal maps)**.** Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras. A bounded linear transformation $\Phi : \mathscr{M} \to \mathscr{M}$ is *normal* in case Φ is continuous with respect to the σ-weak topology on $\mathscr{M}$ and $\mathscr{N}$.

Let $\mathscr{L}(\mathscr{M}, \mathscr{N})$ denote the set of bounded linear transformations from $\mathscr{M}$ to $\mathscr{N}$. It is evident that $\Phi \in \mathscr{L}(\mathscr{M}, \mathscr{N})$ is normal if and only if $\varphi \circ \Phi \in \mathscr{M}_*$ for all $\varphi \in \mathscr{N}_*$, and then decomposing

φ into a linear combination of positive components, it suffices that $\varphi \circ \Phi \in \mathscr{M}_*$ for all $\varphi \in \mathscr{N}_*^+$. $\mathscr{L}_N(\mathscr{M},\mathscr{N})$ denote the subspace of $\mathscr{L}(\mathscr{M},\mathscr{N})$ consisting of normal linear transformations. Then $\mathscr{L}_N(\mathscr{M},\mathscr{N})$ is norm closed in $\mathscr{L}(\mathscr{M},\mathscr{N})$: If $\{\Phi_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathscr{L}_N(\mathscr{M},\mathscr{N})$ converging to $\Phi \in \mathscr{L}(\mathscr{M},\mathscr{N})$, for each $\varphi \in \mathscr{N}_*$, $\{\varphi \circ \Phi_n\}_{n\in\mathbb{N}}$ converges to $\varphi \circ \Phi$ in $\mathscr{N}_*$, a closed subspace of $\mathscr{M}^*$.

Let Φ be a linear transformation on $\mathscr{M}$ such that $\Phi(A) \in \mathscr{N}^+$ for all $A \in \mathscr{M}^+$. Then for all $\phi \in \mathscr{M}^*$, $\phi \circ \Phi$ is a positive linear functional on $\mathscr{M}$, and is therefore bounded. By the Uniform Boundedness Principle, there exists a finite constant c such that $\|\phi \circ \Phi\| \leq c$ for all $\phi \in \mathscr{N}^*$ with $\|\phi\| \leq 1$. But then $\|\Phi\| \leq c$. Hence positive linear transformations, like positive linear functionals, are always bounded.

4.37 LEMMA. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras, and let $\Phi : \mathscr{M} \to \mathscr{N}$ be a positive map. Then Φ is normal if and only if whenever $\mathcal{S}$ be a bounded upward directed set in $\mathscr{M}_{\text{s.a.}}$,*

$$\Phi\left(\bigvee_{A\in\mathcal{S}} A\right) = \bigvee_{A\in\mathcal{S}} \Phi(A)\ . \tag{4.28}$$

Proof. As remarked above, Φ is normal if and only if $\varphi \circ \Phi \in \mathscr{M}_*$ for all $\varphi \in \mathscr{N}_*^+$. Therefore, we must show that (4.28) is valid whenever $\mathcal{S}$ be a bounded upward directed set in $\mathscr{M}_{\text{s.a.}}$ if and only if $\varphi \circ \Phi \in \mathscr{M}_*$ for all $\varphi \in \mathscr{N}_*^+$.

Since Φ is positive, $\Phi(\mathcal{S}) := \{\Phi(A)\ :\ A \in \mathcal{S}\}$ is an upward directed bounded set in $\mathscr{N}_{\text{s.a.}}$. Let $B := \bigvee_{A\in\mathcal{S}} A$ and $C := \bigvee_{A\in\mathcal{S}} \Phi(A)$. Since $B \geq A$ for all $A \in \mathcal{S}$, and since Φ is positive, $\Phi(B) \geq C$, and (4.28) can be restated as $\Phi(B) = C$.

Likewise, since $C \geq \Phi(A)$, for all $A \in \mathcal{S}$, for any positive $\varphi \in \mathscr{N}^*$,

$$\varphi(C) = \varphi\left(\bigvee_{A\in\mathcal{S}} \Phi(A)\right) \geq \bigvee_{A\in\mathcal{S}} \varphi \circ \Phi(A)\ , \tag{4.29}$$

and there is equality when $\varphi \in \mathscr{N}_*$.

