# POSITIVE MAPS ON OPERATOR ALGEBRAS
AND QUANTUM MARKOV SEMIGROUPS
An introductory course with application in quantum mechanics

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1 Fundamentals of the Theory of Banach Algebras

1.1 Basic definitions and notation

1.1 DEFINITION (Banach algebra). A Banach algebra is an algebra A over the complex numbers equipped with a norm ∥·∥ under which it is complete as a metric space, and such that
\[ \|AB\| \leq \|A\|\|B\| \quad \text{for all } A, B \in A . \] (1.1)

1.2 EXAMPLE. For a locally compact Hausdorff space X, we write C0(X) to denote the set of continuous complex valued functions on X that vanish at infinity. We equip it with the supremum norm and the usual algebraic structure of pointwise addition and multiplication. Then A = C0(X) is a commutative Banach algebra. There is a multiplicative identity if and only if X is compact.

1.3 EXAMPLE. Equip Rn with Lebesgue measure, and let A be the Banach space L1(Rn) further equipped with the convolution product
\[ f \ast g(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy . \]

Then A is a commutative Banach algebra that does not have an identity.

1.4 EXAMPLE. Let ℋ be a Hilbert space, and let A = B(ℋ), the set of all continuous linear transformations from ℋ to ℋ, equipped with the composition product and the operator norm
\[ \|A\| = \sup \{ \|A\psi\|_{\mathcal{H}} : \psi \in \mathcal{H}, \|\psi\|_{\mathcal{H}} = 1 \} \]
\[ = \sup \{ \Re(\langle \varphi, a\psi \rangle_{\mathcal{H}}) : \varphi, \psi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}}, \|\psi\|_{\mathcal{H}} = 1 \} , \] (1.2)

where \|·\|_{\mathcal{H}} is the norm on ℋ, and \langle ·, · \rangle_{\mathcal{H}} is the inner product in ℋ. This is a canonical example of a non-commutative Banach algebra.
1.5 **EXAMPLE.** Let \( \mathcal{A} \) be the algebra of \( n \times n \) matrices. The *Frobenius norm*, or *Hilbert-Schmidt norm*, on \( \mathcal{A} \) is the norm \( \| \cdot \|_2 \) given by \[ \| A \|_2 = \left( \sum_{i,j=1}^{n} |A_{i,j}|^2 \right)^{1/2} \] where \( a_{i,j} \) denotes the \( i, j \)th entry of \( a \). By the Cauchy-Schwarz inequality, for all \( A, B \in \mathcal{A} \),
\[
\| AB \|_2 = \left( \sum_{i,j=1}^{n} \left| \sum_{k=1}^{n} A_{i,k} B_{k,j} \right|^2 \right)^{1/2} \leq \left( \sum_{i,j=1}^{n} \left( \sum_{k=1}^{n} |A_{i,k}|^2 \right) \left( \sum_{k=1}^{n} |B_{k,j}|^2 \right) \right)^{1/2} = \| A \|_2 \| B \|_2 ,
\]
and thus (1.1) is satisfied. Note that the algebra of \( n \times n \) matrices with the operator norm is the special case of Example 1.4 in which \( \mathcal{H} = \mathbb{C}^n \), but the Frobenius norm is not the operator norm.

1.2 **The spectrum and the resolvent set**

A Banach algebra \( \mathcal{A} \) that has a multiplicative identity \( 1 \) is said to be *unital*. Otherwise, \( \mathcal{A} \) is *non-unital*.

1.6 **DEFINITION** (Canonical unital extension). Let \( \mathcal{A} \) be any Banach algebra, with or without a unit. Define \( \widetilde{\mathcal{A}} \) to be \( \mathbb{C} \oplus \mathcal{A} \) with the multiplication
\[
(\lambda, A)(\mu, B) = (\lambda \mu, \lambda B + \mu A + AB) ,
\]
and the norm
\[
\| (\lambda, A) \| = |\lambda| + \| A \| .
\]

By the definitions,
\[
\| (\lambda, A)(\mu, B) \| = \| (\lambda \mu, \lambda B + \mu A + AB) \| = |\lambda \mu| + \| \lambda B + \mu A + AB \|
\leq |\lambda| \| \mu \| + |\lambda| \| B \| + |\mu| \| A \| + \| A \| \| B \|
= (|\lambda| + \| A \|)(|\mu| + \| B \|) = \| (\lambda, A) \| \| (\mu, B) \| .
\]

This shows that (1.1) is satisfied, and hence that \( \widetilde{\mathcal{A}} \) is a Banach algebra. Now define \( 1 := (1, 0) \in \widetilde{\mathcal{A}} \). Then \( (1,0)(\lambda, A) = (\lambda, A)(1,0) = (\lambda, A) \) so that \( 1 \) is the identity in \( \widetilde{\mathcal{A}} \).

The map \( A \mapsto (0, A) \) is an isometric embedding of \( \mathcal{A} \) into \( \widetilde{\mathcal{A}} \). None of the elements \( (0, A) \) are invertible in \( \widetilde{\mathcal{A}} \), even when \( \mathcal{A} \) itself has an identity. Indeed, if \( (\lambda, A) \) has an inverse \( (\mu, b) \), then
\[
(1,0) = (\lambda, A)(\mu, B) = (\lambda \mu, \lambda B + \mu A + AB) ,
\]
and this is impossible if \( \lambda = 0 \). However, it will be important in what follows that if \( \mathcal{A} \) has a unit \( 1 \), then \( 1 - A \) is invertible in \( \mathcal{A} \) if and only if \( (1, -A) \) is invertible in \( \widetilde{\mathcal{A}} \).

1.7 **PROPOSITION.** Let \( \mathcal{A} \) be a Banach algebra with unit \( 1 \). Then \( 1 + A \) is invertible in \( \mathcal{A} \) if and only if there exists \( B \in \mathcal{A} \) such that
\[
A + B + AB = A + B + BA = 0 .
\]
Consequently, \( 1 + A \) is invertible in \( \mathcal{A} \) if and only if \( (1, A) \) is invertible in \( \widetilde{\mathcal{A}} \).
Proof. Suppose that there exists $B \in \mathcal{A}$ such that (1.5) is true. Then
\[(1 + B)(1 + A) = 1 + B + A + AB = 1 \quad \text{and} \quad (1 + A)(1 + B) = 1 + B + A + BA = 1.\]
Therefore, $(1 + A)$ is invertible, and $(1 + B) = (1 + A)^{-1}$.

Suppose that $1 + A$ is invertible. Define $B := (1 + A)^{-1} - 1$. Then $(1 + A)B = 1 - (1 + A) = -A$, and hence $A + B + AB = 0$. The proof of $A + B + BA = 0$ is similar. This proves the first part.

Next, $(1, A)$ is invertible in $\mathcal{A}$ if and only if there exists $B \in \mathcal{A}$ such that $(1, A)(1, B) = (1, B)(1, A) = (1, 0)$, and by the definition of the product in $\mathcal{A}$, this is the same as (1.5). □

1.8 DEFINITION (Spectrum and resolvent set). Let $\mathcal{A}$ be a Banach algebra, and let $A \in \mathcal{A}$. If $\mathcal{A}$ has a unit, the spectrum of $A$ in $\mathcal{A}$, $\sigma_{\mathcal{A}}(A)$ is defined to be the set of all $\lambda \in \mathbb{C}$ such that $\lambda 1 - A$ is not invertible. If $\mathcal{A}$ does not have a unit, then $\sigma_{\mathcal{A}}(A)$ is defined to be the spectrum of $(0, A) \in \mathcal{A}$. The resolvent set of $A$ in $\mathcal{A}$, $\rho_{\mathcal{A}}(A)$ is defined to be the complement of $\sigma_{\mathcal{A}}(A)$.

It is useful to have an intrinsic characterization of the spectrum for non-unital $\mathcal{A}$. By Proposition 1.7, for any given $A \in \mathcal{A}$, there is at most one $X \in \mathcal{A}$ such that $A + X + AX = A + X +XA = 0$, \hspace{1cm} (1.6)
and there is one solution exactly when $(1, A)$ is invertible in $\mathcal{A}$, and in that case $(1, X) = (1, A)^{-1}$ in $\mathcal{A}$. This justifies the following definition:

1.9 DEFINITION. Let $\mathcal{A}$ be a non-unital Banach algebra. Define a map $\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by
\[A \circ B = A + B + AB,\] \hspace{1cm} (1.7)
$A \in \mathcal{A}$ is quasi regular in case there exists $A' \in \mathcal{A}$, necessarily unique, such that $A \circ A' = A' \circ A = 0$. In this case, $A'$ is called the quasi inverse of $A$. We reserve the prime to denote the quasi inverse of a quasi regular element of a non-unital Banach algebra.

The definition has been made so that $A \circ B = B \circ A = 0$ if and only if $(1, A)(1, B) = (1, B)(1, A) = 1$.

1.10 LEMMA. Let $\mathcal{A}$ be a non-unital Banach algebra. Let $A, B \in \mathcal{A}$ be quasi regular. Then $A \circ B$ is quasi regular and
\[(A \circ B)' = B' \circ A',\] \hspace{1cm} (1.8)
and for $\lambda \in \mathbb{C} \setminus \{0\}$,
\[\lambda \in \rho_{\mathcal{A}}(A) \iff -\lambda^{-1}A \text{ is quasi regular.}\] \hspace{1cm} (1.9)
In particular,
\[\rho_{\mathcal{A}}(A) = \{ \lambda \in \mathbb{C} \setminus \{0\} : -\lambda^{-1}A \text{ is quasi regular} \} .\] \hspace{1cm} (1.10)
Proof. By Proposition 1.7, $A, B \in \mathcal{A}$ are quasi regular exactly when $(1, A), (1, B)$ are invertible in $\mathcal{A}$, and in this case
\[((1, A)(1, B))^{-1} = (1, B)^{-1}(1, A)^{-1} = (1, B')(1, A') = 1 + B' \circ A'.\]
Therefore, $A \circ B$ is quasi regular, and since $(1, A)(1, B) = (1, A \circ B), B' \circ A'$ is the pseudo inverse $A \circ B$. Next, for $\lambda \in \mathbb{C} \setminus \{0\}$, $(-\lambda, A)$ is invertible in $\mathcal{A}$ if and only if $(1, -\lambda^{-1}A)$ is invertible in $\mathcal{A}$, which is the case if and only if $-\lambda^{-1}A$ is quasi regular. □
Let \( \mathcal{A} \) be a Banach algebra with a identity 1. Then we can still carry out the process of adjoining an identity to form \( \widetilde{\mathcal{A}} \), and can regard each \( A \in \mathcal{A} \) also as an element of \( \widetilde{\mathcal{A}} \). Since no element of \( \mathcal{A} \) is invertible in \( \mathcal{A} \), \( 0 \in \sigma_{\mathcal{A}}(A) \) for all \( A \in \mathcal{A} \). However, for \( \lambda \neq 0 \), \( \lambda 1 - A \) is invertible if and only if \( 1 - \lambda^{-1} A \) is invertible. Likewise, \( (\lambda - A) \) is invertible if and only if \( (1, -\lambda^{-1} A) \) is invertible. Then by Proposition 1.7, \( \lambda 1 - A \) is invertible in \( \mathcal{A} \) if and only if \( (1, 0) - (0, \lambda^{-1} A) \) is invertible in \( \widetilde{\mathcal{A}} \). This shows that for \( \lambda \neq 0 \), \( \lambda \in \sigma_{\mathcal{A}}(A) \iff \lambda \in \sigma_{\mathcal{A}}((0, A)) \). We summarize:

\[
\{0\} \cup \sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{A}}((0, A)).
\] (1.11)

**1.11 Lemma** (Spectral Mapping Lemma). Let \( \mathcal{A} \) be a Banach algebra, and let \( p \) be a polynomial. In case \( \mathcal{A} \) has no identity, we suppose that \( p \) has no constant term. Then for all \( A \in \mathcal{A} \),

\[
p(\sigma_{\mathcal{A}}(A)) = \sigma_{\mathcal{A}}(p(A)) .
\]

Proof. We may suppose that \( p \) is not identically constant. We first suppose that \( \mathcal{A} \) has an identity. Fix \( \lambda \in \sigma_{\mathcal{A}}(A) \). We shall show that \( p(\lambda)1 - p(A) \) is not invertible. The polynomial \( p(0) - p(z) \) has a root at \( z = \lambda \), and hence \( p(\lambda) = p(z) = (\lambda - z)q(z) \) for some polynomial \( q(z) \). Replacing \( z \) by \( A \),

\[
p(\lambda)1 - p(A) = (\lambda - A)q(A) .
\]

Were \( p(\lambda)1 - p(A) \) invertible, we would have \( 1 = (\lambda - a)[q(1)(p(\lambda) - p(A))^{-1}] \), and then since polynomials in \( A \) commute, \( 1 = [q(1)(p(\lambda) - p(A))^{-1}]1(\lambda - A) \). This would mean that \( \lambda 1 - A \) is invertible, with contradicts our hypothesis that \( \lambda \in \sigma_{\mathcal{A}}(A) \). Hence \( p(\lambda) - p(A) \) is not invertible, and hence \( p(\lambda) \in \sigma_{\mathcal{A}}(p(A)) \). This shows that \( p(\sigma_{\mathcal{A}}(A)) \subset \sigma_{\mathcal{A}}(p(A)) \).

Next, fix \( \mu \in \sigma_{\mathcal{A}}(p(A)) \), and factor \( \mu - p(z) = \alpha(\lambda_1 - z) \cdots (\lambda_n - z) \) where \( \alpha \neq 0 \) and \( n \geq 1 \). For each \( j \), \( \mu = p(\lambda_j) \). We have

\[
\mu 1 - p(A) = \alpha(\lambda_1 1 - A) \cdots (\lambda_n 1 - A)
\]

and if each \( \lambda_j 1 - A \) were invertible, then \( \mu 1 - p(A) \) would be invertible, but this is not the case. Hence Hence for some \( j \), \( \lambda_j \in \sigma_{\mathcal{A}}(A) \), and \( \mu = p(\lambda_j) \in \sigma_{\mathcal{A}}(p(A)) \). This shows that \( \sigma_{\mathcal{A}}(p(A)) \subset p(\sigma_{\mathcal{A}}(A)) \), and completes the proof when \( \mathcal{A} \) has an identity. The general case now follows by adjoining an identity and then appealing to (1.11).

### 1.3 Properties of the inverse function

Now let \( \mathcal{A} \) be a Banach algebra with an identity 1. Let \( A \in \mathcal{A} \) be such that \( \|1 - A\| = r < 1 \). By (1.1), \( \| (1 - A)^n \| \leq r^n \) for all \( n \in \mathbb{N} \). For all \( n \in \mathbb{N} \), define \( S_n = \sum_{j=0}^{n} (1 - A)^j \) where, as usual, we interpret \( (1 - A)^0 = 1 \). For all \( n > m \), by the triangle inequality and (1.1),

\[
\| S_n - S_m \| \leq \sum_{j=m+1}^{n} \| (1 - A)^j \| \leq \sum_{j=m+1}^{n} r^j = \frac{r^m - r^n}{r - 1} .
\]

Hence \( \{ S_n \}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( \mathcal{A} \), and by the metric completeness of \( \mathcal{A} \), there exists \( B \in \mathcal{A} \) such that \( \lim_{n \to \infty} \| B - S_n \| = 0 \) and \( \| B \| \leq (1 - r)^{-1} \). Then

\[
BA = \lim_{n \to \infty} S_n A = \lim_{n \to \infty} S_n (1 - (1 - A)) = \lim_{n \to \infty} (1 - (1 - A)^{n+1}) = 1 .
\]
The same reasoning shows that \( AB = 1 \), and so \( A \) is invertible, and \( A^{-1} = B \), so that \( \|A^{-1}\| \leq (1-r)^{-1} \). Let \( \Omega \) denote the set of invertible elements in \( \mathcal{A} \). This brings us to:

1.12 Lemma. Let \( \mathcal{A} \) be a unital Banach algebra. Let \( \Omega \) be the set of invertible elements of \( \mathcal{A} \). Then \( \Omega \) contains every \( A \in \mathcal{A} \) such that \( \|1 - A\| < 1 \), and in this case \( A^{-1} \) is given by the convergent series

\[
A^{-1} = \sum_{j=0}^{\infty} (1 - A)^j \quad \text{and} \quad \|A^{-1}\| \leq \frac{1}{1 - \|1 - A\|}.
\]  

(1.12)

Moreover, if \( |\lambda| > \|A\| \), then \( \lambda 1 - A \) is invertible. In particular, \( \sigma_{\mathcal{A}}(A) \) is contained in the centered closed disk in \( \mathbb{C} \) of radius \( \|A\| \).

Proof. If \( |\lambda| > \|A\| \), then \( \lambda 1 - A = \lambda(1 - \lambda^{-1}A) \) and \( \|1 - (1 - \lambda^{-1}A)\| = |\lambda|^{-1}\|A\| < 1 \), so that \( (1 - \lambda^{-1}A) \) is invertible. The rest has been proved above.

1.13 Lemma. Let \( A_0 \in \Omega \) and let \( A \in \mathcal{A} \) satisfy \( \|A - A_0\| < \|A_0^{-1}\|^{-1} \). Then \( A \in \Omega \) and

\[
\|A^{-1}\| \leq \|A_0^{-1}\| \|A_0 - A\| \leq \|A_0^{-1}\| \frac{1}{1 - \|1 - AA_0^{-1}\|} \leq \frac{\|A_0\| \|A_0^{-1}\|}{\|A_0\| - \|A - A_0\|}.
\]  

(1.13)

In particular, \( \Omega \) is open.

Proof. Since \( \|1 - AA_0^{-1}\| = \|(A_0 - A)A_0^{-1}\| \leq \|A - A_0\| \|A_0^{-1}\| \), if \( \|A - A_0\| < \|A_0^{-1}\|^{-1} \), then \( \|1 - AA_0^{-1}\| < 1 \), and then by Lemma 1.12, \( AA_0^{-1} \in \Omega \). Since \( \Omega \) is closed under multiplication, \( A = (AA_0^{-1})A_0 \in \Omega \). Then \( A^{-1} = A_0^{-1}(AA_0)^{-1} \) and then by the bound in (1.12) and by (1.1) twice, (1.13) is valid. This shows that for all \( A_0 \in \Omega \), the open ball of radius \( \|A_0^{-1}\|^{-1} \) centered at \( A_0 \) is contained in \( \Omega \).

Since \( \Omega \) is open, for all \( A \in \mathcal{A} \), \( \rho_{\mathcal{A}}(A) \) is open, and hence that \( \sigma_{\mathcal{A}}(A) \) is closed. The following simple identity will prove useful in what follows.

1.14 Lemma (Resolvent identity). Let \( \mathcal{A} \) be a unital Banach algebra. For all \( A, B \in \Omega \),

\[
A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}.
\]  

(1.14)

Proof. \( A^{-1} - B^{-1} = A^{-1}(B - B^{-1}) - (A^{-1}A)B^{-1} = A^{-1}(B - A)B^{-1} \).

1.15 Corollary. For \( A, B \) invertible in a unital Banach algebra \( \mathcal{A} \),

\[
\|A - B\| \leq \min\{\frac{1}{2}\|B\|, \|B^{-1}\|^{-1}\} \quad \Rightarrow \quad \|A^{-1} - B^{-1}\| \leq 2\|B^{-1}\|^{2}\|A - B\|.
\]

In particular, the map \( A \mapsto A^{-1} \) is continuous on \( \Omega \).

Proof. By (1.14), \( \|A^{-1} - B^{-1}\| \leq \|A^{-1}\|\|B - A\|\|B^{-1}\| \), and by the hypothesis \( \|A - B\| \leq \min\{\frac{1}{2}\|B\|, \|B^{-1}\|^{-1}\} \), (1.13) is valid, and \( \|A^{-1}\| \leq 2\|B^{-1}\|^{2} \).

For a non-unital Banach algebra \( \mathcal{A} \), \( 0 \) is in the spectrum of every element \( A \) of \( \mathcal{A} \). For unital \( \mathcal{A} \), we have not yet shown that \( \sigma_{\mathcal{A}}(A) \) is non-empty for all \( A \in \mathcal{A} \). We now prove this fundamental fact:
1.16 THEOREM. Let \( \mathcal{A} \) be a Banach algebra. Then for all \( A \in \mathcal{A} \), \( \sigma_{\mathcal{A}}(A) \) is a nonempty closed set contained in the closed disc of radius \( ||A|| \) centered at 0 in \( \mathbb{C} \).

Proof. Suppose that \( (1 - \zeta A) \) is invertible for all \( \zeta \in \mathbb{C} \). Then \( A = 0 \): By the Hahn-Banach Theorem, there is a linear functional \( \varphi \in \mathcal{A}^* \) such that \( ||\varphi|| = 1 \) and \( \phi(A) = ||A|| \).

Define a complex valued function \( g \) on \( \mathbb{C} \) by \( g(\zeta) = \varphi((1 - \zeta A)^{-1}) \). For any \( \zeta \) and \( \zeta + \eta \), by (1.14), \( g(\zeta + \eta) - g(\zeta) = -\eta \varphi[(1 - (\zeta + \eta)A)(1 - (1 - \zeta A)^{-1})] \). From this identity and the continuity of the inverse function, it follows that

\[
\lim_{\eta \to 0} \frac{g(\zeta + \eta) - g(\zeta)}{\eta} = -\varphi[(1 - \zeta A)^{-2}]
\]

which shows that \( g \) is an entire analytic function, and \( g'(0) = \varphi(A) = ||A|| \).

For \( \zeta \neq 0 \), \( ||(1 - \zeta A)^{-1}\| = ||\zeta|^{-1}||((1 - A)^{-1})^{-1}|| \). For \( ||\zeta|^{-1} \leq \min\{\frac{1}{2}||A||, ||A^{-1}||^{-1}\} \), (1.13) yields \( ||\zeta^{-1} - A|| \leq 2||A^{-1}|| \). Hence \( \lim_{\zeta \to \infty} g(\zeta) = 0 \), and \( g \) is bounded.

By Liouville’s Theorem, \( g \) is constant. Hence \( ||A|| = g'(0) = 0 \). That is, if \( A \neq 0 \), there is some \( \zeta \), necessarily non-zero, such that \( (1 - \zeta A) \) is not invertible and then \( \zeta^{-1} \in \sigma_{\mathcal{A}}(A) \). Of course if \( A = 0 \), \( 0 \in \sigma_{\mathcal{A}}(A) \). Hence it is impossible that \( \sigma_{\mathcal{A}}(A) = \emptyset \). We have already seen that \( \sigma_{\mathcal{A}}(A) \) is closed and bounded, hence compact.

The fact that in a Banach algebra \( \mathcal{A} \), every element has a non-empty spectrum has a corollary that we shall make use of shortly:

1.17 COROLLARY (Gelfand-Mazur Theorem). Let \( \mathcal{A} \) be a unital Banach algebra. If \( \mathcal{A} \) is a division algebra, then \( \mathcal{A} \) is isomorphic to \( \mathbb{C} \). More specifically, each element \( A \) of \( \mathcal{A} \) satisfies \( A = \lambda 1 \) for some necessarily unique \( \lambda \in \mathbb{C} \), and \( A \mapsto \lambda \) is an isomorphism with \( \mathbb{C} \).

Proof. Suppose that \( \mathcal{A} \) is a division algebra. By Theorem 1.16, there exists \( \lambda \in \sigma_{\mathcal{A}}(A) \). Thus \( \lambda 1 - A \) is not invertible. Since the only non-invertible element in a division algebra is 0, \( A = \lambda 1 \).

1.18 DEFINITION (Spectral radius). The spectral radius of an element \( A \in \mathcal{A} \), \( \mathcal{A} \) a Banach algebra, is

\[
\nu(A) = \max\{||\lambda|| : \lambda \in \sigma_{\mathcal{A}}(A)\}.
\]

1.19 THEOREM (The Beurling-Gelfand Formula). The spectral radius of an element \( a \) of a Banach algebra \( \mathcal{A} \) is given by

\[
\nu(A) = \lim_{n \to \infty} \|A^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|A^n\|^{1/n}.
\]

In particular, the limit exists.

Proof. Suppose that \( \lambda \in \sigma_{\mathcal{A}}(A) \). By the Spectral Mapping Lemma, \( \lambda^n \in \sigma_{\mathcal{A}}(A^n) \), and then by Theorem 1.16, \( ||\lambda|| \leq \|A^n\| \). Taking the nth root, we obtain \( \nu(A) \leq \|A^n\|^{1/n} \) for all \( n \in \mathbb{N} \).