Suppose that (4.28) is valid whenever $\mathcal{S}$ is a bounded upward directed set in $\mathscr{M}_{\text{s.a.}}$. Then $C = \Phi(B)$, and since there is equality in (4.29) whenever $\varphi \in \mathscr{N}_*$, in this case,

$$\varphi \circ \Phi\left(\bigvee_{A\in\mathcal{S}} A\right) = \varphi(\Phi(B)) = \varphi(C) = \bigvee_{A\in\mathcal{S}} \varphi \circ \Phi(A)\ . \tag{4.30}$$

Then by the deeper part of Theorem 4.35, $\varphi \circ \Phi$ is normal, and thus Φ is normal.

Conversely, suppose that for some bounded upward directed set $\mathcal{S}$ in $\mathscr{M}_{\text{s.a.}}$, $\Phi(B) > C$. Since normal linear functionals separate, there exists $\varphi \in \mathscr{N}_*$ such that $\varphi(\Phi(B)) > \varphi(C)$. Then (4.30) fails for this φ, and $\varphi \circ \Phi$ is not normal. Hence Φ is not normal. $\square$

4.38 COROLLARY. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras. Let $\Phi : \mathscr{M} \to \mathscr{N}$ be normal and positive. Then for all $A \in \mathscr{M}$ and all $B \in \mathscr{N}$, define $\Psi : \mathscr{M} \to \mathscr{N}$ by*

$$\Psi(X) := B^*\Phi(A^*XA)B\ . \tag{4.31}$$

Then Φ is normal and positive.

Proof. The positivity is clear; we prove the normality. Let $\mathcal{S}$ be a bounded directed set in $\mathscr{M}$. Then evidently $\widetilde{\mathcal{S}} := \{A^*XA \ : \ A \in \mathcal{S}\}$ is a bounded upward directed set, and

$$\bigvee_{X\in\mathcal{S}} A^*XA = A^*\left(\bigvee_{X\in\mathcal{S}} X\right)A \ .$$

Then by Lemma 4.37,

$$\bigvee_{X\in\mathcal{S}} \Phi(A^*XA) = \Phi\left(\bigvee_{X\in\mathcal{S}} A^*XA\right) = \Phi\left(A^*\left(\bigvee_{X\in\mathcal{S}} X\right)A\right) \ .$$

Now another application of (4.31), this time with B in place of A, and $\Phi(A^*XA)$ in place of X, completes the proof. □

Among positive maps, the von Neumann algebra morphisms are particularly important: A linear transformation $\Phi : \mathscr{M} \to \mathscr{N}$, where $\mathscr{M}$ and $\mathscr{N}$ are von Neumann algebras, is morphism in case it is a $*$-homomorphism. Then for all $A \in \mathscr{M}^+$, $A = B^*B$, $B \in \mathscr{M}$, and then $\Phi(A) = \Phi(B)^*\Phi(B) \in \mathscr{N}^+$. That is, every morphism is positive.

4.39 LEMMA. *Every isomorphism Φ between von Neumann algebras is normal.*

Proof. Let $\mathcal{S}$ be any bounded directed set in $\mathscr{M}$. Then $\Phi(\mathcal{S}) = \{\Phi(A) \ : \ A \in \mathcal{S}\}$ is a bounded directed set in $\mathscr{N}$. Let $B := \bigvee_{A\in\mathcal{S}} A$ and $C := \bigvee_{A\in\mathcal{S}} \Phi(A)$. Then $B \geq C$, and since $\mathcal{S} = \Phi^{-1}(\Phi(\mathcal{S}))$, $C \geq B$. Thus $C = B$, and the claim follows from Lemma 4.37. □

4.40 THEOREM (Kernels and ranges of normal morphisms)**.** *$\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras, and let $\Phi : \mathscr{M} \to \mathscr{N}$ be a normal morphism. Then* $\ker(\Phi)$ *is a strongly closed ideal in $\mathscr{M}$, and $\Phi(\mathscr{M})$ is a von Neumann subalgebra of $\mathscr{N}$.*

Proof. Since Φ is normal, $\mathscr{J} := \ker(\Phi)$ is σ-weakly closed, and hence is norm closed. Hence $\mathscr{J}$ is a norm closed ideal in $\mathscr{M}$, and hence a C^* subalgebra of $\mathscr{M}$. We must show that it is strongly closed.

The unit ball in $\mathscr{M}$, $B_{\mathscr{M}}$ is σ-weakly closed, and even compact. Hence $B_{\mathscr{J}} = \mathscr{J} \cap B_{\mathscr{M}}$ is σ-weakly closed. Since the weak and σ-weak topologies coincide on bounded sets, $B_{\mathscr{J}}$ is weakly closed, and then since $B_{\mathscr{J}}$ is convex, it is strongly closed.