Therefore, the second equality in (1.16) will follow from the first, and it remains to show that

\[
\limsup_{n \to \infty} \|A^n\|^{1/n} \leq \nu(A).
\]

To this end, pick \( \lambda \) with \( ||\lambda|| > \nu(A) \) so that \( \lambda \in \rho_{\mathcal{A}}(A) \). Let \( \varphi \) be any continuous linear functional on \( \mathcal{A} \). Then, as in the proof of Theorem 1.16, the function \( g \) defined by \( g(\zeta) = \varphi((1 - \zeta A)^{-1}) \) is
Likewise, $(1 - \lambda A)$ is invertible; i.e., $\{0\} \cup \{\zeta : \zeta^{-1} \in \rho_{\mathcal{A}}(A)\}$. Hence $g$ is analytic on the open disc about $0$ with radius $1/\nu(A)$.

For $\zeta$ with $|\zeta| < \|A\|^{-1}$, $(1 - \zeta A)^{-1}$ has the convergent power series $(1 - \zeta A)^{-1} = \sum_{n=0}^{\infty} \zeta^n A^n$. Therefore, by the uniqueness of the power series representation, $g(\zeta) = \sum_{n=0}^{\infty} \zeta^n \varphi(A^n)$ is a convergent power series for all $\zeta$ with $|\zeta| \leq 1/\nu(A)$. It follows that for all such $\zeta$, $\lim_{n \to \infty} \zeta^n \varphi(A^n) = 0$. In particular, there exists a finite constant $C_\varphi$ such that

$$|\zeta^n \varphi(A^n)| \leq C_\varphi \quad \text{for all} \quad n \in \mathbb{N} .$$

(1.18)

For each $n \in \mathbb{N}$ define a linear functional $\Lambda_n$ on $\mathcal{A}^*$, the Banach space dual of $\mathcal{A}$, by $\Lambda_n(\varphi) = \zeta^n \varphi(A^n)$. Then (1.18) says that $\sup_{n \in \mathbb{N}} \{ |\Lambda_n(\varphi)| \} \leq C_\varphi$. The Uniform Boundedness Principle then implies that there exists a finite constant $M$ such that $\|\Lambda_n\| \leq M$ for all $n$, and hence for all $\varphi \in \mathcal{A}^*$ with $\|\varphi\| = 1$,

$$|\zeta^n \varphi(A^n)| \leq M \quad \text{for all} \quad n \in \mathbb{N} .$$

The Hahn-Banach Theorem provides $\varphi \in \mathcal{A}^*$ with $\|\varphi\| = 1$ such that $\varphi(A^n) = \|A^n\|$. Hence $|\zeta^n\|A^n\| \leq M$, and then, $|\zeta^n\|A^n\|^{-1/n} \leq M^{1/n}$. This proves that $|\zeta| \limsup_{n \to \infty} \|A^n\|^{-1/n} \leq 1$. However, $\zeta$ was any complex number with $|\zeta| < 1/\nu(A)$, this proves (1.17), and hence the first equality in (1.16).

The following result is trivial for commutative Banach algebras, and familiar for the algebra of $n \times n$ matrices:

**1.20 THEOREM** (Spectrum of $AB$ and $BA$). If $\mathcal{A}$ is a Banach algebra, then for all $A, B \in \mathcal{A}$,

$$\{0\} \cup \sigma_{\mathcal{A}}(AB) = \{0\} \cup \sigma_{\mathcal{A}}(BA) .$$

(1.19)

*Proof.* Passing to $\tilde{\mathcal{A}}$, we may suppose that $\mathcal{A}$ has an identity. For each $\lambda \neq 0$, we must show that $(\lambda I - AB)$ is invertible if and only if $(\lambda I - BA)$ is invertible. Dividing through by $\lambda$, we may take $\lambda = 1$. Therefore, suppose that $(1 - AB)$ is invertible, and let $Z = (1 - AB)^{-1}$. Then

$$(1 - BA)(1 + BZA) = 1 - BA + BZA - BABZA$$

$$= 1 - BA + B(1 - AB)ZA = 1 - BA + BA = 1 .$$

Likewise, $(1 + BZA)(1 - BA) = 1$, and so $(1 - BA)$ is invertible with inverse $(1 + BZA)$.

*1.21 THEOREM* (Spectral Contraction Theorem). Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and let $\pi : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then for all $A \in \mathcal{A}$,

$$\sigma_{\mathcal{B}}(\pi(A)) \subset \{0\} \cup \sigma_{\mathcal{A}}(A) .$$

(1.20)

*Proof.* Adjoin identities to $\mathcal{A}$ and $\mathcal{B}$, and define $\tilde{\pi} : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$ by $\tilde{\pi}(\pi(A)) = (1, \pi(A))$. This is a homomorphism, and takes the identity in $\tilde{\mathcal{A}}$ to the identity in $\tilde{\mathcal{B}}$. Since adjoining an identity had no effect on non-zero spectrum, we may assume that $\mathcal{A}$ and $\mathcal{B}$ have identities $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ respectively, and that $\pi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. 
Now suppose that $\lambda \in \rho_{\mathcal{A}}(A)$. Then $1_{\mathcal{A}} = (\lambda 1_{\mathcal{A}} - A)(\lambda 1_{\mathcal{A}} - A)^{-1}$. Since $\pi$ is a homomorphism,

\begin{equation}
1_{\mathcal{A}} = \pi(1_{\mathcal{A}}) = (\lambda 1_{\mathcal{A}} - \pi(A))\pi((\lambda 1_{\mathcal{A}} - A)^{-1}) .
\end{equation}

Thus $\pi((\lambda 1_{\mathcal{A}} - A)^{-1})$ is a right inverse of $\lambda 1_{\mathcal{A}} - \pi(A)$, and the same reasoning shows it is also a left inverse. Hence $\lambda \in \rho_{\mathcal{A}}(\pi(A))$. This shows that $\rho_{\mathcal{A}}(A) \subset \rho_{\mathcal{B}}(\pi(A))$, which is equivalent to the statement $\sigma_{\mathcal{A}}(\pi(A)) \subset \sigma_{\mathcal{A}}(A)$, and even shows that when $\mathcal{A}$ and $\mathcal{B}$ have identities and $\pi$ takes the identity in $\mathcal{A}$ to that in $\mathcal{B}$, it is not necessary to adjoin $\{0\}$ on the right side in (1.20) \hfill $\square$

The next theorem gives a useful continuity property of the spectrum.

1.22 THEOREM (Newburgh’s Theorem). Let $\mathcal{A}$ be a Banach algebra and $A \in \mathcal{A}$. Let $U$ be an open subset of $\mathbb{C}$ with $\sigma_{\mathcal{A}}(A) \subset U$. Then there exists $\delta > 0$ such that if $\|B - A\| \leq \delta$,

\[ \sigma_{\mathcal{A}}(B) \subset U. \]

Proof. Adjoining a unit if needed, we may assume that $\mathcal{A}$ is unital. First note that for all $B \in \mathcal{A}$ with $\|B - A\| < 1$, $\|B\| < \|A\| + 1$, and hence for all $\lambda \in \mathbb{C}$ with $\lambda \geq \|A\| + 1$, $\lambda \in \rho_{\mathcal{A}}(B)$. Hence when $\|B - A\| \leq 1$, $\sigma_{\mathcal{A}}(B)$ is contained in the closed centered disc of radius $\|A\| + 1$.

Let $K = U \cap \{ \lambda : |\lambda| \leq \|A\| + 1 \}$ which is a compact subset of $\rho_{\mathcal{A}}(A)$. It suffices to show that there is an $r > 0$ so that for all $\mu \in K$, $(\mu 1 - B)$ is invertible whenever $\|B - A\| < r$.

Let $\lambda \in K$. Then $\lambda \in \rho_{\mathcal{A}}(A)$, and for all $B \in \mathcal{A}$ and $\mu \in \mathbb{C}$ with

\[ |\mu - \lambda| + \|B - A\| < \|(\lambda 1 - A)^{-1}\|^{-1} \Rightarrow \|(\mu 1 - B) - (\lambda 1 - A)\| \leq \|(\lambda 1 - A)^{-1}\|^{-1} , \]

and hence $\mu \in \rho_{\mathcal{A}}(B)$. For each $\lambda \in K$, define $U_{\lambda} = \{ \mu : |\mu - \lambda| < \frac{1}{2} \|(\lambda 1 - A)^{-1}\|^{-1} \}$. Since $K$ is compact, there exists a finite sub-cover $\{ U_{\lambda_1}, \ldots, U_{\lambda_n} \}$. Define

\[ r = \min\left\{ \frac{\sqrt{2}}{2} \|(\lambda_1 1 - A)^{-1}\|^{-1}, \ldots, \frac{\sqrt{2}}{2} \|(\lambda_n 1 - A)^{-1}\|^{-1} \right\} . \]

Then for any $B$ with $\|B - A\| < r$ and any $\mu \in K$, $\mu \in U_{\lambda_j}$ for some $j = 1, \ldots, n$, and then

\[ \|(\mu 1 - B) - (\lambda_j 1 - A)\| \leq |\mu - \lambda_j| + \|B - A\| < \|(\lambda_j 1 - A)^{-1}\|^{-1} . \]

Therefore, $(\mu 1 - B)$ is invertible. Thus, for all $\mu \in K$, whenever $\|B - A\| < r$, $\mu \in \rho_{\mathcal{A}}(B)$. \hfill $\square$

1.4 Characters and the Gelfand Transform

1.23 DEFINITION (Characters). A character of a Banach algebra $\mathcal{A}$ is a non-zero algebraic homomorphism from $\mathcal{A}$ to $\mathbb{C}$. The set of characters of $\mathcal{A}$ is denoted $\Delta(\mathcal{A})$, and the set $\{0\} \cup \Delta(\mathcal{A})$ is denoted $\Delta'(\mathcal{A})$.

Though characters are defined with respect to the algebraic structure alone, they are necessarily continuous:

1.24 LEMMA. If $\mathcal{A}$ is a Banach algebra and $\varphi$ is a character of $\mathcal{A}$, then for all $A \in \mathcal{A}$ $\varphi(A) \in \sigma_{\mathcal{A}}(A)$, and $|\varphi(A)| \leq \|A\|$. Moreover, if $\mathcal{A}$ has an identity 1, then $\varphi(1) = 1$. \hfill $\square$
Proof. Suppose first that \( \mathcal{A} \) contains an identity 1. Since \( \varphi(1) = \varphi(1^2) = (\varphi(1))^2 \), \( \varphi(1) \) solves \( \zeta - \zeta^2 = 0 \), so either \( \varphi(1) = 0 \) or \( \varphi(1) = 1 \). But if \( \varphi(1) = 0 \), then for all \( A \in \mathcal{A} \), \( \varphi(A) = \varphi(1A) = \varphi(1)\varphi(A) = 0 \), and this is excluded by the definition. Hence \( \varphi(1) = 1 \).

Next, if \( \varphi(A) = 0 \), \( A \) is not invertible since if \( A \) has an inverse, \( 1 = \varphi(1) = \varphi(A)\varphi(A^{-1}) \) which is incompatible with \( \varphi(A) = 0 \). However, for any \( A \in \mathcal{A} \), \( \varphi(\varphi(A)1-A) = 0 \), and hence \( \varphi(A)1-A \) is not invertible. Hence \( \varphi(A) \in \sigma_{\mathcal{A}}(A) \), and therefore \( \|\varphi(A)\| \leq \|A\| \).

Now suppose that \( \mathcal{A} \) lacks a unit. Let \( \tilde{\mathcal{A}} \) be the algebra obtained by adjoining an identity, and let \( \tilde{\varphi} \) be the character on \( \tilde{\mathcal{A}} \) given by

\[
\tilde{\varphi}(\lambda, A) = \lambda + \varphi(A),
\]

which is easily seen to be a character. Since \( \sigma_{\mathcal{A}}(A) = \sigma_{\tilde{\mathcal{A}}}((0, A)) \) by definition, and \( \tilde{\varphi}((0, A)) = \varphi(A) \), it follows from the above that \( \varphi(A) \in \sigma(A) \), and then that \( \|\varphi(A)\| \leq \|(0, A)\| = \|A\| \).

Note that if \( \varphi \in \Delta(\mathcal{A}) \), then for all \( A, B \in \mathcal{A} \),

\[
\varphi(AB) = \varphi(A)\varphi(B) = \varphi(B)\varphi(A) = \varphi(BA).
\]

Consequently, any character \( \varphi \) must satisfy \( \varphi(AB - BA) = 0 \) for all \( A, B \). When the algebra \( \mathcal{A} \) is not commutative, this can be a stringent constraint, and there may not exist any characters at all.

1.25 EXAMPLE. Let \( \mathcal{A} \) be the algebra of \( 2 \times 2 \) matrices. The Pauli matrices are

\[
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{and} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

Then with \([A, B]\) denoting the commutator \( AB - BA \),

\[
[\sigma_1, \sigma_2] = i2\sigma_3, \quad [\sigma_2, \sigma_3] = i2\sigma_1 \quad \text{and} \quad [\sigma_3, \sigma_1] = i2\sigma_2.
\]

It follows that for any homomorphism \( \varphi \) of \( \mathcal{A} \) into \( \mathbb{C} \), \( \varphi(\sigma_j) = 0 \) for \( j = 1, 2, 3 \). Next, the identity matrix \( I \) satisfies \( I = \sigma_1^2 \), and so \( \varphi(I) = (\varphi(\sigma_1))^2 = 0 \). Thus, for all \((z_0, z_1, z_2, z_3) \in \mathbb{C}^4\),

\[
\varphi(z_0I + z_1\sigma_1 + z_2\sigma_2 + z_3\sigma_3) = 0.
\]

Since every evidently \( \{I, \sigma_1, \sigma_2, \sigma_3\} \) is linearly independent and \( \mathcal{A} \) is 4 dimensional, \( \mathcal{A} \) is the span of \( \{I, \sigma_1, \sigma_2, \sigma_3\} \), and hence \( \varphi \) vanishes identically on \( \mathcal{A} \). Thus, if \( \mathcal{A} \) is the algebra of \( 2 \times 2 \) matrices, \( \Delta(\mathcal{A}) = \emptyset \) and \( \Delta'(\mathcal{A}) \) is the one-point space \( \{0\} \).

In the rest of this section, commutativity will not play any role in the proofs, and so we shall state the results without reference to commutativity. However, one should keep in mind that without commutativity, \( \Delta(\mathcal{A}) \) may be empty and \( \Delta'(\mathcal{A}) \) may be a one-point space.

1.26 DEFINITION (Gelfand topology). For a Banach algebra \( \mathcal{A} \), the Gelfand topology on \( \Delta'(\mathcal{A}) \) is the relative weak-star topology on \( \Delta'(\mathcal{A}) \) considered as a subset of \( \mathcal{A}^* \), the Banach space dual to \( \mathcal{A} \). That is, the Gelfand topology is the weakest topology on \( \Delta'(\mathcal{A}) \) that makes the functions \( \varphi \mapsto \varphi(A) \) continuous for all \( A \in \mathcal{A} \).
1.27 **Lemma.** Let \( \mathcal{A} \) be a Banach algebra. Then \( \Delta'(\mathcal{A}) \), equipped with the Gelfand topology is a compact Hausdorff space. If \( \mathcal{A} \) does not have an identity, then with the Gelfand topology, \( \Delta(\mathcal{A}) \) is a locally compact Hausdorff space, and \( \Delta'(\mathcal{A}) \) is its one-point compactification. If \( \mathcal{A} \) has an identity, \( \Delta(\mathcal{A}) \) itself is compact and 0 is an isolated point in \( \Delta'(\mathcal{A}) \).

*Proof.* Equip \( \mathcal{A}^* \) with the weak-* topology; i.e., the weakest topology making all of functions \( \varphi \mapsto \varphi(A) \) continuous for all \( A \in \mathcal{A} \). The Banach-Alaoglu Theorem asserts that the unit ball in \( \mathcal{A}^* \) is compact in the weak-* topology. For each \( A,B \in \mathcal{A} \), define a function \( f_{A,B} \) on \( \mathcal{A}^* \) by

\[
f_{A,B}(\varphi) = \varphi(AB) - \varphi(A)\varphi(B).
\]

This is evidently continuous for the weak-* topology. Now note that

\[
\Delta'(\mathcal{A}) = \bigcap_{A,B \in \mathcal{A}} \{ \varphi \in \mathcal{A}^* : f_{A,B}(\varphi) = 0 \}.
\]

This displays \( \Delta'(\mathcal{A}) \) as an intersection of closed sets. Hence \( \Delta'(\mathcal{A}) \) is a closed subset of the unit ball in \( \mathcal{A}^* \), and hence is compact.

For \( \varphi_1, \varphi_2 \in \Delta'(\mathcal{A}) \) with \( \varphi_1 \neq \varphi_2 \), there exists \( A \in \mathcal{A} \) such that \( \varphi_1(A) \neq \varphi_2(A) \). Let \( U_1 \) and \( U_2 \) be disjoint open sets in \( \mathcal{C} \) that contain \( \varphi_1(A) \) and \( \varphi_2(A) \) respectively. Then

\[
\{ \psi \in \Delta'(\mathcal{A}) : \psi(A) \in U_1 \} \quad \text{and} \quad \{ \psi \in \Delta'(\mathcal{A}) : \psi(A) \in U_2 \}
\]

are disjoint open sets in \( \Delta'(\mathcal{A}) \) that contain \( \varphi_1 \) and \( \varphi_2 \) respectively. In particular, for each \( \varphi \in \Delta(\mathcal{A}) \), there exist disjoint open neighborhoods \( V_1 \) of \( \varphi \) and \( V_2 \) of 0, and then since \( V_1 \subset V_2' \), \( V_2' \) is a compact neighborhood of \( \varphi \). Thus, \( \Delta(\mathcal{A}) \) is locally compact and Hausdorff. If \( \mathcal{A} \) has an identity 1, \( \varphi(1) = 1 \) for all \( \varphi \in \Delta(\mathcal{A}) \), while \( 0(1) = 0 \). Consequently, the zero homomorphism is an isolated point of \( \Delta'(\mathcal{A}) \) in this case. \( \square \)

1.28 **Definition** (Gelfand transform). Let \( \mathcal{A} \) be a Banach algebra. The **Gelfand transform** is the map \( \gamma \) from \( \mathcal{A} \) to \( \mathcal{C}(\Delta'(\mathcal{A})) \) given by

\[
(\gamma(A))[\varphi] = \varphi(A) .
\]

That is, \( \gamma(A) \) is the function of evaluation at \( A \), and it is continuous by the definition of the Gelfand topology.

1.29 **Theorem.** Let \( \mathcal{A} \) be a Banach algebra. The Gelfand transform is a norm reducing homomorphism from \( \mathcal{A} \) to \( \mathcal{C}(\Delta'(\mathcal{A})) \). That is, the Gelfand transform is a homomorphism of algebras, and for all \( A \in \mathcal{A} \),

\[
\|\gamma(A)\|_{\mathcal{C}(\Delta'(\mathcal{A}))} \leq \|A\| .
\]

*Proof.* The homomorphism property is evident since for all \( A,B \in \mathcal{A} \) and all \( \varphi \in \Delta'(\mathcal{A}) \),

\[
(\gamma(AB))[\varphi] = \varphi(AB) = \varphi(A)\varphi(B) = (\gamma(A))[\varphi](\gamma(B))[\varphi] .
\]

By Lemma 1.24, \( (\gamma(A))[\varphi] | \varphi(A)| \leq \|A\| \), proving \( \|\gamma(A)\|_{\mathcal{C}(\Delta'(\mathcal{A}))} \leq \|A\| \). \( \square \)

This result, as it stands, does not take us far at all. Even in the commutative case, there may so few characters that the transform may be a trivial homomorphism into a trivial algebra.
1.30 Example. Let $A_0$ be the $n \times n$ matrix, $n > 1$, with $A_{i,j} = \begin{cases} 1 & j = i + 1 \\ 0 & j \neq i + 1 \end{cases}$, and note that $A_0^n = 0$. Let $\mathcal{A}$ denote the subalgebra of the $n \times n$ matrices that are polynomials in $A_0$. That is, every $A \in \mathcal{A}$ has the form

$$A = \sum_{j=0}^{n-1} p_j A_0^j.$$  \hfill (1.22)

This is a commutative algebra with an identity. Let $\varphi \in \Delta(\mathcal{A})$. Then $0 = \varphi(A_0^n) = (\varphi(A_0))^n$ so that $\varphi(A_0) = 0$. Then for $A$ given by (1.22), $\varphi(A) = p_0 \varphi(I) = p_0$. Thus, the only candidate for a character on $\mathcal{A}$ is the map $\varphi_0$ given by $\varphi_0 \left( \sum_{j=0}^{n-1} p_j A_0^j \right) = p_0$. It is readily checked that this is indeed a homomorphism and it is non-zero. Hence $\Delta(\mathcal{A}) = \{ \varphi_0 \}$ and $\Delta'(\mathcal{A}) = \{ \varphi_0 \} \cup \{ 0 \}$. Since $\Delta'(\mathcal{A})$ consists of two isolated points, we may identify $\mathcal{C}(\Delta'(\mathcal{A}))$ with $\mathbb{C}^2$ in the usual way, and then we may write the Gelfand transform as

$$\gamma \left( \sum_{j=0}^{n-1} p_j A_0^j \right) = (p_0, 0),$$

which is indeed a norm reducing homomorphism, but not very interesting.

Before leaving this example, we note that for elements of $\mathcal{A}$, the spectrum is as trivial as Theorem 1.16 allows: For all $A \in \mathcal{A}$, $\sigma(A) = \{ \varphi_0(A) \}$. This is true since when $A$ is given by (1.22), then $A$ is invertible if and only if $p_0 \neq 0$.

1.5 Characters and spectrum in commutative Banach algebras

We have seen in Example 1.30 that in a commutative Banach Algebra $\mathcal{A}$ with an identity, $\Delta'(\mathcal{A})$ may consist of only two isolated points. However, this happens only when for each $A \in \mathcal{A}$, $\sigma_{\mathcal{A}}(A)$ consists of a single point: $\sigma_{\mathcal{A}}(A) = \{ \varphi_0(A) \}$. This is true since when $A$ is given by (1.22), then $A$ is invertible if and only if $p_0 \neq 0$.

1.31 Definition. Let $\mathcal{A}$ be a Banach algebra. An ideal of $\mathcal{A}$ is a subspace of $\mathcal{A}$ such that for all $B \in \mathcal{J}$ and $A \in \mathcal{A}$, $BA \in \mathcal{J}$ and $AB \in \mathcal{J}$. An ideal of $\mathcal{A}$ is proper in case it is not equal to $\mathcal{A}$ itself. An ideal of $\mathcal{A}$ is a closed in case it is topologically closed as a subset of $\mathcal{A}$. If $\mathcal{J}$ is an ideal in $\mathcal{A}$, and $\overline{\mathcal{J}}$ is the norm closure of $\mathcal{J}$, then $\overline{\mathcal{J}}$ is also an ideal in $\mathcal{A}$. If $\mathcal{J}$ is an ideal, an element $U$ of $\mathcal{A}$ is called a unit mod $\mathcal{J}$ in case

$$AU - A \in \mathcal{J} \quad \text{and} \quad UA - A \in \mathcal{J} \quad \text{for all} \quad A \in \mathcal{A}. \hfill (1.23)$$
An ideal \( \mathcal{J} \) is called a modular ideal in case there exists a unit mod \( \mathcal{J} \).

Given a Banach algebra \( \mathcal{A} \) and an ideal \( \mathcal{J} \), there is a natural equivalence relation \( \sim \) on \( \mathcal{A} \) given by

\[
A \sim B \iff A - B \in \mathcal{J}.
\]

Let \( \{A\} \) and \( \{B\} \) denote the equivalence classes of \( A \) and \( B \) respectively. Let \( \tilde{A} \) and \( \tilde{B} \) be any other representative of \( \{A\} \) and \( \{B\} \) respectively. Then for some \( X, Y \in \mathcal{J} \), \( \tilde{A} = A + X \) and \( \tilde{B} = B + Y \). Then

\[
\tilde{A}\tilde{B} = (A + X)(B + Y) = AB + (AY + XB + XY) = AB.
\]

Even more simply one sees that \( \tilde{A} + \tilde{B} \sim A + B \) and for all \( \lambda \in \mathbb{C} \), \( \lambda\tilde{A} \sim \lambda A \). Hence \( \mathcal{A}/\mathcal{J} \), the set of equivalence classes in \( \mathcal{A} \), equipped with the operations

\[
\{A\}\{B\} = \{AB\} \quad \text{and} \quad \{A\} + \{B\} = \{A + B\} \quad \text{and} \quad \lambda\{A\} = \{\lambda A\}
\]

is an algebra, and \( A \mapsto \{A\} \) is a homomorphism of \( \mathcal{A} \) onto \( \mathcal{A}/\mathcal{J} \).

Now introduce a norm on \( \mathcal{A}/\mathcal{J} \) by

\[
\|\{A\}\| = \inf\{ \|\tilde{A}\| : \tilde{A} \sim A \} = \inf\{ \|A - B\| : B \in \mathcal{J} \}.
\]

Note that \( \|\{A\}\| \leq \|A\| \). To see that

\[
\|\{A\}\{B\}\| \leq \|\{A\}\|\|\{B\}\| \quad (1.24)
\]

for all \( \{A\}, \{B\} \in \mathcal{A}/\mathcal{J} \), let \( 0 < \epsilon < \min\{\|\{A\}\|, \|\{B\}\|\} \), and pick \( \tilde{A} \in \{A\} \) and \( \tilde{B} \in \{B\} \) so that \( \{A\} > \|\tilde{A}\| - \epsilon \) and \( \{B\} > \|\tilde{B}\| - \epsilon \). Then

\[
\|\{A\}\{B\}\| = \|\tilde{A}\tilde{B}\| \leq \|\tilde{A}\|\|\tilde{B}\| \leq \|\tilde{A}\|\|\tilde{B}\| \leq (\|\{A\}\| + \epsilon)(\|\{B\}\| + \epsilon)\).
\]

Since \( \epsilon \) can be taken arbitrarily small, (1.24) is proved.

Therefore, \( \mathcal{A}/\mathcal{J} \) will be a Banach algebra with this norm provided it is complete in this norm. Consider a Cauchy sequence \( \{\{A\}_n\}_{n \in \mathbb{N}} \) in \( \mathcal{A}/\mathcal{J} \). A standard argument shows that this sequence always has a limit if \( \mathcal{J} \) is closed. Thus, when \( \mathcal{J} \) is a closed ideal, \( \mathcal{A}/\mathcal{J} \) is a Banach algebra, and the map \( A \mapsto \{A\} \) is a contractive homomorphism of \( \mathcal{A} \) onto \( \mathcal{A}/\mathcal{J} \). This homomorphism is called the natural homomorphism of \( \mathcal{A} \) onto \( \mathcal{A}/\mathcal{J} \).

It is possible for \( \mathcal{A}/\mathcal{J} \) to have an identity even when \( \mathcal{A} \) does not. Suppose that \( \mathcal{J} \) is modular, and that \( U \) is a unit mod \( \mathcal{J} \). Then for all \( A \in \mathcal{A} \), \( \{U\}\{A\} = \{UA\} = \{A\} \) and \( \{A\}\{U\} = \{AU\} = \{A\} \). Thus, \( \{U\} \) is a multiplicative identity in \( \mathcal{A}/\mathcal{J} \). Clearly if \( \mathcal{A} \) has an identity \( 1 \), \( 1 \) is a unit mod \( \mathcal{J} \).

There is a close connections between closed ideals and kernels of continuous homomorphisms of Banach algebras.

**1.32 PROPOSITION.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be Banach algebras, and let \( \pi : \mathcal{A} \to \mathcal{B} \) be a continuous homomorphism. Then \( \mathcal{J} = \ker(\pi) \) is a closed ideal in \( \mathcal{A} \). Conversely, if \( \mathcal{J} \) is a closed ideal in \( \mathcal{A} \), then the map \( A \mapsto \{A\} / \mathcal{J} \), sending \( A \) to its equivalence class mod \( \mathcal{J} \), is a homomorphism of \( \mathcal{A} \) onto \( \mathcal{A}/\mathcal{J} \).
Proof. Suppose that \( \pi : \mathcal{A} \to \mathcal{B} \) be a continuous homomorphism. Then evidently \( \mathcal{I} = \ker(\mathcal{A}) \) is a closed by the continuity of \( \pi \), and it is a subspace by the linearity of \( \pi \). Next, for all \( X \in \mathcal{I} \) and \( A, B \in \mathcal{A} \), \( \pi(AXB) = \pi(A)\pi(X)\pi(B) = \pi(A)0\pi(B) = 0 \). Hence \( AXB \in \ker(\pi) \), and so \( \mathcal{I} \) is an ideal. The converse is clear from the construction of \( \mathcal{A} / \mathcal{I} \) described above.

Now consider a commutative Banach algebra \( \mathcal{A} \). For any \( X_0 \in \mathcal{A} \), we can define \( \mathcal{I}(X_0) \) to be the subset of \( \mathcal{A} \) given by

\[
\mathcal{I}(X_0) = \{ YX_0 : Y \in \mathcal{A} \}.
\]

Then for all \( YX_0 \in \mathcal{I}(X_0) \) and all \( A, B \in \mathcal{A} \), \( AYX_0B = (AYB)X_0 \in \mathcal{I}(x_0) \), and evidently \( \mathcal{A} \) is a subspace of \( \mathcal{A} \). Hence \( \mathcal{I}(X_0) \) is an ideal, and it is called the ideal generated by \( X_0 \).

Let \( X_0 \) be a non-invertible element of \( \mathcal{A} \). Let \( \mathcal{I}(X_0) \) be the ideal generated by \( X_0 \). Then no element of \( \mathcal{I}(X_0) \) is invertible. Indeed, if \( YX_0 \) were invertible, there would exist \( Z \in \mathcal{A} \) such that \( (YX_0)Z = (YZ)X_0 = 1 \), and then (since \( \mathcal{A} \) is commutative), \( YZ \) would be an inverse of \( X_0 \), which is not possible. Hence, for all non-invertible \( X_0 \), \( \mathcal{I}(X_0) \) consists entirely of non-invertible elements. Since the open unit ball about the identity consists of invertible elements, \( \mathcal{I}(X_0) \) does not intersect the open unit ball about the identity \( 1 \). In particular, \( 1 \) does not belong to \( \overline{\mathcal{I}(X_0)} \), the closure of \( \mathcal{I}(X_0) \).

Now we come to a crucial construction of characters in a commutative Banach algebra:

1.33 THEOREM. Let \( \mathcal{A} \) be a commutative Banach algebra with identity \( 1 \). Then for all non-invertible \( X_0 \in \mathcal{A} \), there exists a character \( \varphi \in \Delta(\mathcal{A}) \) such that \( \varphi(X_0) = 0 \).

Proof. Since \( X_0 \) is not invertible, \( \mathcal{I}(X_0) \) is a proper ideal in \( \mathcal{A} \), and in fact, as explained above, the open unit ball about \( 1 \) does not intersect \( \mathcal{I}(X_0) \). Now consider any chain of proper ideals in \( \mathcal{A} \), ordered by inclusion. Since no proper ideal contains the identity, the union of this chain is again a proper ideal. Hence by Zorn's Lemma, there exists a maximal proper ideal \( \mathcal{M} \) containing \( \mathcal{I}(X_0) \). Since no proper ideal can contain any invertible elements, this ideal does not intersect the open unit ball about \( 1 \). Hence its closure \( \overline{\mathcal{M}} \) also contains \( \mathcal{I}(X_0) \) and is proper. Since \( \mathcal{M} \) is maximal among such ideals, \( \mathcal{M} = \overline{\mathcal{M}} \). Hence in a commutative Banach algebra \( \mathcal{A} \) with identity \( 1 \), for each non-invertible \( X_0 \in \mathcal{A} \), there exists a closed proper ideal \( \mathcal{M} \) that contains any ideal in \( \mathcal{A} \) that contains \( \mathcal{I}(X_0) \).

We now claim that the Banach algebra \( \mathcal{B} = \mathcal{A} / \mathcal{M} \) is a division algebra. Suppose not. Then it contains a non-zero, non-invertible element \( \{Y_0\} \mathcal{M} \). Let \( \mathcal{N} \) be the closure of the ideal in \( \mathcal{B} \) generated by \( \{Y_0\} \mathcal{M} \). Let \( \pi_1 \) be the natural homomorphism of \( \mathcal{A} \) onto \( \mathcal{B} \), and let \( \pi_2 \) be the natural homomorphism of \( \mathcal{B} \) onto \( \mathcal{B} / \mathcal{N} \). Then \( \pi_2 \circ \pi_1 \) is a homomorphism of \( \mathcal{A} \) onto \( \mathcal{B} / \mathcal{N} \). By Proposition 1.32, \( \ker(\pi_2 \circ \pi_1) \) is a closed ideal that contains \( \mathcal{M} = \ker(\pi_1) \). The containment is proper since \( \pi_2 \circ \pi_1(0) = 0 \), but \( Y_0 \notin \mathcal{M} \) since \( \{Y_0\} \mathcal{M} \neq 0 \). Finally, \( 1 \notin \ker(\pi_1 \circ \pi_2) \) since \( \{1\} \mathcal{M} \) is a unit \( \mathcal{B} = \mathcal{A} / \mathcal{M} \), and \( \mathcal{N} \) does not contain any invertible elements, so \( \pi_2(\{1\} \mathcal{M}) = \pi_2(\pi_1(1)) \neq 0 \). Thus, \( \ker(\pi_2 \circ \pi_1) \) is a closed proper ideal that strictly contains \( \mathcal{M} \), which is impossible. Hence the hypothesis that \( \mathcal{B} = \mathcal{A} / \mathcal{M} \) contains a non-zero, non-invertible element is false. This shows that \( \mathcal{B} = \mathcal{A} / \mathcal{M} \) is a division algebra, and then the Gelfand-Mazur Theorem tells us that \( \mathcal{B} = \mathcal{A} / \mathcal{M} \) is canonically isomorphic to \( \mathbb{C} \). Hence \( \pi_1 \) may be regarded as a character of \( \mathcal{A} \), and by construction \( X_0 \in \mathcal{I}(X_0) \subset \mathcal{M} = \ker(\pi_1) \).

This theorem has the following important consequence:
1.34 COROLLARY. Let $\mathcal{A}$ be a commutative Banach algebra. For every non-zero $\lambda \in \sigma_{\mathcal{A}}(A)$, there exists $\varphi \in \Delta(\mathcal{A})$ such that $\varphi(A) = \lambda$. In particular, the spectral radius $\nu(A)$ of $A$ is given by

$$\nu(A) = \sup \{ |\varphi(A)| : \varphi \in \Delta(\mathcal{A}) \} . \quad (1.26)$$

Proof. Adjoining an identity has no effect on the non-zero spectrum; we may assume that $\mathcal{A}$ is unital. For $\lambda \in \sigma_{\mathcal{A}}(A)$, $\lambda 1 - A$ is not invertible. By Theorem 1.33, there exists $\varphi \in \Delta(\mathcal{A})$ such that $\varphi(\lambda 1 - A) = 0$. But $\varphi(\lambda 1 - A) = \lambda \varphi(1) - \varphi(A) = \lambda - \varphi(A)$. \hfill \Box

2 Fundamentals of the Theory of $C^*$ Algebras

2.1 $C^*$ algebras

2.1 DEFINITION (Banach $*$-algebra). A Banach $*$-algebra is a Banach algebra $\mathcal{A}$ equipped with a map $*: \mathcal{A} \to \mathcal{A}$, the action of which is written as $A \mapsto A^*$, and which satisfies:

(i) The map $*$ is conjugate linear. That is, for all $A, B \in \mathcal{A}$ and all $z \in \mathbb{C}$, $(zA + B)^* = zA^* + B^*$.

(ii) For all $A, B \in \mathcal{A}$, $(AB)^* = B^*A^*$ .

(iii) The $*$ map is an involution; for all $A \in \mathcal{A}$, $A^{**} = A$.

The map $A \mapsto A^*$ is called the involution in $\mathcal{A}$.

2.2 DEFINITION ($C^*$-algebra). A $C^*$ algebra is a Banach $*$ algebra for which the involution and the norm are related by the requirement that: (iv) For all $a \in \mathcal{A}$,

$$\|AA^*\| = \|A\|\|A^*\| . \quad (2.1)$$

The identity (2.1) is called the $C^*$ algebra identity.

Definition 2.2 comes from a seminal 1943 paper of Gelfand and Neumark [8]. The term $C^*$ algebra was introduced Segal [29] in 1947. Segal originally used the term [29, p. 75] to describe norm closed self adjoint subalgebras of the algebra of bounded linear operator on a Hilbert space; see Example 2.4 below. These later became known as “concrete $C^*$-algebras”, and the $a priori$ more general class of $C^*$ algebras specified in Definition 2.2 became known as “abstract $C^*$-algebras”. The main result of [8] is that, under some additional assumptions (see below), every unital abstract $C^*$ algebra is isometric and $*$-isomorphic with a concrete $C^*$- algebra, The additional assumptions, and the requirement that $\mathcal{A}$ be unital, have since been shown to be unnecessary, and these days the distinction between abstract and concrete $C^*$ algebras is rarely emphasized.

In their paper, Gelfand and Neumark not only assumed the existence of a multiplicative identity in $\mathcal{A}$; they also imposed two other hypotheses that they conjectured to be consequences of (i) - (iii) and (2.1) which was their (iv), but they were unable to show this. The additional hypotheses that they imposed were that

(v) The $*$ map is an isometry; for all $A \in \mathcal{A}$, $\|A^*\| = \|A\|$.

(vi) For all $A \in \mathcal{A}$, $1 + A^*A$ is invertible.

In the second footnote of their paper, they conjecture that (i) - (iv) imply (v) and (vi), remarking that the authors “have not succeeded in proof of this fact”. They then note that postulates (iv) and (v) may be replaced by

$$\|A^*A\| = \|A\|^2 \quad (2.2)$$
Since $\epsilon > 0$ is a commutative $A$, says that up to isometric isomorphism, every commutative $C$ it is evident that $\|\phi,\psi\|$ for all $A \in \mathcal{A}$ says that up to isometric isomorphism, every commutative $C$ algebra is also a $B^*$ algebra, and the latter term is now rarely used.

In the more recent literature on the subject, many authors simply define a $C^*$-algebra to be a Banach $*$-algebra $\mathcal{A}$ in which the norm and the involutions satisfy (2.2) for all $A \in \mathcal{A}$. By what Gelfand and Neumark observed in the second footnote, every $B^*$-algebra is a $C^*$-algebra. In 1967, Vowden [34] completed the proof of the fact that every $C^*$-algebra is also a $B^*$ algebra, and the latter term is now rarely used.

In 1946, Richart defined a $B^*$-algebra to be a Banach $*$-algebra $\mathcal{A}$ in which the norm and the involutions satisfy (2.2) for all $A \in \mathcal{A}$. By what Gelfand and Neumark observed in the second footnote, every $B^*$-algebra is a $C^*$-algebra. In 1967, Vowden [34] completed the proof of the fact that every $C^*$-algebra is also a $B^*$ algebra, and the latter term is now rarely used.

In 1946, Richart defined a $B^*$-algebra to be a Banach $*$-algebra $\mathcal{A}$ in which the norm and the involutions satisfy (2.2) for all $A \in \mathcal{A}$; see e.g., [1]. This has some advantages since, for instance, the isometry property of the involution is an immediate consequence of the other postulates. Here we shall have to work harder for this since we start from the original postulates $(i)$ through $(iv)$, $(iv)$ being (2.1), as originally proposed by Gelfand and Neumark, and present the proof of their conjecture on the redundancy of $(v)$ and $(vi)$ in the the general non-unital case. While the proof of this conjecture (in its non-unital form) had to wait for two dozen years and the contributions of a number of mathematicians, the insights through which this was achieved are quite simple, clean and clear. Like many simple, clean and clear ideas, they will turn out to have other uses, and this more than justifies our choice of starting point.

2.3 EXAMPLE. Let $\mathcal{A} = C_0(X)$ with the structure specified in Example 1.2, together with the involution $*$ defined by $f^*(x) = \overline{f(x)}$ for all $x \in X$ and all $f \in \mathcal{A}$. Then one readily checks that $\mathcal{A}$ is a commutative $C^*$-algebra. A theorem of Gelfand and Neumark to be proved in this chapter says that up to isometric isomorphism, every commutative $C^*$ algebra is of this form.

2.4 EXAMPLE. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{A} = \mathcal{B}(\mathcal{H})$ equipped with the structure specified in Example 1.4, together with the involution $*$ defined by taking $A^*$ to be the Hermitian adjoint of $A$. That is, $A^*$ is the unique operator in $\mathcal{B}(\mathcal{H})$ such that

$$\langle A^*\phi, \psi \rangle = \langle \phi, A\psi \rangle \quad (2.3)$$

for all $\phi, \psi \in \mathcal{H}$. Then properties $(i)$, $(ii)$ and $(iii)$ are evidently satisfied. To see that $(iv)$ is satisfied, choose $\epsilon > 0$ and a unit vector $\psi \in \mathcal{H}$ such that $(1 - \epsilon)\|A\| \leq \|A\psi\|$. Then

$$\langle \psi, A^*A\psi \rangle = \langle A\psi, A\psi \rangle = \|A\psi\|^2 \geq (1 - \epsilon)^2\|A\|^2.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\|A^*A\| \geq \|A\|^2$. Since (2.3) says that $|\langle \psi, A^*\phi \rangle| = |\langle \phi, A\psi \rangle|$, it is evident that $\|A^*\| = \|A\|$ and hence $\|A^*A\| \geq \|A\|^2$ can be written as $\|A^*A\| \geq \|A\|^2\|A\|$. Since $\mathcal{B}(\mathcal{H})$ is a Banach algebra, we have $\|A^*A\| \leq \|A^*\|\|A\|$, and hence $(iv)$ is proved.

We have already proved the isometry property $(v)$ in this example on our way to $(iv)$. It is also not hard to give a direct proof of $(vi)$ in this context; see Remark 3.4 and the discussion just
above it. That proof of (vi), just as this proof of (iv) and (v), makes use of the vectors in the space on which the elements of \( \mathcal{A} \) operate. In the abstract setting of Definition (2.2), the elements of \( \mathcal{A} \) need not be operators on a Hilbert space, and such arguments are not immediately available. However, the fundamental theorem in the 1943 paper of Gelfand and Neumark – as extended over the next two dozen years – says that every C*-algebra is isometrically *-isomorphic to a C*-algebra of operators in a Hilbert space \( \mathcal{H} \).

A difficulty that arises when studying non-unital C*-algebras is that our standard construction for adjoining a unit does not in general yield a norm satisfying (iv). However, we can always embed a Banach *-algebra \( \mathcal{A} \) in a unital Banach *-algebra \( \widetilde{\mathcal{A}} \) by a small extension of our standard construction: Let \( \mathcal{A} \) be a non-unital Banach *-algebra and let \( \widetilde{\mathcal{A}} := \mathbb{C} \oplus \mathcal{A} \) with the product defined in (1.3) and the norm defined in (1.4). For \((\lambda, A) \in \widetilde{\mathcal{A}}\), define \((\lambda, A)^* := (\overline{\lambda}, A^*)\). It is easy to see that \( \widetilde{\mathcal{A}} \) satisfies (i), (ii) and (iii) of Definition 2.1. However, (iv) will not in general be satisfied. When \( \mathcal{A} \) is a Banach *-algebra, \( \widetilde{\mathcal{A}} \) always denotes the unital Banach *-algebra obtained using this construction.

2.5 LEMMA. Let \( \mathcal{A} \) be a unital Banach *-algebra with unit 1. Then \( 1^* = 1 \), and \( A \in \mathcal{A} \) is invertible if and only if \( A^* \) is invertible.

Proof. Evidently \( 1^* = 1^1 \), and then by (ii) and (iii), \( 1 = 1^1 \). Next, suppose that \( A \) is invertible and let \( B \) be its inverse. Then \( 1 = AB = BA \). By (ii) and (iii), and the first part of the proof, \( 1 = 1^* = B^*A^* = A^*B^* \). Thus, \( A^* \) is invertible, and \( B^* \) is the inverse. By (iii) once more, when \( A^* \) is invertible, \( A \) is invertible.

2.6 LEMMA. Let \( \mathcal{A} \) be a Banach *-algebra. Then for all \( A \in \mathcal{A} \),
\[
\sigma_{\mathcal{A}}(A^*) = \sigma_{\mathcal{A}}(A)^*.
\]
(2.4)

In particular, \( \nu(A^*) = \nu(A) \).

Proof. Suppose \( \mathcal{A} \) is non-unital. Let \( \widetilde{\mathcal{A}} \) be unital Banach *-algebra obtain obtained from our standard construction. By definition, \( \sigma_{\mathcal{A}}(A) = \sigma_{\widetilde{\mathcal{A}}}(0, A) \). By Lemma 2.5, \((\lambda, -A)\) is invertible if and only if \((\lambda, -A)^* := (\overline{\lambda}, A^*)\) is invertible, and this proves (2.4). The unital case is even more direct. The final statement follows from the definition of the spectral radius.

2.7 DEFINITION (Self-adjoint, normal). Let \( \mathcal{A} \) be a C*-algebra. Then:

(1) \( A \in \mathcal{A} \) is self-adjoint in case \( A = A^* \).

(2) \( A \in \mathcal{A} \) is normal in case \( AA^* = A^*A \).

We write \( \mathcal{A}_{s.a.} \) to denote the sets of self-adjoint elements in \( \mathcal{A} \).

2.8 LEMMA. Let \( \mathcal{A} \) be a C*-algebra. Then for all normal \( A \in \mathcal{A} \),
\[
\|A\| = \|A^*\| = \nu(A).
\]
(2.5)

Proof. Suppose first that \( A \) is self-adjoint. By (iv), \( \|A^2\| = \|A\|^2 \), and by an obvious induction, for all \( m \in \mathbb{N} \), \( \|A^{2m}\| = \|A\|^{2m} \). By the Beurling-Gelfand Formula,
\[
\nu(A) = \lim_{m \to \infty} (\|A^{2m}\|)^{1/2m} = \|A\|.
\]
Therefore, there exist $U \in \mathcal{A}$ such that a sequence of non-invertible elements converges to an invertible element. Thus, the hypothesis leads to contradiction.

By standard estimates on the exponential power series in $A$, we have $\|e^{i\lambda}\| = 4$ for some $\lambda \in \sigma_{\mathcal{A}}(a)$. Taking an appropriate real multiple of $A$, we may suppose that $|e^{i\lambda}| = 4$ for some $\lambda \in \sigma_{\mathcal{A}}(a)$.

For each $n \in \mathbb{N}$, define the polynomial $p_n(\zeta) = \sum_{j=0}^{n} (i\zeta)^j / j!$. By the Spectral Mapping Lemma, for each $n$, $p_n(\lambda) \in \sigma_{\mathcal{A}}(p_n(A))$. For $n > m$,

$$\|p_n(A) - p_m(A)\| \leq \sum_{j=m+1}^{n} \|A\|^j / j!.$$ 

By standard estimates on the exponential power series in $C$, $\{p_n(A)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{A}$. Therefore, there exist $\tilde{U} \in \mathcal{A}$ with $U = \lim_{n \to \infty} p_n(A)$. Evidently, for all $n$, $(p_n(A))^* = p_n(-A)$, so that once again, by standard estimates for the exponential power series, $U^*U = UU^* = 1$. Now define $W = U - 1$, and note that $W$ is in $\mathcal{A}$. From $U^*U = UU^* = 1$ in $\mathcal{A}$, we have

$$W + W^* + W^*W = W + W^* + WW^* = 0,$$ 

and in particular, $W$ is normal. By (2.7) once more, $\|W^*W\| \leq \|W\| + \|W^*\|$, and then by Lemma 2.8, $\|W\|^2 \leq 2\|W\|$ and hence $\|W\| \leq 2$. Then in $\mathcal{A}$,

$$\lim_{n \to \infty} (p_n(\lambda)1 - p_n(A)) = e^{i\lambda}1 - 1 - W.$$ 

Since $|e^{i\lambda}| = 4$, and $\|W\| \leq 2$, $\|e^{-i\lambda}(1 + W)\| \leq \frac{1}{4}$, and $1 - e^{-i\lambda}(1 + W)$ is invertible. However, for each $n$ $p_n(\lambda)1 - p_n(A)$ is non-invertible. Since the set of invertible elements is open, it cannot be that a sequence of non-invertible elements converges to an invertible element. Thus, the hypothesis that for $A \in \mathcal{A}_{s.a.}$, $\sigma_{\mathcal{A}}(A)$ is not contained in $\mathbb{R}$ leads to contradiction.

2.10 THEOREM. In a Banach $*$-algebra $\mathcal{A}$, the involution is continuous if and only if the set $\mathcal{A}_{s.a.}$ of self-adjoint elements is closed.

Proof. If the involution is continuous, then as the inverse image $\{0\}$ under the continuous function $A \mapsto A - A^*$, $\mathcal{A}_{s.a.}$ is closed.
For the converse, we apply the Closed Graph Theorem, which is also valid for conjugate linear maps. Consider a sequence \(\{A_n\}_{n \in \mathbb{N}}\) such that for some \(B, C \in \mathcal{A}\), \(\lim_{n \to \infty} A_n = B\) and \(\lim_{n \to \infty} A_n^* = C\). We must show that \(C^* = B\). Note that

\[
\lim_{n \to \infty} \left( \frac{A_n + A_n^*}{2} \right) = \frac{B + C}{2} \quad \text{and} \quad \lim_{n \to \infty} \left( \frac{A_n - A_n^*}{2i} \right) = \frac{B - C}{2i}.
\]

Since \(\mathcal{A}_{s.a.}\) is closed, both \(\frac{B+C}{2}\) and \(\frac{B-C}{2i}\) are self-adjoint, and hence

\[
B + C = B^* + C^* \quad \text{and} \quad B - C = C^* - B^*,
\]

and thus \(2B = (B + C) + (B - C) = (B^* + C^*) + (C^* - B^*) = 2C^*\), as was to be shown. \(\square\)

2.11 THEOREM. Let \(\mathcal{A}\) be a C* algebra. Then \(\mathcal{A}_{s.a.}\) is closed, and the involution is continuous.