Let B belong to the strong closure of $\mathscr{J}$, and suppose that $\|B\| \leq 1$. By Kaplansky's Density Theorem, B belongs to the strong closure of $B_{\mathscr{J}}$, and since $B_{\mathscr{J}}$ is closed, to $B_{\mathscr{J}}$. The assumption that $\|B\| \leq 1$ is then eliminated by scaling.

By Theorem 2.54, Φ induces an isometric $*$-isomorphism of $\mathscr{M}/\mathscr{J}$ onto $\Phi(\mathscr{M})$. We must show that $\Phi(\mathscr{M})$ is strongly closed in $\mathscr{N}$. Let C belong to the strong closure of $\Phi(\mathscr{M})$, and suppose $\|C\| \leq 1$. Again by Kaplansky's Density Theorem, every weak neighborhood of C, being also a strong neighborhood of C, contains points in in $\Phi(\mathscr{M})$ of norm no greater than 1, and hence ontains points in $\Phi(B_{\mathscr{M}})$. Since the weak and the σ-weak topology coincide on bounded sets, the same is true for every σ-weak neighborhood of C, and hence C belongs to the σ-weak closure of $\Phi(B_{\mathscr{M}})$. The proof is completed by showing that $\Phi(B_{\mathscr{M}})$ is σ-weakly closed since sacaling eliminates the assumption that $\|C\| \leq 1$. Observe that since $B_{\mathscr{M}}$ is σ-weakly compact, and since continuous images of compact sets are compact, the image of $B_{\mathscr{M}}$ under Φ is σ-weakly compact, and hence σ-weakly closed. □

4.7 The GNS construction, Sherman's Theorem, and some applications

A construction due to Gelfand, Neumark and Segal, known as the GNS construction, associates to every state φ on any C^* algebra $\mathscr{A}$ a representation π of $\mathscr{A}$ on a Hilbert space built out of $\mathscr{A}$ itself and the state φ. The final part of the theorem, referring to von Neumann algebras, is due to Dixmier.

4.41 THEOREM (The GNS construction). *Let $\mathscr{A}$ be a unital C^* algebra with identity* 1, *and let φ be a state on $\mathscr{A}$. Then there exists a Hilbert space $\mathscr{H}$ and a cyclic representation π of $\mathscr{A}$ on $\mathscr{H}$ with a distinguished cyclic unit vector η such that for all $A \in \mathscr{A}$,*

$$\varphi(A) = \langle \eta, \pi(A)\eta \rangle_{\mathscr{H}} \ . \tag{4.32}$$

The representation π is irreducible if and only if φ is a pure state, and in any case, $\pi(\mathscr{A})$ is a C^* subalgebra of $\mathscr{B}(\mathscr{H})$. If $\mathscr{A}$ is a von Neuman algebra and φ is normal, π may be taken to be normal.

Proof. Let $\langle A, B\rangle_\varphi$ be the possibly degenerate inner product on $\mathscr{A}$ defined by $\langle A, B\rangle_\varphi = \varphi(A^*B)$. Define $\mathscr{N} := \{\ B \in \mathscr{A}\ :\ \varphi(B^*B) = 0\ \}$. By Lemma 4.7, $\mathscr{N}$ is a closed left ideal in $\mathscr{A}$.

Now consider the vector space $\mathscr{A}/\mathscr{N}$. With $\sim$ denoting equivalence mod $\mathscr{N}$, we have

$$A \sim A' \quad \text{and} \quad B \sim B' \quad \Rightarrow \quad \langle A, B\rangle_\varphi = \langle A', B'\rangle_\varphi \ ,$$

and hence we may define a *non-degenerate* inner product on $\mathscr{A}/\mathscr{N}$ by $\langle \{A\}, \{B\}\rangle = \langle A, B\rangle_\varphi$. Let $\mathscr{H}$ be the completion of $\mathscr{A}/\mathscr{N}$ in the corresponding Hilbertian norm, and let $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ denote the resulting inner product on $\mathscr{H}$.

For $A \in \mathscr{A}$, let $\pi(A)$ denote the linear operator on $\mathscr{A}/\mathscr{N}$ defined by $\pi(A)\{B\} = \{AB\}$ which is well-defined since $\mathscr{N}$ is a left ideal. By (**??**) once more,

$$\|\pi(A)\{B\}\|^2_{\mathscr{H}} = \varphi(B^*A^*AB) \leq \|A\|^2\varphi(B^*B) = \|A\|^2\|\{B\}\|^2_{\mathscr{H}} \ .$$

Since $\mathscr{A}/\mathscr{N}$ is dense in $\mathscr{H}$, $\pi(A)$ extends to a bounded operator on $\mathscr{H}$ with $\|\pi(A)\| \leq \|A\|$. It is evident that π is a homomorphism of $\mathscr{A}$ into $\mathscr{B}(\mathscr{H})$, and note that for all $X, Y \in \mathscr{A}$,

$$\langle \{X\}, \pi(A)\{Y\}\rangle_{\mathscr{H}} = \varphi(X^*AY) = \varphi((A^*X)^*Y) = \langle \pi(A)\{X\}, \{Y\}\rangle_{\mathscr{H}} \ ,$$

showing that $\pi(A^*) = \pi(A)^*$, and thus π is a $*$-homomorphism.

The representation π is cyclic since for all $A \in \mathscr{A}$, $\{A\} = \{A1\} = \pi(A)\{1\}$, showing that $\eta := \{1\}$ is a cyclic vector for π. Finally, note that $\langle \eta, \pi(A)\eta\rangle_{\mathscr{H}} = \varphi(1^*A1) = \varphi(A)$, and this proves (4.32). The statement concerning irreducibility follows from Theorem 4.17, and the fact that the image of $\pi(\mathscr{A})$ is norm closed in $\mathscr{B}(\mathscr{H})$ follows from Theorem 2.54.

Now suppose that $\mathscr{A}$ is a von Neumann algebra and that φ is normal. Let $\mathscr{H}$, π and η be constructed exactly as described in the paragraphs above. Let $\mathcal{S}$ be a bounded upward filtered set in $\mathscr{A}$, and let $B := \bigvee_{A\in\mathcal{S}} A$. Then $\pi(\mathcal{S}) = \{\pi(A)\, A \in \mathcal{S}$ is a bounded upward filtered set in $\mathscr{B}(\mathscr{H})$, and $\pi(B) \geq \pi(A)$ for all $A \in \mathcal{S}$. For all $A \in \mathcal{S}$, and all $X \in \mathscr{A}$,

$$\langle \{X\}, (\pi(B) - \pi(A))\{X\}\rangle_\varphi = \varphi(X^*(B - A)X) = X^*\varphi X(B - A) \ . \tag{4.33}$$

Since $X^*\varphi X$ is normal,

$$X^*\varphi X(B) = X^*\varphi X\left(\bigvee_{A\in\mathcal{S}} A\right) = \bigvee_{A\in\mathcal{S}} X^*\varphi(A)\ .$$

Therefore, for all $\epsilon > 0$, there exists $A \in \mathcal{S}$ such that $\langle \{X\}, (\pi(B) - \pi(A))\{X\}\rangle_\varphi < \epsilon$, and then since the set of $\{X\}$, $X \in \mathscr{A}$ is dense in $\mathscr{H}$,

$$\pi(B) = \pi\left(\bigvee_{A\in\mathcal{S}} A\right) = \bigvee_{A\in\mathcal{S}} \pi(A)\ .$$

Then π is normal by Lemma 4.37, and the fact that $\pi(\mathscr{A})$ is a von Neumann subalgebra of $\mathscr{B}(\mathscr{H})$ follows from Theorem 4.40. □

4.42 DEFINITION (Canonical cyclic representation)**.** Let $\mathscr{A}$ be a unital C^* algebra, and let φ be a state on $\mathscr{A}$. Then the cyclic representation π of $\mathscr{A}$ on the Hilbert space $\mathscr{H}$ obtained by completion of $\mathscr{A}/\mathscr{N}$ as in the proof of Theorem 4.41 is called the *canonical cyclic representation induced by* φ, and the cyclic vector $\eta := \{1\}$ is its *canonical cyclic vector*, and the map $A \mapsto \{A\}$ is the *canonical embedding of* $\mathscr{A}$ *into* $\mathscr{H}$.