Proof. By Theorem 2.10, it suffices to prove that \(\mathcal{A}_{s.a.}\) is closed. Let \(\{H_n\}_{n \in \mathbb{N}}\) be a convergent sequence in \(\mathcal{A}_{s.a.}\) with \(\lim_{n \to \infty} H_n = H + iK\), \(H, K \in \mathcal{A}_{s.a.}\). Subtracting \(H\) from each \(H_n\), we may suppose that \(H = 0\). Also without loss of generality, we may suppose that \(\|H_n\| \leq 1\) for all \(n\), with the consequence that \(\|K\| \leq 1\). By the Spectral Mapping Theorem, \(\sigma_{\mathcal{A}}(H_n^2 - H_n^4) = \{\lambda^2 - \lambda^4 : \lambda \in \sigma_{\mathcal{A}}(H_n)\}\). Then by Lemma 2.5 and Theorem 2.9,

\[
\|H_n^2 - H_n^4\| = \sup\{\lambda^2 - \lambda^4 : \lambda \in \sigma_{\mathcal{A}}(H_n)\} \leq \sup\{\lambda^2 : \lambda \in \sigma_{\mathcal{A}}(H_n)\} = \|H_n\|^2,
\]

where we have used the fact that \(\sigma_{\mathcal{A}}(H_n) \subset [-1, 1]\) by Lemma 2.5 once more. Taking the limit \(n \to \infty\), we obtain \(\|K^2 + K^4\| \leq \|K\|^2\). Choose \(\lambda_1 \in \sigma_{\mathcal{A}}(K)\) so that \(\lambda_1 = \nu(K^2) = \|K\|^2\). Then \(\lambda_1 + \lambda_1^2 \leq \lambda_1\) so \(\lambda_1 = 0\) and \(K = 0\). \(\square\)

2.2 Commutative C* algebras

2.12 DEFINITION (Hermitian character). Let \(\mathcal{A}\) be a C* algebra. A character \(\varphi\) of \(\mathcal{A}\) is Hermitian in case for all \(A \in \mathcal{A}\),

\[
\varphi(A^*) = \overline{\varphi(A)}.
\]

2.13 LEMMA. All characters of a C* algebra are Hermitian.

Proof. For any \(A \in \mathcal{A}\) define \(X = \frac{1}{2}(A + A^*)\) and \(Y = \frac{1}{2i}(A - A^*)\). Then \(X\) and \(Y\) are self-adjoint, and \(A = X + iY\). For any character \(\varphi\) of \(\mathcal{A}\),

\[
\varphi(A) = \varphi(X + iY) = \varphi(X) + i\varphi(Y) \quad \text{and} \quad \varphi(A^*) = \varphi(X - iY) = \varphi(X) - i\varphi(Y).
\]

By Lemma 1.24 and Theorem 2.9, \(\varphi(X), \varphi(Y) \in \mathbb{R}\), and hence \(\varphi(A^*) = \overline{\varphi(A)}\). \(\square\)

2.14 THEOREM (Commutative Gelfand-Neumark Theorem). Let \(\mathcal{A}\) be a commutative C*-algebra. Then the Gelfand transform is an isometric isomorphism of \(\mathcal{A}\) onto \(\mathcal{C}_0(\Delta(\mathcal{A}))\).

Proof. The Gelfand transform is a *-homomorphism on account of Lemma 2.13: For all \(A \in \mathcal{A}\) and all \(\varphi \in \Delta(\mathcal{A})\):

\[
\gamma(A^*)[\varphi] = \varphi(A^*) = \overline{\varphi(A)} = \gamma(A)^*[\varphi].
\]
By the easy Lemma 1.24, $|\gamma(A)[\varphi]| = |\varphi(A)| \leq \|A\|$, so that $\|\gamma(A)\| \leq \|A\|$. To show that $\|\gamma(A)\| \geq \|A\|$, note that in a commutative $C^*$ algebra, every element is normal, and hence $\|A\| = \nu(A)$ for all $A$. By Corollary 1.34 of Theorem 1.33, there exists a character $\varphi$ with $\nu(A) = |\varphi*(A)| = |\gamma(A)[\varphi]|$. Hence $\|\gamma(A)\| \geq \nu(A) = \|A\|$.

This proves that the Gelfand transform is an isometry, and hence is injective onto a subalgebra of $\gamma(\mathcal{A})$ of $\mathcal{C}_0(\Delta(\mathcal{A}))$. However, $\gamma(\mathcal{A})$ separates points, and does not vanish at any $\varphi \in \Delta(\mathcal{A})$, and is closed under complex conjugation. Hence by the Stone-Wierstrass Theorem, and the fact that $\gamma(\mathcal{A})$ is closed, $\gamma(\mathcal{A}) = \mathcal{C}_0(\Delta(\mathcal{A}))$.

\[\square\]

### 2.3 Spectral invariance and the Abstract Spectral Theorem

Let $\mathcal{A}$ be a Banach algebra with identity, and let $\mathcal{B}$ be a Banach subalgebra. It can happen that some $B \in \mathcal{B}$ is not invertible in $\mathcal{B}$, but is invertible in $\mathcal{A}$.

#### 2.15 Example.

Let $D$ denote the closed unit disc in $\mathbb{C}$, and let $S^1$ denote its boundary, the unit circle. Let $\mathcal{A} = \mathcal{C}(S^1)$, the algebra of continuous functions on $S^1$. Let $\mathcal{B}$ denote the algebra of continuous functions on $D$ that are holomorphic in the interior of $D$. These functions are determined by their values on $S^1$, and their maximum absolute value is attained on $S^1$. Therefore, restriction to $S^1$ is an isometric embedding of $\mathcal{B}$ in $\mathcal{A}$, so we may regard $\mathcal{B}$ as a subalgebra of $\mathcal{A}$.

Let $B$ denote the function $f(e^{i\theta}) = e^{i\theta}$, which evidently belongs to $\mathcal{B}$. Then $1\lambda - B$ is invertible in $\mathcal{A}$ if and only if $\lambda \notin S^1$, in which case the inverse is the function $g(e^{i\theta}) = (\lambda - e^{i\theta})^{-1}$. However, for $\lambda$ in the interior of $D$, $\zeta \mapsto (\lambda - \zeta)^{-1}$ is not holomorphic in the interior of $D$, and so the inverse of $B$ in $\mathcal{A}$ does not belong to $\mathcal{B}$. That is $\sigma_\mathcal{A}(B) = S^1$, but $\sigma_\mathcal{B}(B) = D$.

Now we specialize to $C^*$ algebras, first introducing certain minimal subalgebras:

#### 2.16 Definition.

Let $\mathcal{A}$ be a $C^*$ algebra, and $S$ a subset of $\mathcal{A}$. Then $C(S)$ is the smallest $C^*$ subalgebra of $\mathcal{A}$ that contains $S$.

#### 2.17 Theorem.

Let $\mathcal{A}$ be a $C^*$ algebra, and let $\mathcal{B}$ be a $C^*$ subalgebra of $\mathcal{A}$. Suppose either that $\mathcal{A}$ is unital with unit $1$ and that $1 \in \mathcal{B}$ or else that neither $\mathcal{A}$ nor $\mathcal{B}$ is unital. Then for all $B \in \mathcal{B}$,

$$\sigma_\mathcal{A}(B) = \sigma_\mathcal{B}(B).$$

\[\text{Proof.}\]

It is evident that $\sigma_\mathcal{A}(B) \subset \sigma_\mathcal{B}(B)$, and we need only prove that $\sigma_\mathcal{A}(B) \subset \sigma_\mathcal{B}(B)$.

Suppose first that $\mathcal{A}$ is unital with unit $1$ and that $1 \in \mathcal{B}$. We first prove (2.8) for $B$ self adjoint: Suppose that $B$ is self adjoint and invertible in $\mathcal{A}$. By Theorem 2.9, $\sigma_{C(\{1,B\})}(B) \subset \mathbb{R}$, and consequently, for all $n \in \mathbb{N}$, $B - (i/n)1$ is invertible in $C(\{1,B\})$. Since $\lim_{n \to \infty}(B - (i/n)1) = B$ in $\mathcal{A}$, and the inverse is continuous, $\lim_{n \to \infty}(B - (i/n)1)^{-1} = B^{-1}$ in $\mathcal{A}$. But since $(B - (i/n)1)^{-1} \in C(\{1,B\})$ for all $n$, and since $C(\{1,B\})$ is closed,

$$B^{-1} = \lim_{n \to \infty}(B - (i/n)1)^{-1} \in C(\{1,B\}).$$

Hence, $B$ is invertible within $C(\{1,B\})$.

Now let $B$ be any invertible element of $\mathcal{A}$. Then $B^*$ and $B^*B$ are invertible in $\mathcal{A}$, and also belong to $C(\{1,B\})$. Since $B^*B$ is self adjoint, what we have proved above says that $(B^*B)^{-1} \in C(\{1,B\})$. Define $X = (B^*B)^{-1}B^* \in C(\{1,B\})$. Evidently $XB = 1$. Thus, $B$ has a left inverse in $C(\{1,B\})$. 


The same argument shows that $Y = B^*(BB^*)^{-1}$ is a well defined right inverse of $B$ in $C\{\{1, B\}\}$, and then $X = X(BY) = (XB)Y = Y$ so $X = Y$ is the inverse of $B$ in $C\{\{1, B\}\}$. In particular, for all $\lambda \in \mathbb{C}$, $\lambda I - B$ is invertible in $C\{\{1, B\}\}$ if and only if it is invertible in $\mathcal{A}$. Thus, $\lambda I - B$ is invertible in $\mathcal{A}$ if and only if it is invertible in $C\{\{1, B\}\}$, and this proves (2.8).

Next, consider the case in which neither $\mathcal{A}$ nor $\mathcal{B}$ are unital. Again, let $B \in \mathcal{B}$ be self adjoint. Then $\sigma_{\mathcal{B}}(B) \subset \mathbb{R}$. Let $\mathcal{A}$ and $\mathcal{B}$ be the Banach $*$-algebras obtained by using the standard construction to adjoin a unit; note that it is the same unit 1 that is adjoined to both $\mathcal{A}$ and $\mathcal{B}$. Although $\mathcal{A}$ and $\mathcal{B}$ need not be $C^*$ algebras, this does not matter. Suppose that $(1, B)$ is invertible in $\mathcal{A}$. Then since $\sigma_{\mathcal{B}}((1, B))$ is real, $(1 + it, B)$ is invertible in $C\{\{(1, 0), (1, B)\}\}$ for all $t \in \mathbb{R}$. Taking limits, we conclude as above that the inverse belongs to $C\{\{(1, 0), (1, B)\}\}$.

Now let $(1, B)$ be any invertible element of $\mathcal{A}$. Then $(1, B)^*$ and $(1, B)^*(1, B)$ are invertible in $\mathcal{A}$, and also belong to $C\{\{(1, 0), (1, B)\}\}$. By what we have proved above, $((1, B)^*(1, B))^{-1} \in C\{\{(1, 0), (1, B)\}\}$. The argument proceeds from here just as in the case in the unital case considered at the beginning.

2.18 LEMMA. Let $\mathcal{A}$ be a unital $C^*$ algebra, and let $A \in A$ be normal. Then the map $\varphi \mapsto \varphi(A)$ is a homeomorphism of $\Delta(C\{\{1, A\}\})$ onto $\sigma_{\mathcal{A}}(A)$. Let $\mathcal{A}$ be a non-unital $C^*$ algebra, and let $A \in A$ be normal. Then the map $\varphi \mapsto \varphi(A)$ is a homeomorphism of $\Delta'(C\{\{A\}\})$ onto $\sigma_{\mathcal{A}}(A)$.

Proof. First suppose that $\mathcal{A}$ is unital. Since $A$ and $A^*$ commute, the closure of the linear span of $\{ A^m(A^*)^n : m, n \geq 0 \}$ is a $C^*$ algebra that contains 1 and $A$. Evidently, it is $C\{\{1, A\}\}$. Hence if $\varphi \in \Delta(C\{\{1, A\}\})$, $\varphi$ is determined by its values on $A$ and $A^*$. In fact, since $\varphi$ is necessarily Hermitian, $\varphi$ is determined by its value on $A$. That is, for any $\varphi, \psi \in \Delta(C\{\{1, A\}\})$,

$$\varphi = \psi \iff \varphi(A) = \psi(A). \tag{2.9}$$

We have also seen that for all $\varphi \in \Delta(C\{\{1, A\}\})$, $\varphi(A) \in \sigma_{C\{\{1, A\}\}}(A) = \sigma_{\mathcal{A}}(A)$, and for all $\lambda \in \sigma_{\mathcal{A}}(A) = \sigma_{C\{\{1, A\}\}}(A)$, there is a $\varphi_\lambda \in \Delta(C\{\{1, A\}\})$ such that $\varphi_\lambda(A) = \lambda$. This shows that the map $\varphi \mapsto \varphi(A)$ is a one-to-one map of $\Delta(C\{\{1, A\}\})$ onto $\sigma_{\mathcal{A}}(A)$. This map is also continuous by the definition of the Gelfand topology, and continuous bijections between compact spaces are homeomorphisms.

The case in which $\mathcal{A}$ is non-unital is similar. Since $A$ and $A^*$ commute, the closure of the linear span of $\{ A^m(A^*)^n : m, n \geq 0 , m + n > 0 \}$ is a $C^*$ algebra that contains $A$. Evidently, it is $C\{\{A\}\}$. As before, (2.9) is valid, and $\varphi(A) = 0$ if and only if $\varphi$ is the zero character. Then, as before $\varphi \mapsto \varphi(A) \in \sigma_{C\{\{A\}\}}(A)$ is a one to one continuous map from the compact set $\Delta'(C\{\{A\}\})$ to the compact set $\sigma_{C\{\{A\}\}}(A) = \sigma_{\mathcal{A}}(A)$, and is a homeomorphism. □

We now come to the Abstract Spectral Theorem:

2.19 THEOREM (Abstract Spectral Theorem). Let $\mathcal{A}$ be a $C^*$ algebra with identity 1, and let $A \in \mathcal{A}$ be normal. Identifying $C_{\Delta(C\{\{1, A\}\})}$ and $C(\sigma_{\mathcal{A}}(A))$ through the homeomorphism provided by Lemma 2.18, we may regard the Gelfand transform as a homomorphism of $C\{\{1, A\}\}$ into $C(\sigma_{\mathcal{A}}(A))$. Then the Gelfand transform $\gamma$ is an isometric isomorphism of $C\{\{1, A\}\}$ onto $C(\sigma_{\mathcal{A}}(A))$. For all non-negative integers $m, n$, $\gamma(A^m(A^*)^n)$ is the function on $\sigma_{\mathcal{A}}(A)$ given by

$$\lambda \mapsto \lambda^m \overline{\lambda}^n. \tag{2.10}$$
In case \( \mathcal{A} \) is non-unital, the homeomorphism provided by Lemma 2.18 identifies \( C(\{A\}) \) with \( C_0(\sigma_{\mathcal{A}}(A)\setminus\{0\}) \); i.e., the \( C^* \) algebra of continuous functions \( f \) on \( \sigma_{\mathcal{A}}(A) \) that vanish at 0, and the Gelfand transform \( \gamma \) is an isometric isomorphism of \( C(\{A\}) \) onto \( C_0(\sigma_{\mathcal{A}}(A)\setminus\{0\}) \), and (2.10) is valid for all \( m, n \geq 1 \).

**Proof.** The Commutative Gelfand-Neumark Theorem says that \( \gamma \) is an isometric isomorphism, and if \( \varphi \in \Delta(C(A)) \),

\[
\gamma(A^m(A^*)^n)[\varphi] = \varphi(A)^m((\varphi(A))^*)^n = \lambda^m \lambda^n
\]

for \( \lambda = \varphi(A) \) so that under the identification provided by Lemma 2.18, \( \gamma(A^m(A^*)^m) \) is indeed given by (2.10).

2.20 **DEFINITION.** For a unital \( C^* \) algebra \( \mathcal{A} \) and for any normal element \( A \) of \( \mathcal{A} \), and any \( f \in C(\sigma_{\mathcal{A}}(A)) \), \( f(A) \) is defined by \( \gamma^{-1}(f) \); i.e., \( f(A) \) is the inverse image of \( f \) under the isometric isomorphism of \( C(\{A\}) \) onto \( C(\sigma_{\mathcal{A}}(A)) \) that is provided by the Commutative Gelfand-Neumark Theorem. Likewise, if \( \mathcal{A} \) is not unital, and \( f \) is a continuous function of \( \sigma_{\mathcal{A}}(A) \) with \( f(0) = 0 \), \( f(A) \) is defined by \( \gamma^{-1}(f) \).

2.21 **THEOREM** (Spectral Mapping Theorem). Let \( \mathcal{A} \) be a \( C^* \) algebra, \( A \) a normal element of \( \mathcal{A} \), and \( f \in C(\sigma_{\mathcal{A}}(A)) \). Then

\[
\sigma_{\mathcal{A}}(f(A)) = f(\sigma_{\mathcal{A}}(A))
\]

**Proof.** First suppose that \( \mathcal{A} \) is unital. For all \( \mu \in \mathbb{C} \), the function \( \lambda \mapsto \mu - f(\lambda) \) is invertible in \( C(\sigma_{\mathcal{A}}(A)) \) if and only if \( \mu \) does not belong to the range of \( f \), which is \( f(\sigma_{\mathcal{A}}(A)) \). Then, using the isomorphism provided by the Commutative Gelfand Neumark Theorem, we see that \( \mu 1 - f(A) \) is invertible in \( C(\{1, A\}) \) if and only if \( \mu \notin f(\sigma_{\mathcal{A}}(A)) \), and hence \( \sigma_{C(\{1, A\})}(f(A)) = f(\sigma_{\mathcal{A}}(A)) \). Finally, by Theorem 2.17, the spectrum of \( f(A) \) is the same in all \( C^* \) subalgebras of \( \mathcal{A} \) that contain \( f(A) \) and 1. In particular, \( \sigma_{C(\{1, A\})}(f(A)) = \sigma_{\mathcal{A}}(f(A)) \).

Their argument in the non-unital case is essentially the same, except we must use quasi inverses to characterize the spectrum.

2.22 **THEOREM.** Let \( \mathcal{A} \) be a \( C^* \) algebra with identity 1. Let \( U \subset \mathbb{C} \) be open with \( \overline{U} \) compact. Let \( \mathcal{M}_U \) be given by

\[
\mathcal{M}_U = \{ A \in \mathcal{A} : A A^* = A^* A \text{ and } \sigma_{\mathcal{A}}(A) \subset U \}
\]  

(2.11)

Then \( \mathcal{M}_U \) is an open subset of the normal elements of \( \mathcal{A} \). Moreover, let \( f \) be a continuous complex valued function on \( \overline{U} \), and for all \( A \in \mathcal{M}_U \), define \( f(A) \in \mathcal{A} \) using the Abstract Spectral Theorem. Then the map \( A \mapsto f(A) \) is continuous on \( \mathcal{M}_U \).

**Proof.** The first assertion is an immediate consequence of Newburg's Theorem, Theorem 1.22. For the second, first note that if \( p \) is a monomial, \( p(\lambda) = \lambda^n \). then, by the telescoping sum identity and the obvious estimates, \( \|p(B) - p(A)\| \leq n(\max\{\|A\|, \|B\|\})^{n-1}\|B - A\| \). It follows that for any polynomial \( p \), the map \( A \mapsto p(A) \) is continuous. Now consider any sequence \( \{p_n\} \) of polynomials converging uniformly to \( f \) on \( \overline{U} \). Then for all \( A \in \mathcal{M}_U \),

\[
\|p_n(A) - f(A)\| \leq \sup_{\lambda \in \overline{U}} \{\|p_n(\lambda) - f(\lambda)\|\}.
\]
That is,
\[ \lim_{n \to \infty} \left( \sup_{A \in \mathcal{A}} \{ \| p_n(A) - f(A) \| \} \right) = 0. \]

Thus, the function \( A \mapsto f(A) \) is the norm limit of the continuous functions \( A \mapsto p_n(A) \).

**2.23 COROLLARY.** Let \( A \) be a normal element of a \( C^* \) algebra \( \mathcal{A} \). Let \( f \) be continuous on \( \sigma_{\mathcal{A}}(A) \). Let \( g \) be continuous on a compact set \( K \) containing \( \sigma_{\mathcal{A}}(A) \) in its interior. Then \( f(g(A)) = f \circ g(A) \).

*Proof.* Fix \( \epsilon > 0 \), and choose a polynomial \( q \) such that
\[ \sup_{\lambda \in K} |f(\lambda) - q(\lambda)| < \epsilon. \quad (2.12) \]
Let \( \{p_n\}_{n \in \mathbb{N}} \) be a sequence of polynomials converging uniformly to \( g \) on \( \sigma_{\mathcal{A}}(A) \). By Theorem 2.22, for all \( n \) sufficiently large, \( \sigma_{\mathcal{A}}(p_n(A)) \subset K^\circ \). It is evident that \( q \circ p_n(A) = q(p_n(A)) \). By (2.12), and this identity,
\[ \| f(p_n(A)) - q(p_n(A)) \| = \| f(p_n(A)) - q \circ p_n(A) \| \leq 2\epsilon \]
for all \( n \) sufficiently large, and then by Theorem 2.22, \( \| f(g(A)) - q \circ g(A) \| < 2\epsilon \). By (2.12) once more, \( \sup_{\lambda \in \sigma_{\mathcal{A}}(A)} |f \circ g(\lambda) - q \circ g(\lambda)| < \epsilon \), and hence \( \| q \circ g(A) - f \circ g(A) \| < \epsilon \). Altogether, \( \| f(g(A)) - f \circ g(A) \| < 3\epsilon \), and since \( \epsilon > 0 \) is arbitrary, the proof is complete. \( \square \)

**2.24 DEFINITION.** Let \( \mathcal{A} \) be a \( C^* \) algebra. The *positive* elements of \( \mathcal{A} \) are the self adjoint elements \( A \) such that \( \sigma_{\mathcal{A}}(A) \subset [0, \infty) \). The set of all positive elements of \( \mathcal{A} \) is denoted \( \mathcal{A}^+ \).

If \( A \in \mathcal{A}^+ \), we may use the Abstract Spectral Theorem to define \( \sqrt{A} \), and then \( A = (\sqrt{A})^2 = (\sqrt{A})^*(\sqrt{A}) \). One of the 1943 conjectures of Gelfand and Neumark was that for all \( B \in \mathcal{A} \), \( B^*B \) is positive. This conjecture was finally proved in 1952 and 1953 through the contributions of Fukamiya and Kaplansky, for the case of unital \( C^* \) algebras. (Gelfand and Neumark only considered unitary \( C^* \)-algebras.) The history is interesting: Kaplansky had managed to prove that if the sum of two positive elements is necessarily positive, then \( B^*B \) is necessarily positive. However, he was unable to show that \( \mathcal{A}^+ \) was closed under addition. He published nothing, but showed his proof to many people. When Fukamiya proved the closure of \( \mathcal{A}^+ \) under addition in 1952, the reviewer of Fukamiya’s paper in Math Reviews knew of Kaplansky’s proof, and Kaplansky gave him permission to publish it in the review. Finally, in 1956, Bonsall removed the hypothesis that \( \mathcal{A} \) be unital.

**2.25 THEOREM** (Fukamiya’s Theorem). Let \( \mathcal{A} \) be a unital \( C^* \) algebra. Then \( \mathcal{A}^+ \) is a closed, pointed convex cone. That is:
\begin{enumerate}
  \item For all \( \lambda \in \mathbb{R}^+ \), and all \( A \in \mathcal{A} \), \( \lambda A \in \mathcal{A}^+ \), and for all \( A, B \in \mathcal{A}^+ \), \( A + B \in \mathcal{A}^+ \). \\
  \item \( -\mathcal{A}^+ \cap \mathcal{A}^+ = \{0\} \).
\end{enumerate}
(The first part says that \( \mathcal{A}^+ \) is a convex cone; the second part says that this cone is pointed.)
Proof. Let \( B_{sa} \) denote the closed unit ball in \( \mathcal{A} \). Fukamiya observed that \( \mathcal{A}^+ \cap B_{sa} \) consists precisely of the self-adjoint elements \( A \) with both \( A \) and \( 1 - A \) in \( B_{sa} \). To see this suppose that \( A \in \mathcal{A}^+ \cap B_{sa} \). Since \( A \) is self adjoint, \( \nu(A) = \|A\| \leq 1 \), and so \( \sigma_{sa}(A) \subset [0,1] \). By (an easy case of) the Spectral Mapping Lemma, \( \sigma_{sa}(1 - A) \subset [0,1] \), and hence \( \|1 - A\| = \nu(1 - A) \leq 1 \).