4.43 COROLLARY. *Let $\mathscr{A}$ be a unital C^* algebra. For every non-zero $A \in \mathscr{A}$, there is a representation π of $\mathscr{A}$ such that $\|\pi(A)\| = \|A\|$.*

Proof. By Lemma 4.11, there exists $\varphi \in S_{\mathscr{A}}$ such that $|\varphi(A^*A)| = \|A\|^2$. Let π be the GNS representation of $\mathscr{A}$ associated to φ, and η the associated distinguished cyclic unit vector. Then

$$\|\pi(a)\eta\|^2_{\mathscr{H}} = \langle \eta\pi(A^*A)\eta\rangle_{\mathscr{H}} = \varphi(A^*A) = \|A\|^2\ ,$$

showing that $\|\pi(A)\| \geq \|A\|$, and since it is automatic that $\|\pi(A)\| \leq \|A\|$, $\|\pi(A)\| = \|A\|$. □

We now arrive at the Non-Commutative Gelfand-Neumark Theorem:

4.44 THEOREM (Non-Commutative Gelfand-Neumark Theorem)**.** *Every C^* algebra $\mathscr{A}$ with an identity is isometrically $*$-isomorphic to a C^* algebra of operators.*

Proof. For each $\varphi \in S_{\mathscr{A}}$, let π_φ be the representation of A on $\mathscr{B}(\mathscr{H}_\varphi)$ provided by the GNS construction. Define $\mathscr{H} = \bigoplus_{\varphi\in S_{\mathscr{A}}} \mathscr{H}_\varphi$, and define $\pi = \bigoplus_{\varphi\in S_{\mathscr{A}}} \pi_\varphi$. For each $A \in \mathscr{A}$, there exists $\varphi \in S_{\mathscr{A}}$ such that $\|\pi_\varphi(A)\| = \|A\|$, and hence $\pi(A)\| = \|A\|$. □

With $\pi = \bigoplus_{\varphi\in S_{\mathscr{A}}} \pi_\varphi$, by the von Neumann Double Commutant Theorem, $(\pi(\mathscr{A}))''$ is a von Neumann algebra in which $\pi(\mathscr{A})$ is strongly dense. This von Neumann algebra, known as the *enveloping von Neumann algebra*, turns out to be the bidual of $\mathscr{A}$ considered as a Banach space. This fact is extremely useful for studying linear maps Φ from one C^* algebra $\mathscr{A}$ to another, $\mathscr{B}$. The bidual $\Phi^{**} : \mathscr{A}^{**} \to \mathscr{B}^{**}$ has the same norm as Φ. Moreover, we can identify $\mathscr{A}$ with a subspace of $\mathscr{A}^{**}$ using the canonical embedding, which identifies $A \in \mathscr{A}$ with Λ_A, the evaluation functional $\Lambda_A(\psi) = \psi(A)$ on $\mathscr{A}^*$. Then for all $\psi \in \mathscr{B}^*$,

$$\Phi^{**}(\Lambda_A)(\psi) = \Lambda_A(\Phi^*\psi) = (\Phi^*\psi)(A) = \psi(\Phi(A)) = \Lambda_{\Phi(A)}(\psi)\ .$$

That is, identifying $\mathscr{A}$ with its image under the canonical embedding, $\Phi^{**}|_{\mathscr{A}} = \Phi$. Once we know that the bidual of a C^* algebra is isometrically isomorphic to to its enveloping von Neumann algebra, we have a norm preserving extension of $\Phi : \mathscr{A} \to \mathscr{B}$ to a map, that we still denote by Φ^{**}, from $(\pi(\mathscr{A}))''$ to $(\pi(\mathscr{B}))''$. In the investigation of Φ^{**}, and hence Φ, we may then use the rich structure that come along with von Neumann algebras; e.g., the measurable functional calculus and the consequent fact that von Neumann algebras are generated by the projections that they contain. Hence the following theorem is fundamentally important.

4.45 THEOREM (Sherman's Theorem)**.** *Let $\mathscr{A}$ be a C^* algebra, and let $\mathscr{H} = \bigoplus_{\varphi\in S_{\mathscr{A}}} \mathscr{H}_\varphi$, and $\pi = \bigoplus_{\varphi\in S_{\mathscr{A}}} \pi_\varphi$ be given as in Theorem 4.44. Let $\mathscr{M} := (\pi(A))''$. Let $\mathscr{A}^{**}$ be the bidual of $\mathscr{A}$ considered as a Banach space. Then A^{**} is isometrically isomorphic to $\mathscr{M}$.*