Conversely, suppose \( A \) is self-adjoint and both \( A \) and \( 1 - A \) are in \( B_{sa} \). Since \( A \) is self adjoint and \( \|A\| \leq 1 \), \( \sigma_{sa}(A) \subset [-1,1] \). Then by the Spectral Mapping Lemma, \( \sigma_{sa}(1 - A) \subset [0,2] \). However, if \( \|1 - A\| \leq 1 \), then \( \sigma_{sa}(1 - A) \subset [-1,1] \), and altogether we have that \( \sigma_{sa}(1 - A) \subset [0,1] \), and then by the identity \( A = 1 - (1 - A) \) and the Spectral Mapping Lemma once more, \( \sigma_{sa}(A) \subset [0,1] \), so that \( A \in \mathcal{A}^+ \). That is,

\[
\mathcal{A}^+ \cap B_{sa} = \{ A \in \mathcal{A}_{s.a.} : A, 1 - A \in B_{sa} \}.
\]

(2.13)

Let \( A, B \in B_{sa} \). By Minkowski’s inequality, \( \|(A + B)/2\| \in B_{sa} \) and

\[
\left\| 1 - \frac{A + B}{2} \right\| \leq \frac{1}{2}(\|1 - A\| + \|1 - B\|).
\]

(2.14)

If furthermore, \( A, B \in \mathcal{A}^+ \), then we also have that \( \|1 - A\| \leq 1 \) and \( \|1 - B\| \leq 1 \), and then from (2.14), \( \|1 - (A + B)/2\| \leq 1 \). Thus, \( C := (A + B)/2 \) is self adjoint and both \( C \) and \( 1 - C \) belong to \( B_{sa} \). Therefore, \( (A + B)/2 \in \mathcal{A}^+ \).

Since the closure of \( \mathcal{A}^+ \) under positive multiples is clear, it then clear that \( \mathcal{A}^+ \) is a cone. If \( A \in \mathcal{A}^+ \) and \( -A \in \mathcal{A}^+ \), then \( \sigma_{sa}(A) \subset (-\infty, 0] \cap [0,\infty) = \{0\} \), so that \( \|A\| = \nu(a) = 0 \), and hence \( A = 0 \). The norm closure of \( \mathcal{A}^+ \cap B_{sa} \) is evident from (2.13) together with Theorem 2.11, and then the clousre of \( \mathcal{A}^+ \) follows by a simple scaling argument.

\[\Box\]

2.26 THEOREM (Bosall’s Theorem). The condition in Fukamiya’s Theorem that \( \mathcal{A} \) be unital is superfluous.

Proof. Let \( \mathcal{A} \) be a non-unital \( C^* \) algebra. It suffices to show that \( \mathcal{A}^+ \) is closed under addition, and closed in norm, since the unital nature of \( \mathcal{A} \) was used only in proving these fundamental parts. We begin with closure under addition. Since \( A \in \mathcal{A}^+ \) if and only if \( A \) is self adjoint and \( \lambda, -A \) is invertible in \( \mathcal{A} \) for all \( \lambda < 0 \), \( A \in \mathcal{A}^+ \) if and only if \( A \) is self adjoint and \( tA \) is quasi regular for all \( t > 0 \). Now let \( A, B \in \mathcal{A}^+ \). Then \( A + B \) is self adjoint, and to show that \( A + B \) in \( \mathcal{A}^+ \), it suffices to show that for all \( t > 0 \), \( t(A + B) = (tA + tB) \) is quasi regular. Hence it suffices to show that the sum of quasi regular elements in \( \mathcal{A}^+ \) is quasi regular.

Let \( A \in \mathcal{A}^+ \). We first show that \( A' \), the quasi inverse of \( A \), satisfies \( -A' \in \mathcal{A}^+ \) and \( \|A'\| < 1 \). Indeed, by the Abstract Spectral Theorem in \( C(\{A\}) \), \( A' = f(A) \) where

\[
f(t) = \frac{-t}{1 + t},
\]

since in \( \mathcal{C}_0(\sigma_{sa}(A) \setminus \{0\}) \), \( t + f(t) + tf(t) = t + f(t) + f(t)t = 0 \). By the Spectral Mapping Theorem,

\[
\sigma_{sa}(A') = \{-\lambda/(1 + \lambda) : \lambda \in \sigma_{sa}(A)\} \subset (-1,0].
\]

Hence \( \|A\| = \nu(A) < 1 \), and \( -A \in \mathcal{A}^+ \).

Now let \( A, B \in \mathcal{A}^+ \). Then since \( \|A'\|, \|B'\| < 1 \), \( \|A'B'\| < 1 \) and hence \( 1, -A'B' \) is invertible in \( \mathcal{A} \). That is, \(-A'B' \) is quasi regular. Now a simple calculation shows that \( A \circ (-A'B') \circ B = A + B \), and by Lemma 1.10, \( A + B \) is quasi regular.
For the closure in norm, let \( \{ A_n \}_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{A}^+ \) converging to \( B \). For each \( \epsilon > 0 \), \( n \in \mathbb{N} \), \( (\epsilon, A_n) \) is invertible in \( \mathcal{A}^+ \), and by the above, with \( f(t) = -t/(1+t) \), \( (\epsilon, A_n)^{-1} = \frac{1}{\epsilon} (1, f(\epsilon^{-1} A_n)) \), and hence \( \| (\epsilon, A_n)^{-1} \| \leq \frac{1}{\epsilon} \| f(\epsilon^{-1} A_n) \| \leq \frac{2}{\epsilon} \). Since \( \| (\epsilon, B) - (\epsilon, A_n) \| = \| B - A_n \| \), by (1.13) and Lemma 1.13, since \( \| B - A_n \| < \epsilon/2 \) for all sufficiently large \( n \), \( (\epsilon, B) \) is invertible. Since \( \epsilon > 0 \) is arbitrary, \( B \in \mathcal{A}^+ \).

2.27 THEOREM (Fukamiya-Kaplansky-Bonsall Theorem). Let \( \mathcal{A} \) be a \( C^* \) algebra. For all \( A \in \mathcal{A} \), \( A^* A \in \mathcal{A}^+ \).

Proof. We first show that if \( A^* A \in -\mathcal{A}^+ \), then \( A^* A = 0 \). Since \( A^* A \) and \( AA^* \) have the same spectrum, Fukamiya’s Theorem says that \( A^* A + AA^* \in -\mathcal{A}^+ \). However, writing \( A = X + iY \) with \( X \) and \( Y \) self adjoint,

\[
A^* A + AA^* = 2(X^2 + Y^2) \in \mathcal{A}^+ ,
\]

where once again we have used Fukamiya’s Theorem, and the Spectral Mapping Lemma. Since \( \mathcal{A}^+ \) is a pointed cone, this means that \( A^* A + AA^* = 0 \). But then \( A^* A = (A^* A + AA^*) - AA^* = -AA^* \in \mathcal{A}^+ \). Again since \( \mathcal{A}^+ \) is pointed, this means that \( A^* A = 0 \), as claimed.

Define continuous functions \( f, g : \mathbb{R} \to \mathbb{R} \) by \( f(t) = \max \{ t, 0 \} \) and \( g(t) = t - f(t) \). Our goal is to show that for all \( B \in \mathcal{A} \), \( Z := g(B^* B) = 0 \), since then \( B^* B = f(B^* B) \in \mathcal{A}^+ \).

Note that \( f(t)g(t) = 0 \) for all \( t \), and hence with \( Y := f(B^* B), YZ = 0 \). Then

\[
(BZ)^*(BZ) = ZB^*BZ = Z(Y + Z)Z = Z^3 \in -\mathcal{A}^+ .
\]

The first part of the proof says that \( (BZ)^*(BZ) = Z^3 = 0 \), and hence \( Z = 0 \). \( \square \)

2.28 DEFINITION. Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( C^* \) algebras. A linear transformation \( \Phi : \mathcal{A} \to \mathcal{B} \) is positive in case \( \Phi(A) \in \mathcal{B}^+ \) for all \( A \in \mathcal{A}^+ \). Define \( \mathcal{L}^+(\mathcal{A}, \mathcal{B}) \) to be the set of positive linear transformations from \( \mathcal{A} \) to \( \mathcal{B} \). The term positive map is a synonym for positive linear transformation.

2.29 COROLLARY. Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( C^* \) algebras. Then \( \mathcal{L}^+(\mathcal{A}, \mathcal{B}) \) is a pointed cone in \( \mathcal{L}(\mathcal{A}, \mathcal{B}) \), the space of linear transformations from \( \mathcal{A} \) to \( \mathcal{B} \).

Proof. It is evident that \( \mathcal{L}^+(\mathcal{A}, \mathcal{B}) \) is closed under multiplication by non-negative scalars, and it is closed under addition as an immediate consequence of Theorem 2.27. If \( \Phi \in \mathcal{L}^+(\mathcal{A}, \mathcal{B}) \) and \( -\Phi \in \mathcal{L}^+(\mathcal{A}, \mathcal{B}) \), then for all \( A \in \mathcal{A}^+ \), \( \Phi(A) \in \mathcal{B}^+ \cap (-\mathcal{B}^+) = \{0\} \) by Theorem 2.25 and Theorem 2.26. Since every element of \( \mathcal{A} \) is a linear combination of at most 4 elements of \( \mathcal{A}^+ \), \( \Phi(A) = 0 \) for all \( A \in \mathcal{A} \). This proves that \( \mathcal{L}^+(\mathcal{A}, \mathcal{B}) \) is a pointed cone. \( \square \)

The next corollary of Theorem 2.27 gives us an important class of positive maps.

2.30 COROLLARY. Let \( \mathcal{A} \) be a \( C^* \) algebra, and let \( B \in \mathcal{A} \). Then the linear transformation \( \Phi : \mathcal{A} \to \mathcal{A} \) given by \( \Phi(A) := B^* A B \) is positive.

Proof. Let \( A \in \mathcal{A}^+ \). Then since \( f(t) := \sqrt{t} \) is continuous and vanishes at 0, we may use the Abstract Spectral Theorem to define \( A^{1/2} \in \mathcal{A}^+ \) with \( A = A^{1/2} A^{1/2} \). Then

\[
\Phi(A) = B^* A^{1/2} A^{1/2} B = (A^{1/2} B)^*(A^{1/2} B) \in \mathcal{A}^+ ,
\]

where we used Theorem 2.27 in the final step. \( \square \)
In the next section we shall make use of the following lemma:

**2.31 Lemma (Vowden’s Lemma).** Let $\mathcal{A}$ be a C* algebra. For all $A, B \in \mathcal{A}^+$ such that $B - A \in \mathcal{A}^+$, $\|B\| \geq \|A\|$.

When $\mathcal{A}$ has a unit, this is an easy consequence of Fukamiya observation that $\mathcal{A}^+ \cap B_{\mathcal{A}}$ consists precisely of the self-adjoint elements $X$ with both $X$ and $1 - X$ in $B_{\mathcal{A}}$: We may suppose without loss of generality that $\|B\| = 1$. Then $1 - B$ is positive. By Theorem 2.27, $1 - A = (1 - B) + (B - A)$ is positive, but if $\|A\| > 1$, there exists $\lambda \in \sigma_{\mathcal{A}}(S)$ with $\lambda > 1$, and then $1 - A$ is not positive. Hence $\|A\| \leq 1 = \|B\|$.

Vowden has given a proof of the same result without assuming the existence of an identity; his proof makes use of the Abstract Spectral Theorem and the main theorems of this section:

*Proof of Lemma 2.31.* Assume $\|B\| = 1$. We must show that $\|A\| \leq 1$. If we show that $A^2 - A^3 \geq 0$, then since for $\lambda > 1$, $\lambda^2 - \lambda^3 < 0$, no such $\lambda$ can belong to $\sigma_{\mathcal{A}}(A)$, and then $\sigma_{\mathcal{A}}(A) \subset [0, 1]$, and hence $\|A\| \leq 1$.

We now show that $A^2 - A^3 \in \mathcal{A}^+$. For any $C \in \mathcal{A}_{a.a.}$, define $X = A - CA$. Then $X^*X = A^2 + A(C^2 - 2C)A$. If we can choose $C$ so that $C^2 - 2C + B = 0$, then

$$X^*X = A^2 - ABA = A^2 - A^3 - A(B - A)A.$$  

By Corollary 2.30, and Theorem 2.25, $A^2 - A^3 = X^*X + A(B - A)A \in \mathcal{A}^+$.

To produce $C$, note that the quadratic polynomial $x^2 - 2x + b = 0$ has the roots $x = 1 \pm \sqrt{1 - b}$. Therefore, define the function $f(t)$ on $[0, 1]$ by $f(t) = 1 - \sqrt{1 - t}$. Although the formula for $f$ makes explicit reference to 1, $f$ is continuous and $f(0) = 0$. Therefore, $f(B)$ is defined for any $B \in \mathcal{A}$ with $\sigma_{\mathcal{A}}(B) \subset [0, 1]$, even when $\mathcal{A}$ is not unital. Now define $C = f(B)$, and then $C^2 - 2C + B = 0$. \(\square\)

2.5 Adjoining a unit to a non-unital C* algebra

We will make use of the following approximation lemma. Since it has many uses, we state it in a more general form than we need at present to avoid repeating ourselves later on.

**2.32 Theorem.** Let $\mathcal{A}$ be a C* algebra, and let $\{A_1, \ldots, A_m\}$ be any finite subset of $\mathcal{A}$. Then there is a sequence $\{E_n\}_{n \in \mathbb{N}} \subset B_{\mathcal{A}} \cap \mathcal{A}^+$ such that for each $j = 1, \ldots, m$,

$$\lim_{n \to \infty} \|A_j E_n - A_j\| = 0. \tag{2.15}$$

Furthermore:

1. Let $\mathcal{J}$ be a closed ideal in $\mathcal{A}$, and suppose that $\{A_1, \ldots, A_m\} \subset \mathcal{J}$. Then we may take $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{J}$, and still have that (2.15) is valid for each $j = 1, \ldots, m$.

2. If $\{A_1, \ldots, A_m\}$ is closed under the involution, we may choose $\{E_n\}_{n \in \mathbb{N}}$ to have the additional property that both (2.15) and

$$\lim_{n \to \infty} \|E_n A_j - A_j\| = 0 \tag{2.16}$$

are valid for $j = 1, \ldots, m$. \(\square\)
2.33 REMARK. Theorem 2.32 provides a kind of “approximate identity” in a $C^*$ algebra, and one can make precise sense of this notion. However, in applications, Theorem 2.32 can be applied directly without going through this (short) detour.

Proof of Theorem 2.32. Define $X := \sum_{j=1}^m A_j^* A_j \in \mathcal{A}^+$. Consider the sequence of continuous functions $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f_n(t) = \min\{nt, 1\}$. Note that

$$g_n(t) := t(1 - f_n(t))^2 = \begin{cases} t(1 - nt)^2 & t \leq 1/n \\
0 & t > 1/n \end{cases}.$$  

Moreover, $g_n$ is continuous, $g_n(0) = 0$ and $\sup_{t \geq 0} |g_n(t)| \leq 1/n$. By the Abstract Spectral Theorem, $\|g_n(X)\| \leq 1/n$. For each $j = 1, \ldots, m$,

$$(A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j) = f_n(X) A_j^* A_j f_n(X) + A_j^* A_j - A_j^* A_j f_n(X) - f_n(X) A_j^* A_j.$$  

Summing on $j$, $\sum_{j=1}^m (A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j) = g_n(X)$, and therefore,

$$\left\| \sum_{j=1}^m (A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j) \right\| = \|g_n(X)\| \leq \frac{1}{n}.$$  

Since $g_n(X) \in \mathcal{A}^+$, and since for each $j = 1, \ldots, m$, $g_n(X) - (A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j) \in \mathcal{A}^+$ (by Theorem 2.26 once more), Lemma 2.31 yields

$$\max_{j=1,\ldots,m} \left\{ \| (A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j) \| \right\} \leq \frac{1}{n}.$$  

By the $C^*$ algebra identity, $\| (A_j f_n(X) - A_j)^*(A_j f_n(X) - A_j) \| = \| (A_j f_n(X) - A_j)^* \| \| A_j f_n(X) - A \|$, and then by the continuity of the involution, there is a $c > 0$ such that for each $j = 1, \ldots, m$,

$$\frac{1}{n} \geq c \| A_j f_n(X) - A_j \|^2.$$  

Thus, $\lim_{n \to \infty} \| A_j f_n(X) - A_j \| = 0$. By the Abstract Spectral Theorem, $\|f_n(X)\| \leq 1$ and $f_n(X) \in \mathcal{A}^+$ for all $n$. Hence we may take $E_n = f_n(X)$, and then we have proved that $\lim_{n \to \infty} \| A_j E_n - A_j \| = 0$. This proves the first statement. To prove (1), simply note that if $\{ A_1, \ldots, A_m \} \subset \mathcal{J}$, then $X \in \mathcal{J}$, and $E_n = f_n(X)$ is the norm limit of polynomials in $\mathcal{J}$, and then since $\mathcal{J}$ is closed, $E_n \in \mathcal{J}$.

To prove (2), suppose that for each $j \in \{ 1, \ldots, m \}$, there is some $k \in \{ 1, \ldots, m \}$ so that $A_j^* = A_k$. Then, $\| E_n A_j - A_j \| = \| E_n A_k^* - A_k^* \| = \| (A_k E_n - A_k)^* \|$. Since the involution is continuous, (2.16) then follows from (2.15). \hfill \Box

2.34 LEMMA. Let $\mathcal{A}$ be a $C^*$ algebra. For all $A \in \mathcal{A}$,

$$\sup_{\| B \| \leq 1} \{ \| A B \| \} = \| A \| = \sup_{\| B \| \leq 1} \{ \| B A \| \}.$$  

(2.17)

Proof. For $\| B \| \leq 1$, $\| A B \| \leq \| A \|$. For $A \neq 0$, define $B := \| A^* \|^{-1} A^*$. Then $\| B \| = 1$ and

$$\| A B \| = \frac{\| A A^* \|}{\| A^* \|} = \frac{\| A \| \| A^* \|}{\| A^* \|} = \| A \|.$$  

Altogether, $\| A \| = \sup_{\| B \| \leq 1} \{ \| A B \| \}$. The proof of the second identity is essentially the same. \hfill \Box
Now suppose that $\mathcal{A}$ is not unital, and let $\mathcal{B}(\mathcal{A})$ be the Banach space of bounded linear transformations on $\mathcal{A}$. We embed $\mathbb{C} \oplus \mathcal{A}$ into $\mathcal{B}(\mathcal{A})$ as follows: For $(\lambda, A) \in \mathbb{C} \oplus \mathcal{A}$, define $T_{(\lambda, A)} \in \mathcal{B}(\mathcal{A})$ by

$$T_{(\lambda, A)}B = \lambda B + AB$$

for all $B \in \mathcal{A}$. To see that $(\lambda, A) \mapsto T_{(\lambda, A)}$ is one-to-one, suppose that for some $(\lambda, A) \in \mathbb{C} \oplus \mathcal{A}$, $T_{(\lambda, A)} = 0$. If $\lambda = 0$, this is impossible unless also $A = 0$ since if $T_{(0, A)} = 0$, then $AB = 0$ for all $B$ and then by Lemma 2.34, $A = 0$. However, if $T_{(\lambda, A)} = 0$ and $\lambda \neq 0$, replacing $A$ by $-\lambda^{-1}A$, we have that $T_{(-1, A)} = 0$, but then for all $B \in \mathcal{A}$, $B = AB$. Taking $B = A^*$, $A^* = AA^*$, and hence $A$ is self-adjoint. Then $B^* = B^*A$ for all $B \in \mathcal{A}$, and hence we have that $B = AB = BA$ for all $B \in \mathcal{A}$. This is impossible since $\mathcal{A}$ lacks a multiplicative identity. Hence $(\lambda, A) \mapsto T_{(\lambda, A)}$ is one-to-one from $\mathbb{C} \oplus \mathcal{A}$ into $\mathcal{B}(\mathcal{A})$. Define a norm $\| \cdot \|$ on $\mathbb{C} \oplus \mathcal{A}$ by $\|(\lambda, A)\| = \|T_{(\lambda, A)}\|_{\mathcal{B}(\mathcal{A})}$. That is,

$$\|(\lambda, A)\| = \sup_{\|B\| \leq 1} \{\|\lambda B + AB\|\}. \quad (2.18)$$

By Lemma 2.34,

$$\|(0, A)\| = \sup_{\|B\| \leq 1} \{\|AB\|\} = \|A\|, \quad (2.19)$$

and so $A \mapsto T_{(0, A)}$ is an isometry from $\mathcal{A}$ into $\mathcal{B}(\mathcal{A})$. In particular, the image of $\mathcal{A}$ under this embedding is closed. The image of $\mathbb{C} \oplus \mathcal{A}$ under this embedding is the sum of the image of $\mathcal{A}$ and a one dimensional subspace. Since the sum of a closed subspace and a finite dimensional subspace is always closed, the image of $\mathbb{C} \oplus \mathcal{A}$ under this embedding is closed in $\mathcal{B}(\mathcal{A})$, and hence complete since $\mathcal{B}(\mathcal{A})$ is a Banach space. It follows that $\mathbb{C} \oplus \mathcal{A}$ is complete in the norm $\| \cdot \|$. Since $\mathcal{B}(\mathcal{A})$ is a Banach algebra, $\mathbb{C} \oplus \mathcal{A}$ is a Banach algebra in this norm; it inherits the multiplicative property from $\mathcal{B}(\mathcal{A})$, and with the standard involution on $\mathbb{C} \oplus \mathcal{A}$, it is a Banach $*$-algebra. We have proved:

**2.35 Lemma.** Let $\mathcal{A}$ be a non-unital $C^*$ algebra. Then $\mathbb{C} \oplus \mathcal{A}$, equipped with the standard multiplication defined in (1.3), and the norm $\| \cdot \|$ defined in (2.18) is a Banach algebra. If we equip $\mathbb{C} \oplus \mathcal{A}$ with the involution $(\lambda, A)^* = (\overline{\lambda}, A^*)$, then it becomes a Banach $*$ algebra, and the map $A \mapsto (0, A)$ is an isometric $*$-isomorphism of $\mathcal{A}$ into this Banach $*$-algebra.

**2.36 Lemma.** In the notation introduced above, we have the alternate formula

$$\|(\lambda, A)\| = \sup_{\|C\| \leq 1} \{\|\lambda C + CA\|\}. \quad (2.20)$$

*Proof.* Choose $\epsilon > 0$, and choose $B \in \mathcal{A}$, $\|B\| \leq 1$ such that

$$(1 - \epsilon)\|(\lambda, A)\| \leq \|\lambda B + AB\|.$$

Then by the second identity in (2.17), there exists $C \in \mathcal{A}$, $\|C\| \leq 1$, so that

$$(1 - \epsilon)\|\lambda B + AB\| \leq \|\lambda C + CA\|B\| \leq \|\lambda C + CA\|.$$

Since $\epsilon > 0$ is arbitrary, $\sup_{\|C\| \leq 1} \{\|\lambda C + CA\|\} \geq \|(\lambda, A)\| = \sup_{\|B\| \leq 1} \{\|\lambda B + AB\|\}$. The same sort of reasoning then shows that $\sup_{\|B\| \leq 1} \{\|\lambda B + AB\|\} \leq \sup_{\|C\| \leq 1} \{\|\lambda C + CA\|\}$. \qed

We are now ready to prove:
2.37 THEOREM (Vowden’s Theorem). Let $\mathcal{A}$ be a non-unital $C^*$ algebra. Then $\mathbb{C} \oplus \mathcal{A}$, equipped with the standard multiplication defined in (1.3), the norm $\| \cdot \|$ defined in (2.18) and the involution $(\lambda, A)^* = (\overline{\lambda}, A^*)$ is a unital $C^*$ algebra, and the map $A \mapsto (0, A)$ is an isometric \textdaggerdash isomorphism of $\mathcal{A}$ into this unital $C^*$-algebra. In particular, every $C^*$ algebra $\mathcal{A}$ is isometrically and $\star$-isomorphically embedded in a unital $C^*$ algebra.

Proof. We need only prove the $C^*$ algebra identity $\| (\lambda, A)^* (\lambda, A) \| = \| (\lambda, A)^* \| \| (\lambda, A) \|$, and since we already know that $\| (\lambda, A)^* (\lambda, A) \| \leq \| (\lambda, A)^* \| \| (\lambda, A) \|$, it remains to prove that

$$\| (\lambda, A)^* (\lambda, A) \| \geq \| (\lambda, A)^* \| \| (\lambda, A) \|. \quad (2.21)$$

Choose $\epsilon > 0$. By Lemma 2.36, we may choose $B, C \in \mathcal{A}$ with $\| B \|, \| C \| \leq 1$ such that $\| (\lambda, A)^* (\lambda, A) \| \leq (1 + \epsilon) \| \lambda B + BA^* \|$ and $\| (\lambda, A) \| \leq (1 + \epsilon) \| \lambda C + AC \|$. By Theorem 2.32 applied to $\{B, C\}$, for every $\delta > 0$, there exists $E \in \mathcal{A}^+$, with $\| E \| \leq 1$ such that $\| B E - B \|, \| E C - C \| \leq \delta$. For appropriate $\delta$ we then have

$$\| (\lambda, A) \| \leq (1 + \epsilon)^2 \| (\lambda E + AE) C \| \leq (1 + \epsilon)^2 \| \lambda E + AE \| .$$

and

$$\| (\lambda, A)^* \| \leq (1 + \epsilon)^2 \| B (\overline{\lambda} E + E A^*) \| \leq (1 + \epsilon)^2 \| (\lambda E + AE)^* \| .$$

Altogether we have

$$\| (\lambda, A)^* \| \| (\lambda, A) \| \leq (1 + \epsilon)^4 \| (\lambda E + AE)^* \| \| \lambda E + AE \| . \quad (2.22)$$

Then by the $C^*$ algebra identity in $\mathcal{A}$,

$$\| (\lambda E + AE)^* \| \| \lambda E + AE \| = \| (\lambda E + AE)^* (\lambda E + AE) \| = \| (\lambda E + AE)^* (\lambda E + AE) \| \leq \| E (|\lambda|^2 E + (\overline{\lambda} A + \lambda A^* + A^* A) E) \| \leq \| |\lambda|^2 E + (\overline{\lambda} A + \lambda A^* + A^* A) E \| \leq \| (\lambda, A)^* (\lambda, A) \| .$$

Since $\epsilon > 0$ is arbitrary, combining this with (2.22) yields the result.

Vowden’s Theorem has a number of consequences. Our first application of it, in the next section, will be the one that motivated Vowden – to finally prove that the involution is an isometry in full generality. This had been done earlier by Glimm and Kadison in the unital case, and Vowden’s theorem immediately extends this to the general case.

2.6 The Russo-Dye Theorem and the Isometry Property

2.38 DEFINITION (Unitary). Let $\mathcal{A}$ be a unital $C^*$ algebra with multiplicative identity $1$. Then $U \in \mathcal{A}$ is unitary in case

$$U^* U = U U^* = I . \quad (2.23)$$
Note in particular that every unitary $U$ is normal. Therefore, by Lemma 2.8, $\|U^*\| = \|U\|$, and by the $C^*$ algebra norm identity, $1 = \|1\| = \|U^*U\| = \|U^*||U\| = \|U\|^2$. Therefore, $\|U\| = 1$ for all unitaries.

Noe let $A \in B_{\mathcal{A}}$. Define $(A^*A)^{1/2}$ using the Abstract Spectral Theorem. If $A$ is invertible, so are $A^*$ and $A^*A$, and then $(A^*A)^{-1}(A^*A)^{1/2}$ is the inverse of $(A^*A)^{1/2}$ which we denote by $(A^*A)^{-1/2}$. For such $A$, define

$$
|A| := (A^*A)^{1/2} \quad \text{and} \quad V := A(A^*A)^{-1/2}.
$$

Then $V^*V = (A^*A)^{-1/2}A^*A(A^*A)^{-1/2} = 1. \text{ Since } V \text{ is the product of invertible elements of } \mathcal{A}, \text{ } V \text{ is invertible, and hence its left inverse } V^* \text{ is also its inverse. That is } V^*V = VV^* = 1. \text{ In particular, } V \text{ is unitary. This proves the greater part of the following lemma:}$

**2.39 LEMMA.** Let $\mathcal{A}$ be a unital $C^*$ algebra, and let $A \in \mathcal{A}$. If $A$ is invertible, then there is a unitary $V$ and a positive operator $H$ such that $A = VH$. Moreover, $H$ and $V$ are uniquely determined, and are given by $H = |A| := (A^*A)^{1/2}$ and $V = A|A|^{-1}$.

**Proof.** It remains to prove the uniqueness. Suppose that $A = WK$ where $K \in \mathcal{A}^+$ and $W$ is unitary. Then $A^*A = K^2$, and hence $K = (A^*A)^{1/2} = |A|$, and then $W = V$ follows immediately. $\square$

The factorization $A = VH$ provided by Lemma 2.39 is called the polar factorization of $A$.

**2.40 LEMMA.** Let $\mathcal{A}$ be a unital $C^*$ algebra. Then for all invertible $A \in \mathcal{A}$, $\|A^*\| = \|A\|$.

**Proof.** Let $A$ be invertible in $\mathcal{A}$, and let $A = VH$ be its polar factorization. Then

$$
\|A\| = \|VH\| \leq \|V\|\|H\| = \|H\| = \|(A^*A)^{1/2}\| = \|A^*A\|^{1/2} = (\|A^*\||A\|)^{1/2}.
$$

It follows that $\|A\| \leq \|A^*\|$. Since $A^*$ is also invertible, the same reasoning yields $\|A^*\| \leq \|A\|$. $\square$

**2.41 LEMMA** (Gardner’s Lemma). Let $\mathcal{A}$ be a unital $C^*$ algebra. For all invertible $A$ in $B_{\mathcal{A}}$, the unit ball of $A$, there exist two unitaries $U_1$ and $U_2$ such that

$$
A = \frac{1}{2}(U_1 + U_2).
$$

(2.24)

**Proof.** Let $A = VH$ be the polar factorization of $A$. Then since $A$ is invertible $\|H\| = \|(A^*A)^{1/2}\| = \|A^*A\|^{1/2} = \|A\| \leq 1$ by Lemma 2.40. By the Spectral Mapping Theorem, $\sigma_{\mathcal{A}}(H^2) \subset [0,1]$. Then we may define, using the Abstract Spectral Theorem, $W := H + i\sqrt{1-H^2}$. Then

$$
W^*W = WW^* = H^2 + (1 - H^2) = 1,
$$

and hence $W$ and $W^*$ are unitary. Evidently, $H = \frac{1}{2}(W + W^*)$ and then with $U_1 = VW$ and $U_2 = VW^*$, $U_1$ and $U_2$ are unitary and (2.24) is satisfied. $\square$

**2.42 THEOREM** (Kadison-Pederson). Let $n \in \mathbb{N}$, $n \geq 2$. If $A \in \mathcal{A}$ satisfies $\|A\| \leq 1 - \frac{2}{n}$, then there are unitaries $\{U_1, \ldots, U_n\}$ such that

$$
A = \frac{1}{n}\sum_{j=1}^{n} U_j.
$$

(2.25)
Proof. Let $B \in \mathcal{A}$, $\|B\| < 1$ and let $V$ be unitary. Then $\|V^*B\| \leq \|V^*\|\|B\| < 1$. Therefore $1 + V^*B$ is invertible, with the consequence that $V + B$ is invertible. That is, for all $A$ with $\|B\| < 1$ and all unitaries $V$, $V + B$ is invertible. Then $\frac{1}{2}(V + B)$ is an invertible element of $B_{\mathcal{A}}$, and by Gardener’s Lemma, there are unitaries $U_1$ and $U_2$ such that $\frac{1}{2}(V + B) = \frac{1}{2}(U_1 + U_2)$. That is, given any $B \in \mathcal{A}$ with $\|B\| < 1$, and any unitary $V$, there are two other unitaries $U_1$ and $U_2$ such that

$$B + V = U_1 + U_2. \tag{2.26}$$

We now claim that for any $n > 2$, there are unitaries $U_1, \ldots, U_n$ such that

$$\frac{n-2}{n}B = \frac{1}{n} \sum_{j=1}^{n} U_j \tag{2.27}$$

We get the $n = 3$ case of (2.27) from (2.26) if we define $U_3 = -V$. Make the inductive hypothesis that for some $n \geq 3$, (2.27) is valid. Then for any unitary $V$, using both (2.26) and (2.27) after multiplying the latter through by $n$,

$$(n - 1)B = B + (n-2)B = B + \sum_{j=1}^{n} U_j = \sum_{j=1}^{n-1} U_j + B + U_n = \sum_{j=1}^{n-1} U_j + V_1 + V_2$$

with $U_1, \ldots, U_n, V_1, V_2$ unitary. Dividing through by $n + 1$, (2.27) is proved. If $\|A\| < (1 - \frac{2}{n})$, $B := (1 - \frac{2}{n})^{-1}A$ satisfies $\|B\| < 1$. Expressing (2.27) for this $B$ in terms of $A$ yields (2.25) \hfill \square

2.43 COROLLARY (Russo-Dye Theorem). Let $\mathcal{A}$ be a unital $C^*$ algebra. Then $B_{\mathcal{A}}$, the closed unit ball of $\mathcal{A}$, is the norm closure of the convex hull of the unitary elements of $B_{\mathcal{A}}$.

2.44 THEOREM (Isometry property of the involution). Let $\mathcal{A}$ be a $C^*$ algebra, not necessarily unital. Then for all $A \in \mathcal{A}$, $\|A^*\| = \|A\|$.

Proof. First suppose that $\mathcal{A}$ is unital. For $A \in \mathcal{A}$ with $\|A^*\|\|A\| = 1$ if $\|A\| \neq \|A^*\|$, then either $\|A\| < 1$, or $\|A^*\| < 1$. Replacing $A$ by $A^*$ if needed, we may suppose that $\|A\| < 1$ and then $\|A^*\| > 1$. Pick $n \in \mathbb{N}$, so that $\|A\| \leq (1 - 2/n)$. Then there are unitaries $U_1, \ldots, U_n$ such that $A = \frac{1}{n} \sum_{j=1}^{n} U_j$. Then

$$\|A^*\| = \left\| \frac{1}{n} \sum_{j=1}^{n} U_j^* \right\| \leq \frac{1}{n} \sum_{j=1}^{n} \|U_j^*\| = 1.$$ 

This contradiction, and a simple scaling argument, proves the result in this case.

If $\mathcal{A}$ is not unital, Vowden’s Theorem says that $\mathcal{A}$ is a $C^*$ subalgebra of a unital $C^*$-algebra $\mathcal{A}_1$. By what we just proves, the involution is an isometry in $\mathcal{A}_1$, and hence in $\mathcal{A}$. \hfill \square

With the isometry property finally in hand, we may prove that the norm in a $C^*$ algebra $\mathcal{A}$ is uniquely determined by the $*$-algebra itself.

2.45 THEOREM (Uniqueness of the norm). Let $\mathcal{A}$ be a $C^*$ algebra. Then for all $A \in \mathcal{A}$,

$$\|A\|^2 = \nu(A^*A).$$

Proof. By Theorem 2.44, $\|A^*A\| = \|A^*\|\|A\| = \|A\|^2$, and by Lemma 2.8, $\nu(A^*A) = \|A^*A\|$. \hfill \square
2.7 Homeomorphisms of $C^*$ algebras

2.46 THEOREM. Let $\mathcal{A}$ be a $C^*$ algebra, and let $\mathcal{J}$ be a norm-closed ideal in $\mathcal{A}$. Then $\mathcal{J}$ is closed under the involution.

Proof of Theorem 2.46. Let $A \in \mathcal{J}$. By part (1) of Theorem 2.32, there exists a sequence $\{E_n\}_{n \in \mathbb{N}}$ of positive elements of $\mathcal{J}$, with $\|E_n\| \leq 1$ for all $n$, such that $\lim_{n \to \infty} \|AE_n - A\| = 0$. Since the involution is an isometry, $\|E_nA^* - A^*\| = \|AE_n - A\|$, and $E_nA^* \in \mathcal{J}$, $\lim_{n \to \infty} \|E_nA^* - A^*\| = 0$. Then since $\mathcal{J}$ is closed, $A^* \in \mathcal{J}$.

Now let $\mathcal{A}$ be a $C^*$ algebra, and let $\mathcal{J}$ be a closed ideal in $\mathcal{A}$. As usual, let $\{A\}$ denote equivalence class of $A$ mod $\mathcal{J}$, and let $\|\{A\}\|$ denote the quotient norm of $\{A\}$; that is,

$$\|\{A\}\| = \inf\{ \|A - B\| : B \in \mathcal{J} \}.$$ (2.29)

Then $\mathcal{A}/\mathcal{J}$ is a Banach algebra with the quotient norm. By Theorem 2.46, $A - B \in \mathcal{J} \iff A^* - B^* \in \mathcal{J}$, and therefore we may define an involution on on $\mathcal{A}/\mathcal{J}$ by $\{A\}^* = \{A^*\}$. Evidently this involution is an isometry, and so for all $A \in \mathcal{A}$, $\|\{A\}^*\{A\}\| \leq \|\{A\}\|^2$. To show that $\mathcal{A}/\mathcal{J}$ is a $C^*$ algebra with this involution, we need only show that for all $A \in \mathcal{A}$,

$$\|\{A\}\|^2 \leq \|\{A\}^*\{A\}\|.$$ (2.28)

It will be convenient to have a unit. If $\mathcal{A}$ is not unital, let $\mathcal{A}_1$ be the $C^*$ algebra obtained by adjoining a unit as in Vowden’s Theorem. Then $\mathcal{J}$, identified with its canonical embedding into $\mathcal{A}_1$, is still a closed ideal in $\mathcal{A}_1$, and for all $(\lambda, A), (\mu, B)$ in $\mathcal{A}_1$, $(\lambda, A) \sim (\mu, B)$ if and only if $(\lambda - \mu, A - B) \in \mathcal{J}$. Evidently this is the case if and only if $\lambda = \mu$ and $A \sim B \in \mathcal{A}$. In particular for all $A \in \mathcal{A}$, the equivalence class of $(0, A)$ in $\mathcal{A}_1$ is precisely the set of $(0, B)$ with $B \sim A$ in $\mathcal{A}_1$, and $\|\{(0, A)\}\| = \|\{A\}\|$. Hence it suffices to prove (2.29) in unital algebras.

2.47 LEMMA. Let $\mathcal{A}$ be a unital $C^*$ algebra, and let $\mathcal{J}$ be a closed ideal in $\mathcal{A}$. For all $A \in \mathcal{A}$, the quotient norm of $\{A\}$ is given by

$$\|\{A\}\| = \inf\{ \|A - AE\| : E \in \mathcal{J} \text{ and } E \in \mathcal{A}^+ \cap B_{\mathcal{A}} \}.$$ (2.29)

Proof. Whenever $E \in \mathcal{J}$, $A \sim (A - AE)$ so that $\|\{A\}\|$ is no greater than the right hand side of (2.29). To prove the equality, pick $\epsilon > 0$ and $B \in \mathcal{J}$ so that $\|\{A\}\| \geq \|A - B\| - \epsilon$. By Theorem 2.32 applied to $\{B\}$, we can choose $E \in \mathcal{A}^+ \cap B_\mathcal{A}$ so that $\|B - BE\| < \epsilon$. By (2.13), $\|1 - E\| \leq 1$, and so

$$\|A - B\| \geq \|A - B\|\|1 - E\| \geq \|(A - B)(1 - E)\| = \|(A - AE) - (B - BE)\| \geq \|A - AE\| - \|B - BE\| \geq \|A - AE\| - \epsilon.$$ 

Hence $\|\{A\}\| \geq \|A - AE\| - 2\epsilon$, and since $\epsilon > 0$ is arbitrary, (2.29) is proved.

Now to prove (2.28), pick $\epsilon > 0$ and $E = E^* \in \mathcal{J}$ with $E \in \mathcal{A}^+ \cap B_\mathcal{A}$ so that

$$\|A^*A(1 - E)\| \leq \|(1 + \epsilon)\|\{A^*A\}\| = (1 + \epsilon)\|\{A\}^*\{A\}\|.$$ 

Then, again using $\|1 - E\| \leq 1$,

$$\|\{A\}\|^2 \leq \|A(1 - E)\|^2 = \|(1 - E)A^*A(1 - E)\| \leq \|(1 - E)\|\|A^*A(1 - E)\| \leq \|A^*A(1 - E)\| \leq \|A^*A(1 - E)\| \leq (1 + \epsilon)\|\{A\}^*\{A\}\|.$$ (2.30)

Since $\epsilon > 0$ is arbitrary, (2.28) is proved. We have shown:
2.48 THEOREM. Let $\mathcal{A}$ be a $C^*$ algebra, and let $\mathcal{J}$ be a norm closed ideal in $\mathcal{A}$, then $\mathcal{J}$ is closed under the involution, and the definition $\{A\}^* = \{A^*\}$ defines an involution on $\mathcal{A} / \mathcal{J}$ so that, equipped with the quotient norm, $\mathcal{A} / \mathcal{J}$ is a $C^*$ algebra.

2.49 LEMMA. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$ algebras, and let $\pi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism. Then $\pi$ is a contraction; i.e., $\|\pi(A)\| \leq \|A\|$ for all $A \in \mathcal{A}$. If moreover $\pi$ is one-to-one, $\pi$ is an isometry.

Proof. For all $A \in \mathcal{A}$, by the Spectral Contraction Theorem, $\nu(\pi(A)^*\pi(A)) = \nu(\pi(A^*A) \leq \nu(A^*A)$. Since for self adjoint elements of a $C^*$ algebra, the norm is the spectral radius, $\|\pi(A)^*\pi(A)\| \leq \|A^*A\| = \|A\|^2$. Thus, $\|\pi(A)\|^2 \leq \|A\|^2$, and this proves that $\pi$ is a contraction.

Notice that if $\nu(\pi(A^*A)) = \nu(A^*A)$, the argument gives $\|\pi(A)\| = \|A\|$. Hence it remains to show that if $\pi$ is one-to-one, $\pi$ cannot decrease the spectral radius of any self adjoint element of $\mathcal{A}$.

Indeed, let $A = A^* \in \mathcal{A}$, and suppose that $\nu(\pi(A)) < \nu(A)$. Then there is a non-zero continuous bounded function $f$ supported on $[-\nu(A), \nu(A)]$ that vanishes identically on $[-\nu(\pi(A)), \nu(\pi(A))]$, since $f$ may be approximated by polynomials, $\pi(f(A)) = f(\pi(A))$. However, since $f$ vanishes identically on the spectrum of $\pi(A)$, $f(\pi(A)) = 0$. Thus, $f(A)$ is in the kernel of $\pi$, which is a contradiction. Hence, when $\pi$ is one-to-one, it preserves the spectral radius of self adjoint elements.

\[\square\]

We summarize with the following theorem:

2.50 THEOREM (Homomorphisms of $C^*$ algebras). Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$ algebras, and let $\pi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism. Then $\pi$ is a contraction, $\pi(\mathcal{A})$ is a $C^*$-subalgebra of $\mathcal{B}$, and $\pi$ induces an isometric isomorphism of $\mathcal{A} / \ker(\pi)$ onto $\pi(\mathcal{A})$.

3 The strong, the weak and the $\sigma$-weak operator topologies

3.1 The polarization identity in $\mathcal{B}(\mathcal{H})$

Three topologies other than the norm topology are fundamental to what follows. After recalling some basic facts, we introduce the first two of these. Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let $\mathcal{B}(\mathcal{H})$ denote the $C^*$-algebra of bounded linear operators on $\mathcal{H}$. (We may write $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\| \cdot \|_{\mathcal{X}}$ for specificity if there is more than one hilbert space under consideration.) For $A \in \mathcal{B}(\mathcal{H})$, $\text{ran}(A)$ denote the closure of the range of $A$, and $\ker(A)$ denotes the null space of $A$. If $\mathcal{X}$ is a subspace of $\mathcal{H}$, $\mathcal{X}^\perp$ is the orthogonal complement of $\mathcal{X}$, which is a closed subspace of $\mathcal{H}$.

3.1 LEMMA (Polarization identify). For all $A \in \mathcal{B}(\mathcal{H})$ and all $\zeta$ and $\xi$ in $\mathcal{H}$,

$$
\langle \zeta, Ax \rangle = \frac{1}{4} \left[ \langle (\zeta + \xi), A(\zeta + \xi) \rangle - \langle (\zeta - \xi), A(\zeta - \xi) \rangle 
- \frac{i}{4} \left[ \langle (\zeta + i\xi), A(\zeta + i\xi) \rangle - \langle (\zeta - i\xi), A(\zeta - i\xi) \rangle \right] \right].
$$

Proof. We compute

$$
\frac{1}{4} \left[ \langle (\zeta + \xi), A(\zeta + \xi) \rangle - \langle (\zeta - \xi), A(\zeta - \xi) \rangle \right] = \frac{1}{2} \left[ \langle \zeta, Ax \rangle + \langle \xi, Ax \rangle \right].
$$
and
\[
\frac{1}{4} \left[ \langle (\zeta + i\xi), A(\zeta + i\xi) \rangle - \langle (\zeta - i\xi), A(\zeta - i\xi) \rangle \right] = \frac{i}{2} \left[ \langle \zeta, A\xi \rangle - \langle \xi, A\xi \rangle \right].
\]

\[\square\]

3.2 COROLLARY. Let \( A \in \mathcal{B}(\mathcal{H}) \) be such that \( \langle \xi, A\xi \rangle \in \mathbb{R} \) for all \( \xi \in \mathcal{H} \). Then \( A = A^* \).

Proof. Since each of the four inner products on the right in the polarization identity are real, \( \langle \xi, A\xi \rangle = \overline{\langle \xi, A\xi \rangle} \). By the definition of the inner product, \( \langle \xi, A\xi \rangle = \langle A\xi, \xi \rangle \), and so \( \langle \xi, A\xi \rangle = \langle A\xi, \xi \rangle = \langle \xi, A^*\xi \rangle \).

\[\square\]

3.3 DEFINITION. An element of \( \mathcal{B}(\mathcal{H}) \) is positive in case \( \langle \xi, A\xi \rangle \geq 0 \) for all \( \xi \in \mathcal{H} \), and is strictly positive in case \( \langle \xi, A\xi \rangle > 0 \) when \( \xi \neq 0 \). The positive elements of \( \mathcal{B}(\mathcal{H}) \) are denoted by \( \mathcal{B}(\mathcal{H})^+ \).

Since \( \mathcal{B}(\mathcal{H}) \) is a unital \( C^* \) algebra, we have another definition of what it means for \( A \in \mathcal{B}(\mathcal{H}) \) to be positive, namely that \( A \) is self adjoint and the spectrum of \( A \) lies in \([0, \infty)\). In other words, that \( A \) is self adjoint, and \( \epsilon I + A \) is invertible for all \( \epsilon > 0 \). We have just seen that when \( A \) is positive in the sense of Definition 3.3, then \( A \) is self adjoint. Next, for any \( \epsilon > 0 \), define \( B = \epsilon I + A \).

Then for all \( \xi \in \mathcal{H} \), \( \langle \xi, B\xi \rangle \geq \epsilon \|\xi\|^2 \). Fix and \( \zeta \in \mathcal{H} \), and define the function
\[
\Phi_\eta(\xi) = -2\Re(\langle \xi, \zeta \rangle) + \langle \xi, B\xi \rangle.
\]
A simple variation on the proof of the Projection Lemma shows that there exists \( \xi_0 \) in \( \mathcal{H} \) such that \( \Phi_\eta(\xi_0) < \Phi_\eta(\xi) \) for all \( \xi \neq \xi_0 \). Differentiating, for all \( \eta \in \mathcal{H} \),
\[
0 = \frac{d}{dt} \Phi_\eta(\xi_0 + t\eta)|_{t=0} = -2\Re(\langle \eta, \zeta \rangle) + 2\Re(\langle \eta, B\xi_0 \rangle),
\]
where we have used the self adjointness of \( B \) in the final term. Replacing \( \eta \) by \( i\eta \), we conclude that \( \langle \eta, \zeta \rangle \geq \langle \eta, B\xi_0 \rangle \) for all \( \eta \), and hence \( B\xi_0 = \zeta \). If also \( B\xi_1 = \zeta \), then \( 0 = \langle \xi_1 - \xi_0, B(\xi_1 - \xi_0) \rangle \geq \epsilon \|\xi_1 - \xi_0\|^2 \), and so \( \xi_1 - \xi_0 \).

Finally,
\[
\epsilon \|\xi_0\|^2 \leq \langle \xi_0, B\xi_0 \rangle = \langle \xi_0, \zeta \rangle \leq \|\xi_0\|\|\zeta\|,
\]
and so \( \|\xi_0\| \leq \epsilon^{-1}\|\zeta\| \). Thus \( B \) is invertible, \( B^{-1}\zeta = \xi_0 \), and \( \|B^{-1}\| \leq \epsilon^{-1} \).

The converse is clear: If \( A \) is self adjoint and has positive spectrum, then \( A^{1/2} \) is well-defined, self adjoint and \( A = A^{1/2}A^{1/2} \). Then \( \langle \xi, A\xi \rangle = \|A^{1/2}\xi\|^2 \geq 0 \).