Proof. We have seen that every element of $\mathscr{A}^*$ is a linear combination of at most 4 states on $\mathscr{A}$. For $\varphi \in S_{\mathscr{A}}$, let ξ_φ be the unit vector in $\mathscr{H}_\varphi$ such that for all $A \in \mathscr{A}$, $\varphi(A) = \langle \xi_\varphi, \pi(A)\xi_\varphi\rangle$; we identify ξ_φ with a vector in $\mathscr{H}$ in the obvious manner, and define $\widehat{\varphi}$ to be the state on $\mathscr{M}$ given by $\widehat{\varphi}(T) = \langle \xi_\varphi, T\xi_\varphi\rangle$. Note that $\|\varphi\| = \sup_{A\in B_{\mathscr{A}}}\{|\varphi(A)|\} = \sup_{A\in B_{\mathscr{A}}}\{|\widehat{\varphi}(\pi(A))|\}$, and by Kaplansky's Density Theorem, $\pi(B_{\mathscr{A}})$ is dense in $B_{\mathscr{M}}$ in the strong, and hence weak, topology on $B_{\mathscr{M}}$. Since $\widehat{\varphi}$ is weakly continuous,

$$\|\varphi\| = \sup_{A\in B_{\mathscr{A}}} \{|\widehat{\varphi}(\pi(A))\} = \|\widehat{\varphi}\| \ .$$

Being weakly continuous, $\widehat{\varphi}$ is σ-weakly continuous, and hence may be identified with an element of $\mathscr{M}_*$. Therefore, $\varphi \mapsto \widehat{\varphi}$ is an isometric isomorphism from $\mathscr{A}^*$ into $\mathscr{M}_*$, and it is surjective since for any $\varphi \in \mathscr{M}_*$, $\varphi|_{\mathscr{A}} \in \mathscr{A}^*$, and the extension of $\varphi|_{\mathscr{A}} \in \mathscr{A}^*$ to $\mathscr{M}$ is φ itself, by the weak density of $\pi(\mathscr{A})$ in $\mathscr{M}$, and the weak continuity of $\varphi|_{\mathscr{A}}$. Then $\mathscr{A}^{**}$ is isometrically isomorphic to $(\mathscr{M}_*)^* = \mathscr{M}$. □

Going forward, it will be useful to identify $\mathscr{A}$ with $\pi(\mathscr{A})$, and to write $\mathscr{A}''$ to denote the universal enveloping von Neumann algebra of $\mathscr{A}$.

4.46 COROLLARY. *Let $\mathscr{A}$ and $\mathscr{B}$ be two C^* algebras, and let $\Phi : \mathscr{A} \to \mathscr{B}$ be a bounded linear transformation. Then there is a norm preserving extension of Φ, denoted by Φ^{**}, that maps $\mathscr{A}''$ to $\mathscr{B}''$.*

Proof. The proof has already been given in the remarks preceding the statement of Sherman's Theorem. □

We now come to an important application of this corollary, namely Tomiyama's Theorem. We first define some classes of maps between C^* algebras with which we shall be concerned in what follows.

4.47 DEFINITION. Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras. A linear transformation $\Phi : \mathscr{A} \to \mathscr{B}$ is *positive* in case $\Phi(A) \geq 0$ whenever $A \geq 0$. When $\mathscr{A}$ and $\mathscr{B}$ are unital then Φ is unital in case $\Phi(1_{\mathscr{A}}) = 1_{\mathscr{B}}$. Φ is Hermitian in case $\Phi(A)^* = \Phi(A^*)$ for all $A \in \mathscr{A}$. Finally, suppose that $\mathscr{B}$ is a C^* subalgebra of $\mathscr{A}$. Then Φ is a *projection* in case $\Phi(B) = B$ for all $B \in \mathscr{B}$.

Before turning to Tomiyama's Theorem, we first prove a Lemma of Kadison [19]. In the rest of this section, it is more the proof of Kadison's Lemma than the statement that will be of use to us. The proof is a variant of the "Ahren's Trick"; see [21, p. 24] for more discussion.

4.48 LEMMA (Kadison's Lemma)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be C^* algebras. Let $\Psi : \mathscr{A} \to \mathscr{B}$ be a unital linear transformation such that $\|\Psi(X)\| \leq \|X\|$ when X is normal. Then Ψ is Hermitian.*

Proof. Let $A \in \mathscr{A}$, $A = A^*$, $\|A\| = 1$. Write $\Psi(A) = B + iC$, $B, C \in \mathscr{B}_{\text{s.a}}$. If $C \neq 0$, there is some $\lambda \in \mathbb{R}\backslash\{0\}$ in $\sigma(C)$. Replacing A by $-A$ if needed, we may suppose that $\lambda > 0$. For all $t > 0$,

$$\|A + it1\| \leq (1+t^2)^{1/2} = t(1+t^{-2})^{1/2} \leq t + (2t)^{-1} \ ,$$

and hence $\|A + it1\| \leq t + \lambda$ for $t \in (0, (2\lambda)^{-1})$. Since $\lambda \in \sigma(C)$, $t + \lambda \leq \|C + t\| \leq \|B + iC + it\|$. But $\|B + iC + it\| = \|\Psi(A + it1)\| \leq \|A + it\|$ since $A + it1$ is normal. Altogether we have $\|A + it1\| < \|A + it1\|$, and this contradiction proves the claim. □