3.4 REMARK. The fact that for all \( \epsilon > 0 \) and all \( A \in \mathcal{B}(\mathcal{H}) \), the operator \( \epsilon I + A \) is invertible and \( \|(\epsilon I + A)^{-1}\| \leq \epsilon^{-1} \) is the Hilbert-space version of the Lax-Milgram Theorem. Since for any \( A \in \mathcal{B}(\mathcal{H}) \), and any \( \xi \in \mathcal{H} \), \( \langle \xi, A^*A\xi \rangle = \langle A\xi, A\xi \rangle \geq 0 \), \( A^*A \) is positive (in any of the equivalent senses), and then by what we have just seen 1 + \( A^*A \) is invertible. Hence, in \( \mathcal{B}(\mathcal{H}) \), hypothesis (vi) of Gelfand and Neumark is valid, but this simple proof of it uses the vectors on which elements of \( \mathcal{B}(\mathcal{H}) \) operate, and not only the algebra itself.
3.2 The strong and weak operator topologies

The strong and weak operator topologies were introduced by von Neumann [21] in 1930:

3.5 DEFINITION (Strong and weak operator topologies). The strong operator topology on $\mathcal{B}(\mathcal{H})$ is the weakest topology such that for each $\xi \in \mathcal{H}$, the function $A \mapsto A\xi$ from $\mathcal{B}(\mathcal{H})$ to $\mathcal{H}$ is continuous with the usual norm topology on $\mathcal{H}$. The weak operator topology on $\mathcal{B}(\mathcal{H})$ is the weakest topology such that for each $\xi,\zeta \in \mathcal{H}$, the function $A \mapsto \langle \zeta, A\xi \rangle$ is continuous from $\mathcal{B}(\mathcal{H})$ to $\mathbb{C}$.

It follows from the definitions that a basic set of neighborhoods of 0 for the strong operator topology is given by the sets

$$U_{\epsilon,\xi_1,\ldots,\xi_n} = \{ A \in \mathcal{B}(\mathcal{H}) : \| A\xi_j \| < \epsilon \quad \text{for} \quad j = 1,\ldots,n \}$$

(3.1)

where $\epsilon > 0$ and $\xi_1,\ldots,\xi_n \in \mathcal{H}$. Likewise, it follows that a basic set of neighborhoods of 0 for the weak operator topology is given by the sets

$$V_{\epsilon,\xi_1,\ldots,\xi_n} = \{ A \in \mathcal{B}(\mathcal{H}) : |\langle \xi_j, A\xi_j \rangle| < \epsilon \quad \text{for} \quad j = 1,\ldots,n \}$$

(3.2)

$\epsilon > 0$ and $\xi_1,\ldots,\xi_n \in \mathcal{H}$. Both topologies are evidently Hausdorff. Since for all $A \in \mathcal{B}(\mathcal{H})$ and all $\xi \in \mathcal{H}$, $|\langle \xi, A\xi \rangle| = |\langle \xi, A^*\xi \rangle|$, $V_{\epsilon,\xi_1,\ldots,\xi_n} = V_{\epsilon,\xi_1,\ldots,\xi_n}^{*}$, and hence $A \mapsto A^*$ is continuous in the weak operator topology.

3.6 LEMMA. The weak operator topology on $\mathcal{B}(\mathcal{H})$ is the weakest topology such that for each $\xi, \zeta \in \mathcal{H}$, the function $A \mapsto \langle \zeta, A\xi \rangle$ is continuous from $\mathcal{B}(\mathcal{H})$ to $\mathbb{C}$.

Proof. This follows immediately from the polarization identity. \hfill \square

Since for each $\xi \in \mathcal{H}$, $A \mapsto A\xi$ is continuous in the norm topology on $\mathcal{B}(\mathcal{H})$, the norm topology is at least as strong as the strong operator topology. Similarly, for all $\xi \in \mathcal{H}$, the function $A \mapsto \langle \xi, A\xi \rangle$ is continuous in the strong operator topology, and hence the strong operator topology is at least as strong as the weak operator topology.

The following proposition shows that the norm topology is strictly stronger than the strong operator topology, which is in turn strictly stronger than the weak operator topology.

3.7 PROPOSITION (Continuity of the norm and adjoint). Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then:

1. The function $A \mapsto \| A \|$ from $\mathcal{B}(\mathcal{H})$ to $\mathbb{R}_+$ is continuous in the norm topology, but is only lower semicontinuous in the strong and weak operator topologies.

2. The function $A \mapsto A^*$ is continuous from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$ in the norm and the weak operator topologies, but not in the strong operator topology.

Proof. Let $\{ \xi_j \}$ be an orthonormal sequence in $\mathcal{H}$. For $n \in \mathbb{N}$, let $P_n$ denote the orthogonal projection onto the span of $\{ \xi_1,\ldots,\xi_n \}$. For all $\xi \in \mathcal{H}$, $\lim_{n \to \infty} \| P_n^\perp \xi \| = 0$ by Bessel’s inequality, so that $\lim_{n \to \infty} P_n^\perp = 0$ in the strong operator topology. Since for $n \neq m$, $\| P_n^\perp - P_n^\perp \| = 1$, the sequence $\{ P_n \}$ is not even Cauchy in the norm topology. Hence the norm is discontinuous in the strong operator topology, and then also in the weak operator topology.
To see that the norm is lower semicontinuous in the weak operator topology, we must show that the sub-level sets \( \{ A \in \mathcal{B}(\mathcal{H}) : \| A \| \leq t \} \) are weakly closed for each \( t > 0 \). Suppose that \( B \) is in the weak operator topology closure of \( \{ A \in \mathcal{B}(\mathcal{H}) : \| A \| \leq t \} \). Then for each pair \( \eta, \xi \) of unit vectors in \( \mathcal{H} \), and each \( n \geq 2 \) there is an \( A_n \in \{ A \in \mathcal{B}(\mathcal{H}) : \| A \| \leq t \} \) such that \( B - A_n \in \{ X \in \mathcal{B}(\mathcal{H}) : |\langle \eta, X\xi \rangle| < 1/n \} \). If \( \eta \) and \( \xi \) are such that \( |\langle \eta, B\xi \rangle| \geq (1 - 1/n)\| B \| \), then
\[
\| B \| \leq (1 - 1/n)^{-1} |\langle \eta, B\xi \rangle| \leq \frac{n}{n - 1} \left( |\langle \eta, A_n\xi \rangle| + \frac{1}{n} \right) \leq \frac{n}{n - 1} + \frac{1}{n}.
\]
Since \( n \geq 2 \) is arbitrary, \( \| B \| \leq t \). This proves the closure in the weak operator topology, and hence also in the strong operator topology.

For the second part, the continuity of the involution is obvious in the norm topology and we have seen it is true in the weak operator topology. To see that the involution is not continuous for the strong operator topology when \( \mathcal{H} \) is infinite dimensional, note that every such Hilbert space contains a copy of \( \ell_2 \), the Hilbert space of all square summable functions from \( \mathbb{N} \) to \( \mathbb{C} \). We may suppose without loss of generality that \( \mathcal{H} = \ell_2 \). Define the shift operator \( a \in \mathcal{B}(\mathcal{H}) \) by
\[
(A\zeta)_j = \begin{cases} 
\zeta_{j-1} & j \geq 2 \\
0 & j = 1 
\end{cases}
\]
Evidently, for all \( \zeta \), \( \| A\zeta \| = \| \zeta \| \). The adjoint is given by \((A^*\zeta)_j = \zeta_{j+1}\) for all \( j \in \mathbb{N} \). Therefore, \( \| A^*\zeta \|^2 = \sum_{j=2}^{\infty} |\zeta_j|^2 = \| \zeta \|^2 - |\zeta_1|^2 \). It follows that for all \( \zeta \in \mathcal{H} \),
\[
\lim_{n \to \infty} \|(A^n)^*\zeta\| = 0 \quad \text{while} \quad \| A^n\zeta \| = \| \zeta \| .
\]
Hence the sequence \((A^n)^*\} \) converges to zero in the strong operator topology, but the sequence \( \{ A^n \} \) does not.

\[ \square \]

3.8 THEOREM (Continuous linear functions for the strong operator topology). Let \( \mathcal{H} \) be a Hilbert space, and let \( \varphi \) be a linear functional on \( \mathcal{B}(\mathcal{H}) \) that is continuous in the strong operator topology. Then there exists \( n \in \mathbb{N} \) and two sets of vectors \( \{ \zeta_1, \ldots, \zeta_n \} \) and \( \{ \xi_1, \ldots, \xi_n \} \) such that for all \( A \in \mathcal{B}(\mathcal{H}) \),
\[
\varphi(A) = \sum_{j=1}^{n} \langle \zeta_j, A\xi_j \rangle .
\] (3.3)

Evidently, every such linear functional is weakly continuous, and hence every strongly continuous linear functional is weakly continuous. Consequently, a convex subset of \( \mathcal{B}(\mathcal{H}) \) is strongly closed if and only if it is weakly closed.

Proof. If \( \varphi \) is strongly continuous, then \( \varphi^{-1}(\{ \lambda : |\lambda| < 1 \}) \) contains a neighborhood of 0 in \( \mathcal{B}(\mathcal{H}) \). Thus, there exists an \( \epsilon > 0 \) and a set of \( n \) vectors \( \xi_1, \ldots, \xi_n \), which without loss of generality we may assume to be orthonormal, such that if \( \| A\xi_j \| < \epsilon \) for \( j = 1, \ldots, n \), \( |\varphi(A)| < 1 \). Note that if \( A\xi_j = 0 \) for \( j = 1, \ldots, n \), then \( t \) \( t\| A\xi_j \| < \epsilon \) for \( j = 1, \ldots, n \), and consequently \( t|\varphi(A)| < 1 \). It follows that
\[
A\xi_j = 0 \quad \text{for} \quad j = 1, \ldots, n \quad \Rightarrow \quad \varphi(A) = 0 .
\] (3.4)
For any $A \in \mathcal{B}(\mathcal{H})$, define $\widehat{A}$ by $\widehat{A} = \sum_{j=1}^{n} A\xi_j(\xi_j, \cdot)$. Evidently $(A - \widehat{A})\xi_j = 0$ for $j = 1, \ldots, n$, and hence by (3.4),

$$
\varphi(A) = \varphi(\widehat{A}) = \sum_{j=1}^{n} \varphi((A\xi_j)(\xi_j, \cdot)).
$$

(3.5)

For each fixed $j$, and any $\eta \in \mathcal{H}$, consider the rank-one operator $\eta(\xi_j, \cdot)$. Then $\eta \mapsto \varphi(\eta(\xi_j, \cdot))$ is a bounded linear functional on $\mathcal{H}$, and therefore by the Riesz Representation Theorem, there is a vector $\zeta_j \in \mathcal{H}$ such that $\langle \zeta_j, \eta \rangle = \varphi(\eta(\xi_j, \cdot))$ for all $\eta \in \mathcal{H}$. Combining this with (3.5) yields (3.3). The final statement is a standard application of the Hahn-Banach Theorem.

3.9 Lemma. When $\mathcal{H}$ is infinite dimensional, the product map, $(A,B) \mapsto AB$ from $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$ is not jointly continuous in the strong operator topology, through its restriction to bounded subsets of $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$.

Proof. Were the product map continuous, it would be continuous at the origin, and then since for all $\xi \in \mathcal{H}$, $U := \{X \in \mathcal{B}(\mathcal{H}) : \|X\xi\| < 1\}$ is strongly open, there would exist strongly open neighborhoods $V, W$ of 0 in $\mathcal{B}(\mathcal{H})$ such that for all $Y \in V$ and all $Z \in W$, $YZ \in U$. This is impossible when $\mathcal{H}$ is infinite dimensional: $V$ would contain the set $\{Y_0 \eta_j : j = 1, \ldots, m\}$ for some set $\{\eta_1, \ldots, \eta_m\} \subset \mathcal{H}$. Choose a unit vector $\eta_0 \in \{\eta_1, \ldots, \eta_m\}^\perp$. Then for all $\zeta \in \mathcal{H}$, the operator $Y_0 := \zeta(\eta_0, \cdot) \in V$ and $Y_0\eta_j = 0$ for each $j$. Since $W$ is open in the norm topology, $W$ contains the open ball of some radius $r > 0$, and hence there is some $Z_0 \in W$ such that $\|Y_0Z_0\xi\| = c > 0$, and then $\|Y_0Z_0\xi\| = c\|\zeta\|$ which can be arbitrarily large since $\zeta$ is arbitrary.

Now let $S$ be a bounded subset of $\mathcal{B}(\mathcal{H})$; suppose that $\|A\| \leq r$ for all $A \in S$. Again, for any $\xi \in \mathcal{H}$, let $U$ defined to be $U := \{X \in \mathcal{B}(\mathcal{H}) : \|X\xi\| < 1\}$. Let $A_0 \in S$ and $B_0 \in \mathcal{B}(\mathcal{H})$. It suffices to show that there exist strongly open neighborhoods $V, W$ of 0 in $\mathcal{B}(\mathcal{H})$ such that for all $Y \in V \cap S$ and all $Z \in W$, $(A_0 + Y)(B_0 + Z) \in U + A_0B_0$, since then taking intersections we get the general strong basic neighborhood of 0. For any $Z \in \mathcal{B}(\mathcal{H})$ and any $Y \in S$,

$$
\|(A_0 + Y)(B_0 + Z)\xi - A_0B_0\xi\| \leq \|YB_0\xi\| + \|A_0Z\xi\| + \|YZ\xi\| \\
\leq \|YB_0\xi\| + \|A_0\|\|Z\xi\| + \|Y\|\|Z\xi\| \leq \|YB_0\xi\| + 2r\|Z\xi\|
$$

We may take $V := \{Y : \|Y(B_0\xi\| < \frac{1}{3}\}$ and $W := \{Z : \|Z\xi\| < \frac{1}{3}r\}$. 

3.3 The Baire functional calculus

If $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert spaces, a unitary transformation $U$ from $\mathcal{H}$ to $\mathcal{K}$ is a linear bijection from $\mathcal{H}$ onto $\mathcal{K}$ such that for all $\xi \in \mathcal{H}$, $\|U\xi\|_\mathcal{K} = \|\xi\|$. Evidently, the inverse of $U$ is also unitary and is equal to $U^*$. In this case the map

$$
A \mapsto UAU^*
$$

is an isometric $*$-isomorphism of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{K})$. 

\qed
Let $A \in \mathcal{B}(\mathcal{H})$ be self adjoint, and let $\eta$ be a non-zero vector in $\mathcal{H}$. Define

$$\mathcal{H}_\eta = \text{Span}(\{A^n\eta|_{n\geq 0}\}) .$$

That is, $\mathcal{H}_\eta$ is the norm closure of the span of the polynomials in $\mathcal{A}$, and indeed of the polynomials with rational coefficients - a countable set. Hence $\mathcal{H}_\eta$ is separable even if $\mathcal{H}$ is not.

Evidently, $\mathcal{H}_\eta$ is invariant under $\mathcal{C}((1, A])$. Our goal is to describe the closure of $\mathcal{C}((1, A])$ in the strong operator topology. Recall the the spectrum of $A$ in $\mathcal{C}((1, A])$ is the same as the spectrum of $A$ in $\mathcal{B}(\mathcal{H})$, and in this section we simply write $\sigma(A)$ to denote this spectrum.

Let $(X, \mathcal{F}, \mu)$ be a measure space, and consider the Hilbert space $\mathcal{H} := L^2(X, \mathcal{F}, \mu)$. An operator $T \in \mathcal{B}(\mathcal{H})$ is a multiplication operator, and a very simple one at that. Let $\mathcal{A}$ be self adjoint. Let $\eta$ be any non-zero vector in $\mathcal{H}$. Let $\mathcal{H}_\eta$ be defined by (3.6). Then there is a Borel measure $\mu_\eta$ of total mass $\|\eta\|^2$ such that the map $f \mapsto f(A)\eta$, $f \in \mathcal{C}(\sigma(A))$ satisfies

$$\|f(A)\eta\|^2 = \int_{\sigma(A)} |f(\lambda)|^2 d\mu_\eta(\lambda) .$$

(3.7)

Let $\mathcal{K}_{\eta}$ denote the Hilbert space $L^2(\sigma(A), \mathcal{B}(\sigma(A)), \mu_\eta)$. The isometry $f \mapsto f(A)\eta$ extends to a unitary transformation $U_\eta$ mapping $\mathcal{K}_{\eta}$ onto $\mathcal{H}_\eta$. Define $\hat{A}$ to be the operator on $\mathcal{H}_\eta$ given by $\hat{A} = U_\eta^*AU_\eta$. Then for all $\psi \in \mathcal{H}_\eta$, $\mathcal{C}(\sigma(A))$ satisfies

$$\hat{A}\psi(\lambda) = \lambda\psi(\lambda) .$$

(3.8)

For a bounded Borel function $f$ on $\sigma(A)$, define the operator $f(\hat{A})$ on $\mathcal{H}_\eta$ by

$$f(\hat{A})\psi(\lambda) = f(\lambda)\psi(\lambda)$$

(3.9)

for all $\phi \in \mathcal{H}_\eta$. Then when $f \in \mathcal{C}(\sigma(A))$, $U_\eta f(\hat{A})U_\eta^* = f(A)$, where $f(A)$ is given by the Abstract Spectral Theorem.

**Proof.** Define a linear functional $\mu_\eta$ on $\mathcal{C}(\sigma(A))$ through

$$\mu_\eta(f) = \langle \eta, f(A)\eta \rangle .$$

(3.10)

Then $\mu$ is a positive linear functional with $\mu_\eta(1) = \|\eta\|^2$. By the Reisz-Markoff Theorem, there is a positive Borel measure on $\sigma(A)$ of total mass $\|\eta\|^2$, also denoted by $\mu_\eta$, so that for all $f \in \mathcal{C}(\sigma(A))$,

$$\mu_\eta(f) = \int_{\sigma(A)} f d\mu_\eta .$$

(3.11)

Combining (3.10) and (3.11), we conclude that for all $f \in \mathcal{C}(\sigma(A))$, $\langle \eta, f(A)\eta \rangle_{\mathcal{H}} = \int_{\sigma(A)} f d\mu_\eta$

Define an operator $U_\eta : \mathcal{C}(\sigma(A)) \rightarrow \mathcal{H}_\eta$ by $U_\eta : f \mapsto f(A)\eta$ where $f(A)$ is defined using the Abstract Spectral Theorem. Then, using the $\ast$-homomorphism property of $f \mapsto f(A)$,

$$\langle U_\eta f, U_\eta f \rangle = \langle f(A)\eta, f(A)\eta \rangle = \langle \eta, |f|^2(A)\eta \rangle = \int_{\sigma(A)} |f(\lambda)|^2 d\mu_\eta .$$
Therefore, $U_\eta$ is an isometry from on a dense set in $\mathcal{H}_\eta$ whose range, which includes $p(A)\eta$ for all polynomials $p$, is dense in $\mathcal{H}_\eta$. Hence $U_\eta$ extends to a unitary operator from $\mathcal{H}_\eta$ onto $\mathcal{H}_\eta$.

For any polynomial $p$, define $q(\lambda) = \lambda p(\lambda)$, Then

$$U_\eta q = Ap(A)\eta = AU_\eta p \quad \text{and hence} \quad U_\eta^* AU_\eta p = q.$$ 

This proves (3.8) when $\psi$ is a polynomial, and since polynomial are dense in $\mathcal{H}$, (3.8) holds for all $\psi \in \mathcal{H}$.

More generally, for $f \in \mathcal{C}(\sigma(A))$, and $p$ any polynomial,

$$U_\eta f p = f(A)p(A)\eta = f(A)U_\eta p \quad \text{and hence} \quad U_\eta^* f(A)U_\eta p = f(A)p = f(\hat{A})p ,$$

using the definition (3.9) of the multiplication operator $f(\hat{A})$. Again, since polynomials are dense in $\mathcal{H}$, this proves that $U_\eta f(\hat{A})U_\eta^* = f(A)$ when $f \in \mathcal{C}(\sigma(A))$.

Recall that the Baire functions on a topological space $(X, \mathcal{O})$ are the smallest class of (complex valued) functions that is closed under the operation of taking pointwise limits of sequences, and which contains all of the continuous functions. The Baire class 1 functions are the functions $f$ on $X$ such that for some sequence $\{f_n\}$ of continuous functions on $X$, $f(x) = \lim_{n \to \infty} f_n(x)$ for all $x \in X$. The Baire class 2 functions are the functions $f$ on $X$ such that for some sequence $\{f_n\}$ of Baire class 1 functions on $X$, $f(x) = \lim_{n \to \infty} f_n(x)$ for all $x \in X$, and so forth. The full class of Baire functions $X$ is obtained by transfinite induction, but we shall not need this. For our purposes, Baire class 1 functions suffice.

Let $A$ be a self adjoint operator in $\mathfrak{B}(\mathcal{H})$. Note that the Baire class 1 functions form a *-algebra on $\sigma(A)$ with the usual operations. Let $f$ be any uniformly bounded Baire class 1 function on $\sigma(A)$; say $|f(\lambda)| \leq K$ for all $\lambda \in \sigma(A)$. Since $f$ is Borel measurable, the operator on $f(\hat{A})$ may be defined by (3.9).

Let $\{f_n\}$ be a sequence of continuous functions on $\sigma(A)$ converging pointwise to $f$. Without loss of generality, we may suppose that for all $n$, $|f_n(\lambda)| \leq K$ for all $\lambda \in \sigma(A)$. Then

$$\lim_{n \to \infty} \int_{\sigma(A)} |f_n(\lambda) - f(\lambda)|^2 d\mu_\eta = 0$$

by the Lebesgue Dominated Convergence Theorem, and so

$$0 = \lim_{n \to \infty} \|U_\eta f_n - U_\eta f\|_{\mathcal{H}} = \lim_{n \to \infty} \|f_n(A)\eta - U_\eta f\|_{\mathcal{H}} .$$

In particular, for all non-zero $\eta \in \mathcal{H}$,

$$\lim_{n \to \infty} f_n(A)\eta = U_\eta f , \quad (3.12)$$

the limit exists in the norm topology on $\mathcal{H}$, and the limit depends only on $f$, $\eta$ and $A$, and not on the approximating sequence $\{f_n\}$.

**3.11 DEFINITION.** Let $A$ be a self adjoint operator in $\mathfrak{B}(\mathcal{H})$, and let $f$ be a uniformly bounded Baire class 1 function on $\sigma(A)$. For all non-zero $\eta \in \mathcal{H}$, define

$$f(A)\eta := U_\eta f , \quad (3.13)$$

and for $\eta = 0$, set $f(A)\eta = 0$. 

As the strong limit of a sequence of uniformly bounded linear operators on $\mathcal{H}$, $f(A) \in \mathcal{B}(\mathcal{H})$, and moreover, it belongs to the strong closure of $\mathcal{C}(\{1, A\})$.

3.12 THEOREM. For all uniformly bounded Baire class 1 functions $f$ on $\sigma(A)$, the map

$$\eta \mapsto f(A)\eta$$

defined in (3.13) is a bounded linear transformation on $\mathcal{H}$ that is in the strong operator closure of $\mathcal{C}(\{1, A\})$. The map $f \mapsto f(A)$ from the $*$-algebra of uniformly bounded Baire class 1 function on $\sigma(A)$ into $\mathcal{B}(\mathcal{H})$ is a norm-reducing $*$-homomorphism, and moreover it is order preserving: If $f$ and $g$ are real valued functions with $f(\lambda) \geq g(\lambda)$ for all $\lambda \in \sigma(A)$, then $f(\lambda) - g(\lambda) \geq 0$ in $\mathcal{B}(\mathcal{H})$.

Proof. The first statement has already been proved. Let $f, g$ be uniformly bounded Baire class 1 functions $f$ on $\sigma(A)$. Let $\{f_n\}, \{g_n\}$ be uniformly bounded sequence of continuous functions on $\sigma(A)$ converging pointwise to $f$ and $g$ respectively. Since $\lim_{n \to \infty} f_n g_n(\lambda) = f(\lambda)g(\lambda)$ for all $\lambda$, and since $f_n(A)g_n(A) = f_n g_n(A)$ for all $n$, $f_n g(A) = f g(A)$. Next, let $g$ and $h$ be the real and imaginary parts, respectively, of a uniformly bounded Baire class 1 function, so that $f(\lambda) = g(\lambda) + ih(\lambda)$ for all $\lambda \in \sigma(A)$. Let $\{h_n\}, \{g_n\}$ be uniformly bounded sequence of continuous functions on $\sigma(A)$ converging pointwise to $h$ and $g$ respectively. Then $f(n) = g(n) + ih(n) = \lim_{n \to \infty} (h_n(A) + ig_n(A))$, and $\overline{f}(A) = g(A) - ih(A) = \lim_{n \to \infty} (h_n(A) - ig_n(A))$. Evidently, $\overline{f}(A) = (f(A))^*$. The linearity of $f \mapsto f(A)$ is even simpler, and is left as an exercise. Hence $f \mapsto f(A)$ is a $*$-homomorphism from the space of Baire class 1 functions $f$ on $\sigma(A)$, equipped with the usual operations, to $\mathcal{B}(\mathcal{H})$.