4.49 THEOREM (Tomiyama's Theorem)**.** *Let $\mathscr{A}$ be a unital C^* algebra, and let $\mathscr{B}$ be a unital C^* subalgebra of $\mathscr{A}$. Let Φ be a norm* 1 *projection from $\mathscr{A}$ to $\mathscr{B}$. Then Φ is unital and positive, and*

$$\Phi(B_1 A B_2) = B_1 \Phi(A) B_2 \quad \text{for all } B_1, B_2 \in \mathscr{B} \ , \ A \in \mathscr{A} \tag{4.34}$$

and for all $A \in \mathscr{A}$,

$$\Phi(A)^* \Phi(A) \leq \Phi(A^* A) \tag{4.35}$$

Proof of Tomiyama's Theorem. By Sherman's Theorem and its Corollary, Φ^{**} may be regarded as a norm one linear map from $\mathscr{A}''$ to $\mathscr{B}''$, and since $\Phi^{**}|_{\mathscr{A}} = \Phi$, its restriction to $\mathscr{B}$ is the identity. More is true: $\Phi^{**}(B) = B$ for all $B \in \mathscr{B}''$.

To see this, note that if $\Lambda \in \mathscr{B}^{**} \subset \mathscr{A}^{**}$, and $\psi \in B^*$, then $(\Phi^{**}\Lambda)(\psi) = \Lambda(\Phi^*(\psi)) = \Lambda(\psi \circ \Phi) = \Lambda(\psi)$. Hence Φ^{**} is the identity on B^{**}, and thus Φ^{**} is a norm one projection of $\mathscr{A}''$ onto $\mathscr{B}''$. Therefore, it suffices to prove the theorem when $\mathscr{A}$ and $\mathscr{B}$ are von Neumann algebras; we now assume this is the case.

First, since $1 \in \mathscr{B}$ and Φ is the identity on $\mathscr{B}$, Φ is unital. We next show that Φ is Hermitian. Arguing as in Kadison's Lemma, let $A \in \mathscr{A}_{\text{s.a.}}$ and write $\Phi(A) = B + iC$ where A and B are self adjoint, and suppose that some $\lambda > 0$ belongs to the spectrum of C. Then for $t > 0$, since $\Phi(1) = 1$,

$$t + \lambda \leq \|C + t1\| \leq \|B + iC + it1\| = \|\Phi(A + it1)\| \leq \|A + it1\| \leq (t^2 + \|A\|^2)^{1/2} \leq t + \frac{\|A\|^2}{2t} \ .$$

This is impossible for large t. Hence C has no positive spectrum. Repeating the argument with $-A$ in place of A, we conclude that C has no negative spectrum, and hence $C = 0$.

The next step is to show that Φ is positive; here Tomiyama cites the argument argument of Kadison [19]. Suppose that $A \in \mathscr{A}^+$, but that $\Phi(A)$ has spectrum in $(-\infty, 0)$. We may assume without loss of generality that $0 \leq A \leq 1$. Then $\|1 - A\| \leq 1$, but $\Phi(1 - A) = 1 - \Phi(A)$ is self adjoint and has spectrum in $(1, \infty)$, so that $\|\Phi(1 - A)\| > 1$. This contradiction proves that Φ is positive.

Now let $0 \leq A \leq 1$, $A \in \mathscr{A}$, and let P be a projection in $\mathscr{B}$, Then $P \geq PAP$ and hence $P \geq \Phi(PAP)$. It follows that $P\Phi(PAP)P = \Phi(PAP)$. Therefore, for all $X \in \mathscr{A}$,

$$P\Phi(PXP)P = \Phi(PXP) \ . \tag{4.36}$$

Now pick $X \in \mathscr{A}$, $\|X\| \leq 1$, and define $Y := PX(1 - P)$. Our immediate goal is to show that

$$P\Phi(Y)P = 0 \ . \tag{4.37}$$

First note that for all $t \in \mathbb{R}$,

$$\|Y + tP\| = \|(Y^* + tP)(Y + tP)\|^{1/2} = \|(1 - P)X^*PX(1 - P) + t^2P\|^{1/2} \leq (1 + t^2)^{1/2} \ .$$

Therefore, $\|\Phi Y + tP\| \leq (1 + t^2)^{1/2}$ for all $t \in \mathbb{R}$.

However,

$$\|\Phi Y + tP\| \geq \|P(\Phi Y)P + tP\| \geq \left\| \frac{P(\Phi Y)P + P(\Phi Y)^*P}{2} + tP \right\| \ .$$

Let λ belong to the spectrum of $\frac{1}{2}(P(\Phi Y)P + P(\Phi Y)^*P)$. If $\lambda > 0$, then for $t > 0$, $\|\Phi Y + tP\| \geq \lambda + t$. If $\lambda < 0$, and $t < 0$, $\|\Phi Y + tP\| \geq -\lambda - t$. Either way, there is a contradiction for suitable values of t. Hence $\frac{1}{2}(P(\Phi Y)P + P(\Phi Y)^*P) = 0$. The same reasoning shows that $\frac{1}{2i}(P(\Phi Y)P - P(\Phi Y)^*P) = 0$, and thus (4.37) is proved.

A similar argument applied to $\|Y + t(1 - P)\|$ proves that

$$(1 - P)(\Phi Y)(1 - P) = 0 \ . \tag{4.38}$$

At this point we have $\Phi Y = (1 - P)\Phi Y P + P\Phi Y(1 - P)$. Therefore, for $t > 0$,

$$\Phi Y + t(1 - P)\Phi Y P = P\Phi Y(1 - P) + (t + 1)(1 - P)\Phi Y P \ .$$

Since the norm of block matrix $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ is given by $\max\{\|A\|, \|B\|\}$, for all sufficiently large t,

$$\|\Phi Y + t(1 - P)\Phi Y P\| = (t + 1)\|(1 - P)\Phi Y P\| \ .$$

However, since Φ is a contractive projection,

$$\begin{aligned} \|\Phi Y + t(1 - P)\Phi Y P\| &\leq \|Y + t(1 - P)\Phi Y P\| \\ &= \|PX(1 - P) + t(1 - P)\Phi Y P\| \\ &= \max\{\|PX(1 - P)\|, t\|(1 - P)\Phi Y P\|\} \ . \end{aligned}$$

For large t, $(t + 1)\|(1 - P)\Phi Y P\| \leq t\|(1 - P)\Phi Y P\|$, and hence $\|(1 - P)\Phi Y P\| = 0$.

At this point we have $(1 - P)\Phi Y P = 0$, $(1 - P)(\Phi Y)(1 - P) = 0$ and $P(\Phi Y)P = 0$. It follows that

$$\Phi Y = P\Phi Y(1 - P) \ . \tag{4.39}$$

Now define $Z := (1 - P)XP$, which is just like the definition of Y, except with P and $1 - P$ interchanged. By Symmetry, we have

$$\Phi Z = (1 - P)\Phi Z P \ . \tag{4.40}$$

Now for any $X \in \mathscr{A}$, we have, using (4.36) once with P, and once with $(1 - P)$ in place of P,

$$\begin{aligned} \Phi X &= \Phi(PXP + PX(1 - P) + (1 - P)XP + (1 - P)X(1 - P)) \\ &= P\Phi(PXP)P + \Phi Y + \Phi Z + (1 - P)\Phi((1 - P)X(1 - P))(1 - P) \end{aligned}$$

Therefore, by (4.39) and (4.40) $(1-P)\Phi XP = (1-P)\Phi ZP = \Phi Z = \Phi((1-P)XP)$ and then, also using (4.36), $P\Phi XP = P\Phi(PXP)P = \Phi(PXP)$. Summing, $(\Phi X)P = \Phi(XP)$. Since a von Neumann algebra is generated by the projection it contains, for all $A \in \mathscr{B}$, $\Phi(X)A = \Phi(XA)$, and taking adjoints, since Φ preserves self-adjointness, $A\Phi X = \Phi(AX)$. This proves (4.34).

Next, since $(X-\Phi X)^*(X-\Phi X) \geq 0$ and since Φ preserves positivity, using (4.34) twice,

$$0 \leq \Phi(X^*X - X^*\Phi X - \Phi X^*X + (\Phi X)^*\Phi X) = \Phi(X^*X) - (\Phi X)^*\Phi X \ ,$$

and this proves (4.35) □

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