Finally, if $f$ and $g$ are real Baire class 1 functions $f$ on $\sigma(A)$, and $f - g \geq 0$ on $\sigma(A)$, define $h = \sqrt{f - g}$, and note that $h$ is a real Baire class 1 functions $f$ on $\sigma(A)$. By what we have just proved, $f(A) - g(A) = (h(A))^2 = (h(A))^*h(A) \geq 0$.

The $*$-homomorphism provided by Theorem 3.12 need not be an isomorphism. The following example is useful elsewhere: Let $\lambda_0 \in \sigma(A)$ and consider the function $1_{\lambda_0}$ given by $1_{\lambda_0}(\lambda) = 1$ for $\lambda = \lambda_0$ and zero otherwise. Then for all $\lambda, \lambda_01_{\lambda_0}(\lambda) = \lambda 1_{\lambda_0}(\lambda)$. By the $*$-homomorphism property,

$$\lambda_01_{\lambda_0}(A) = A1_{\lambda_0}(A) .$$

It follows that any non-zero vector in the range of $1_{\lambda_0}(A)$ is an eigenvector of $A$ with eigenvalue $\lambda_0$, and conversely any such eigenvector $\xi$ is in the range of $1_{\lambda_0}(A)$ as one sees by considering a continuous approximation $\{f_n\}$ to $1_{\lambda_0}$. Now consider $\mathcal{H} = L^2([0, 1])$ with respect to Lebesgue measure. Let $A$ be the multiplication operator $A\psi(t) = t\psi(t)$. Then it is easy to see that $A$ is self adjoint and $\sigma(A) = [0, 1]$, but $A$ has no eigenvectors at all. Hence for each $\lambda_0 \in [0, 1], 1_{\lambda_0}(A)$ is the zero operator.

Next, let $E$ be an interval of the form $(a, b)$, $[a, b]$, $(a, b]$ or $[a, b)$. Then $1_E$ is easily seen to be a Baire class 1 function on $\mathbb{R}$, and hence on $\sigma(A)$ for any self adjoint $A$. Hence $1_E(A)$ is well defined and belongs to the strong operator topology closure of $\mathcal{C}(\{1, A\})$. Moreover, by the $*$-isomorphism established above, $1_E(A) = (1_E(A))^*$ and $(1_E(A))^2 = 1_E(A)$. Hence $1_E(A)$ is an orthogonal projection. Moreover, if $f$ is any Baire class 1 function with support in $E$, for all $\eta \in \mathcal{H}$,

$$1_E(A)(f(A)\eta) = (1_E f(A))\eta = f(A)\eta ,$$

while if the support of $F$ lies in $E^c$,

$$1_E(A)(f(A)\eta) = (1_E f(A))\eta = 0 .$$
3.13 DEFINITION. Let $A \in \mathcal{B}(\mathcal{H})$ be self adjoint. The set of operators of the form $1_E(A)$, where $E \subset \mathbb{R}$ is such that $1_E$ is a Baire class 1 function, is the set of spectral projections of $A$

In particular, let $E = (-\infty, 0) \cup (0, \infty)$. Then $1_E(A)$ is a spectral projection, and by what we have noted above, $1_E(A)\eta = 0$ if and only if $A\eta = 0$. That is, $1_E(A)$ is the orthogonal projection onto the orthogonal complement of $\ker(A)$, or what is the same, the orthogonal projection onto the closure of the range of $A$. Combining the result obtained in this section, we have:

3.14 LEMMA. $A \in \mathcal{B}(\mathcal{H})$ be self adjoint. Then the orthogonal projection onto the closure of the range of $A$ belongs to the strong operator topology closure of $\mathcal{C}\{\{1, A\}\}$.

Let $P_1$ and $P_2$ be two orthogonal projections in $\mathcal{B}(\mathcal{H})$ with ranges $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Then $P_1 \lor P_2$ denotes the orthogonal projection onto $\mathcal{K}_1 + \mathcal{K}_2$, and $P_1 \land P_2$ denotes the orthogonal projection onto $\mathcal{K}_1 \cap \mathcal{K}_2$.

3.15 THEOREM. Let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the strong operator topology. Then $\mathcal{A}$ is the norm closure of the span of the orthogonal projections contained in $\mathcal{A}$. Moreover, if $P_1$ and $P_2$ are two orthogonal projections in $\mathcal{A}$, then $P_1 \lor P_2$ and $P_1 \land P_2$ both belong to $\mathcal{A}$.

Proof. Let $A \in \mathcal{A}$ be self adjoint. Fix $n \in \mathbb{N}$, and define $E_j = [j/h, (j + 1)/n)$. Let $k \in \mathbb{N}$, $k \geq \|A\|$. Define $f_n(\lambda) = \sum_{j=-k}^{k} \frac{j}{n} 1_{E_j}(\lambda)$. Define $f(\lambda) = \lambda$. Then $\|f_n(\lambda) - f(\lambda)\| \leq 1/n$, and $A = f(A)$. Hence $\|A - f_n(A)\| \leq 1/n$, and $A$ is a finite linear combination of the orthogonal projections $E_j(A)$, which belong to $\mathcal{A}$ on account of its closure in the strong operator topology. Hence every self adjoint element of $\mathcal{A}$, is the norm closure of the span of the orthogonal projections contained in $\mathcal{A}$, and since $\mathcal{A}$ is a * algebra, every element in it is the sum of two self adjoint elements. This proves the first part.

For the second part, note that the spectrum of $P_1 \lor P_2$ lies in $\{0, 1, 2\}$. Let $f$ be any continuous real valued function with $f(0) = 0$, $f(1) = f(2) = 1$. Then $P_1 \lor P_2 = f(P_1 + P_2)$, and $f(P_1 + P_2) \in \mathcal{A}$. The proof for $P_1 \land P_2$ is similar.

3.4 The polar decomposition

3.16 DEFINITION (Operator absolute value). Let $\mathcal{H}$ be a Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$. Then the operator absolute value of $A$ is the operator $|A|$ defined by

$$|A| = \sqrt{A^*A},$$

where the square root is taken using the Abstract Spectral Theorem.

3.17 REMARK. One should not be misled by the notation: It is not in general true that $|AB| = |A||B|$, or that $|A^*| = |A|$ or even that $|A + B| \leq |A| + |B|$.

3.18 LEMMA. For all $A \in \mathcal{B}(\mathcal{H})$, there is a unique partial isometry $U$ in $\mathcal{B}(\mathcal{H})$ such that $A = U|A|$, and $\text{ran}(U) = \text{ran}(A)$ and $\text{ran}(U^*) = \ker(A)^\perp$. Moreover, $U$ belongs to the strong closure of $\mathcal{C}\{\{1, A, A^*\}\}$. 
For each $t > 0$, define the operator $U_t := A(t1 + |A|)^{-1}$, noting that $1 + |A|$ is invertible. For $s, t > 0$, the Resolvent Identity (1.14) yields

$$U_t - U_s = (s - t)A[(t1 - |A|)^{-1} - (t1 - |A|)^{-1}] .$$

Hence for any $\xi \in \mathcal{H}$, and $0 < s < t$,

$$\| (U_t - U_s)\xi \|^2 = (s - t)^2 (\xi, |A|^2(t1 - |A|)^{-2}(s1 - |A|)^{-2}\xi)_{\mathcal{H}}$$

$$= (t - s)^2 \int_{\sigma(|A|)} \frac{\lambda^2}{(t + \lambda)^2(s + \lambda)^2} d\mu_\xi$$

$$\leq \int_{\sigma(|A|)\setminus\{0\}} \frac{t^2}{(t + \lambda)^2} d\mu_\xi$$

Since $0 \leq t^2/(t + \lambda)^2 \leq 1$ for all $\lambda > 0$, and since $\lim_{t \to 0} t^2/(t + \lambda)^2 = 0$ for all $\lambda > 0$, the Lebesgue Dominated Convergence Theorem yields $\lim_{t \to 0} \left( \sup \{\| (U_t - U_s)\xi \|^2 \} \right) = 0$. Thus, the strong limit $U = \lim_{t \to 0} U_t$ exists. Note that $U|A| = \lim_{t \to 0} U_t|A| = A\lim_{t \to 0} f_t(|A|)$ where $f_t(\lambda) = \lambda/(t + \lambda)$. Since $\lim_{t \to 0} f_t(\lambda) = 1_{(0,\infty)}(\lambda)$ for all $\lambda > 0$, it follows from Theorem 3.12 that $\lim_{t \to 0} f_t(|A|) = 1_{(0,\infty)}(|A|) = 1 - 1_{[0]}(|A|)$. Since $1_{[0]}(|A|)$ is the projector onto $\ker(|A|) = \ker(A)$,

$$U|A| = A .$$

(3.15)

Next note that $U^*U = \lim_{t \to \infty} f_t^2(|A|)$ with $f_t(\lambda) = \lambda/(t + \lambda)$ once more. It follows that

$$U^*U = 1_{(0,\infty)}(|A|)$$

(3.16)

which is the orthogonal projection onto $\ker(A)\perp$. It follows from (3.15) that $\text{ran}(U) = \text{ran}(A)$, and hence $U$ is a partial isometry from $\ker(A)\perp$ onto $\text{ran}(A)$.

Taking the adjoint of (3.15), we obtain $A^* = |A|U^*$ and hence $AA^* = U^*AU^*$. Squaring both sides and observing that $AU^*U = A$, which follows from (3.16), $AA^*)^2 = U(A^*A)^2U^*$. An induction now yields $(AA^*)^n = U(A^*A)^nU^*$ for all $n$, and then taking a polynomial approximation to the square root, we conclude that

$$U|A|U^* = |A^*| ,$$

(3.17)

and then since $A^* = |A|U^* = U^*U|A|U^*$,

$$A^* = U^*(U|A|U^*)$$

(3.18)

is the polar decomposition of $A^*$.

3.19 THEOREM. Let $\mathcal{A}$ be a unital *-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the strong operator topology. Then for all $A \in \mathcal{A}$, the components $|A|$ and $U$ of the polar decomposition $A = U|A|$ both belong to $\mathcal{A}$. Moreover, for all $A \in \mathcal{A}$, the orthogonal projections onto $\ker(A)\perp$ and $\text{ran}(A)$ both belong to $\mathcal{A}$.

Proof. Since the strong operator topology is weaker than the norm topology, $\mathcal{A}$ is a $C^*$ algebra, and hence $|A| = \sqrt{A^*A} \in \mathcal{A}$. By Lemma 3.18, $U \in \mathcal{A}$. Since $\mathcal{A}$ is a *-algebra, $U^*U$ and $UU^*$ both belong to $\mathcal{A}$, by Lemma 3.18, these are, respectively, the orthogonal projections onto $\ker(A)\perp$ and $\text{ran}(A)$.

$\square$
Three theorems on strong closure

3.20 DEFINITION. A set $S$ of self adjoint operators in $\mathcal{B} (\mathcal{H})$ is an upward directed set in case for all $A, B \in S$, there exists $C \in S$ such that $C \geq A$ and $C \geq B$. Likewise, it is a downward directed set in case for all $A, B \in S$, there exists $C \in S$ such that $C \leq A$ and $C \leq B$.

The next theorem gives a condition for set of self adjoint operators in $\mathcal{B} (\mathcal{H})$ to have a maximal element in its strong operator topology closure. Since $S$ is downward directed if and only if $-S$ is upward directed, there is a trivial restatement in terms of downward directed sets.

3.21 THEOREM (Vigier’s Theorem). Let $S$ be an upward directed set in $\mathcal{B} (\mathcal{H})_{s.a.}$ that is norm bounded in $\mathcal{B} (\mathcal{H})$. Then there exists a unique $A_{\text{max}}$ in the strong operator topology closure of $S$ such that $A_{\text{max}} \geq A$ for all $A \in S$.

Proof. Define a function $q$ on $\mathcal{H}$ by $q(\zeta) = \sup \{ \langle \zeta, B\zeta \rangle : B \in S \}$, and then define a function $b$ on $\mathcal{H} \times \mathcal{H}$ by polarization:

$$b(\eta, \xi) = \frac{1}{4} [q(\eta + \xi) - q(\eta - \xi)] - \frac{i}{4} [q(\eta + i\xi) - q(\eta - i\xi)].$$

For any finite set of vectors in $\mathcal{H}$, and any $\epsilon > 0$, there exists $A \in S$ such that $q(\zeta) - \epsilon \leq \langle \zeta, A\zeta \rangle \leq q(\zeta)$ for all $\zeta$ in the set, and hence on any finite set, $b(\eta, \xi)$ can be uniformly approximated by a function of the form $\langle \eta, A\xi \rangle$, $A \in S$. It follows that $b$ is a bounded sesquilinear form on $\mathcal{H}$, and hence there is an operator $B \in \mathcal{B} (\mathcal{H})$, whose norm is no more than the norm bound on $S$, such that $b(\eta, \xi) = \langle \eta, B\xi \rangle$ for all $\eta, \xi \in \mathcal{H}$. For all $\xi \in \mathcal{H}$, $\langle \xi, B\xi \rangle = q(\xi) \geq \langle \xi, A\xi \rangle$ for all $A \in S$. That is, $B - A \geq 0$ for all $A \in S$.

Finally, let $c$ denote the norm bound on $S$. Given $\epsilon > 0$ and $\{\xi_1, \ldots, \xi_n\} \subset \mathcal{H}$, we may choose $A \in S$ such that for each $j = 1, \ldots, n$,

$$\langle \xi_j, A\xi_j \rangle \geq q(\xi_j) - \frac{\epsilon}{2c} = \langle \xi_j, B\xi_j \rangle - \frac{\epsilon}{2c}.$$

For each $j$, there is a unit vector $\eta_j$ such that

$$\| (B - A)\xi_j \| = \langle \eta_j, (B - A)\xi_j \rangle = \langle (B - A)^{1/2} \eta_j, (B - A)^{1/2} \xi_j \rangle \leq \| B - A \| \langle \xi_j, (B - A)\xi_j \rangle.$$

Then for each $j$, $\| (B - A)\xi_j \| \leq \epsilon$, and hence $B$ belongs to the strong closure of $S$. \hfill \Box

3.22 DEFINITION. If $S$ is a bounded upward directed set of self adjoint operators in $\mathcal{B} (\mathcal{H})$, the unique operator $B$ in the strong closure of $S$ such that $B \geq A$ for all $A \in S$, is called the least upper bound of $S$, and is denoted $\bigvee_{A \in S} A$. Likewise if $S$ is a bounded downward directed set of self adjoint operators in $\mathcal{B} (\mathcal{H})$, the unique operator $B$ in the strong closure of $S$ such that $B \leq A$ for all $A \in S$ is called the greatest lower bound of $S$, and is denoted $\bigwedge_{A \in S} A$.

Vigier’s theorem has a number of consequences. Here is the first of these.

3.23 COROLLARY. Let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B} (\mathcal{H})$ that is closed in the strong operator topology. Then $\mathcal{A}$ is unital.
Proof. Let $\mathcal{S}$ denote the set of all orthogonal projections contained in $\mathcal{A}$. Evidently $\|P\| \leq 1$ for all $P \in \mathcal{S}$, and by Theorem 3.15, for all $P, Q \in \mathcal{S}$, $P \vee Q \in \mathcal{S}$. Then by Theorem 3.21, there exists $P_{\text{max}} = \text{l.u.b.}(\mathcal{S})$ in the closure of $\mathcal{S}$ in the strong operator topology, and hence $\in \mathcal{A}$, such that $P_{\text{max}} \geq P$ for all $P \in \mathcal{S}$. $P_{\text{max}}$ is evidently an orthogonal projection whose range includes the range of any other orthogonal projection $P$ in $\mathcal{A}$. Then by Theorem 3.15 once more, $P_{\text{max}} = AP_{\text{max}} = A$ for all $A \in \mathcal{A}$.

Note that the multiplicative identity $P_{\text{max}}$ provided by Corollary 3.23, need not by the identity in $\mathcal{B}(\mathcal{H})$. However, every operator $A$ in $\mathcal{A}$ annihilates the range of $P_{\text{max}}^\perp$, and hence nothing is lost if we restrict $\mathcal{A}$ to the range of $P_{\text{max}}$, and on this subspace of $\mathcal{H}$, $P_{\text{max}}$ is the identity operator.

3.24 DEFINITION. Let $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$. The commutant $S'$ of $\mathcal{S}$ is the subset of $\mathcal{B}(\mathcal{H})$ given by

$$S' = \{ A \in \mathcal{B}(\mathcal{H}) : AB - BA = 0 \text{ for all } B \in \mathcal{S} \}.$$ We write $[A, B] = AB - BA$ to denote the commutator of $A$ and $B$.

3.25 LEMMA. Let $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$. The commutant $S'$ of $\mathcal{S}$ has the following properties:

1. $S'$ is a closed in the weak operator topology on $\mathcal{B}(\mathcal{H})$.
2. $S'$ is a unital subalgebra of $\mathcal{B}(\mathcal{H})$.
3. If $\mathcal{S}$ is closed under the involution, $S'$ is a weakly closed unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

Proof. For $\zeta, \xi \in \mathcal{H}$, $B \in \mathcal{S}$, define a linear functional $\varphi_{\zeta, \xi, B}$ on $\mathcal{B}(\mathcal{H})$ by

$$\varphi_{\zeta, \xi, B}(A) = \langle \zeta, ([A, B])\xi \rangle_{\mathcal{H}} = \langle \zeta, A(B\xi) \rangle_{\mathcal{H}} - \langle (B^*\zeta), A\xi \rangle_{\mathcal{H}}.$$ Since $\varphi_{\zeta, \xi, B}$ is weakly continuous, $\varphi_{\zeta, \xi, B}^{-1}(\{0\})$ is weakly closed. Then since

$$S' = \bigcap \{ \varphi_{\zeta, \xi, B}^{-1}(\{0\}) : \zeta, \xi \in \mathcal{H}, B \in \mathcal{S} \},$$ (1) is proved. (2) is evident, and then (3) follows from the fact that $[A, B]^* = [B^*, A^*]$ together with (1) and (2).

3.26 THEOREM (von Neumann Double Commutant Theorem). Let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that contains the identity. Then $\mathcal{A}''$ is the weak operator topology closure of $\mathcal{A}$.

Proof. Since $\mathcal{A}$ is convex, the weak and strong operator topology closures of $\mathcal{A}$ coincide. Hence it suffices to show that for all $A \in \mathcal{A}''$, every strong neighborhood of $A$ contains some $B \in \mathcal{A}$. That is, it suffices to show that for all $n \in \mathbb{N}$ and all $\{\eta_1, \ldots, \eta_n\} \subset \mathcal{H}$, and all $\epsilon > 0$, there is some $B \in \mathcal{A}$ such that $\|(B - A)\eta_j\| < \epsilon$ for all $j = 1, \ldots, n$.

Let $\mathcal{H} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$, the direct sum of $n$ copies of $\mathcal{H}$. The elements of $\mathcal{B}(\mathcal{H})$ are $n \times n$ matrices $[B_{i,j}]$ with entries in $\mathcal{B}(\mathcal{H})$.

Let $\mathcal{A}$ be the algebra of all operators on $\mathcal{H}$ of the form $[A\delta_{i,j}]$ with $A \in \mathcal{A}$. Evidently, its commutator $\mathcal{A}'$ consists of all $[B_{i,j}]$ with each $B_{i,j} \in \mathcal{A}'$. Thus, for all $A \in \mathcal{A}''$, $[A\delta_{i,j}] \in \mathcal{A}''$.

Let $\eta = \eta_1 \oplus \cdots \oplus \eta_n$, and define $\mathcal{K} = \mathcal{A}\overline{\eta}$, which is a closed subspace of $\mathcal{H}$ that is invariant under $\mathcal{A}$. Let $P$ be the orthogonal projection onto $\mathcal{K}$. Since $\mathcal{K}$ is invariant under $\mathcal{A}$, $P \in \mathcal{A}'$, and
hence $PB = BP$ for all $B \in \mathcal{A}''$. That is, $\mathcal{A}$ is invariant under $\mathcal{A}''$. In particular, for all $A \in \mathcal{A}''$, $\mathcal{A}$ is invariant under $[A \delta_{i,j}]$.

Since $\mathcal{A}$ contains the identity, $\eta \in \mathcal{A}$, so that $A\eta \oplus \cdots \oplus A\eta_n \in \mathcal{A}$. Therefore, by the definition of $\mathcal{A}$ as the closure of $A\eta$, for all $\epsilon > 0$, there exists $B \in \mathcal{A}$ such that

$$
\|B\eta_1 \oplus \cdots \oplus B\eta_n - A\eta_1 \oplus \cdots \oplus A\eta_n\|_{\mathcal{A}}^2 \leq \epsilon^2.
$$

$\Box$

Let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, and let $\mathcal{M}$ be its closure in the strong operator topology. If $\mathcal{A}$ is unital, then the von Neumann Double Commutant Theorem implies that $\mathcal{M}$ is also a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, even though the product map is not strongly continuous. The next theorem will show that $\mathcal{M}$ is a $*$-subalgebra even when $\mathcal{A}$ is not unital.

$B_{\mathcal{A}}$, the unit ball in $\mathcal{A}$, is contained in $B_{\mathcal{M}}$, the unit ball in $\mathcal{M}$. Since, the norm function is lower semicontinuous in the strong topology see Proposition 3.7, $B_{\mathcal{B}(\mathcal{H})}$ is strongly closed, and hence $B_{\mathcal{M}}$ is strongly closed. However, again on account of the lower semicontinuity, it is not immediately clear that $B_{\mathcal{A}}$ is strongly dense in $B_{\mathcal{M}}$. It is not obvious that there are no elements $B$ of $B_{\mathcal{M}}$ with a weak neighborhood $U$ that only contains elements $A \in \mathcal{A}$ with $\|A\|$ strictly larger – or even much larger – than one. However, this, and more, is true.

**3.27 THEOREM** (Kaplansky’s Density Theorem). Let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, and let $\mathcal{M}$ be its closure in the strong operator topology. Then $B_{\mathcal{A}}$ is strongly dense in $B_{\mathcal{M}}$, and the self-adjoint part of $B_{\mathcal{A}}$ is strongly dense in the self-adjoint part of $B_{\mathcal{M}}$.

**Proof.** Let $\mathcal{A}_{s,a}$ denote the space of self adjoint elements in $\mathcal{A}$, and let $\mathcal{M}_{s,a}$ denote the space of self adjoint elements in $\mathcal{M}$. As a first step, we show that the strong closure of $\mathcal{A}_{s,a}$ is $\mathcal{M}_{s,a}$.

$\mathcal{B}(\mathcal{H})_{s,a}$ is the null space of the real linear map $X \mapsto X - X^*$. Since the involution is weakly continuous, $\mathcal{B}(\mathcal{H})_{s,a}$ is weakly, and hence strongly closed. Since $\mathcal{M}_{s,a} = \mathcal{M} \cap \mathcal{B}(\mathcal{H})_{s,a}$, $\mathcal{M}_{s,a}$ is strongly closed. Therefore, the strong closure of $\mathcal{A}_{s,a}$, $\overline{\mathcal{A}_{s,a}}$, satisfies $\overline{\mathcal{A}_{s,a}} \subset \mathcal{M}_{s,a}$. It then remains to show that $\mathcal{A}_{s,a}$ is weakly dense in $\mathcal{M}_{s,a}$ since, being convex, its weak closure is the same as its strong closure.

Let $Y \in \mathcal{M}_{s,a}$. If $V_\epsilon, \xi_1, \ldots, \xi_n$ contains some $X \in \mathcal{A}$, i.e., there is an $X \in \mathcal{A}$ such that $|\langle \xi_j, (Y - X)\xi_j \rangle| < \epsilon$ for all $j = 1, \ldots, n$. Then $Y$ is self adjoint, $|\langle \xi_j, (Y - X^*)\xi_j \rangle| < \epsilon$ for all $j = 1, \ldots, n$, and then by convexity, with $Z = \frac{1}{2}(X + X^*)$, $|\langle \xi_j, (Y - Z)\xi_j \rangle| < \epsilon$ for all $j = 1, \ldots, n$, and $Z \in \mathcal{A}_{s,a}$. Hence $Y$ is in the weak closure of $\mathcal{A}_{s,a}$. This completes the proof that $\overline{\mathcal{A}_{s,a}} = \mathcal{M}_{s,a}$.

Let $B_{\mathcal{A}_{s,a}}$ denote the unit ball in $\mathcal{A}_{s,a}$, and let $B_{\mathcal{M}_{s,a}}$ denote the unit ball in $\mathcal{M}_{s,a}$. We now show that the strong closure of $B_{\mathcal{A}_{s,a}}$ is $B_{\mathcal{M}_{s,a}}$, which is the second part of the theorem.

Without loss of generality, we may suppose that $\mathcal{A}$ is norm closed; i.e., that $\mathcal{A}$ is a $C^*$ algebra. Consider the function $f : [-1,1] \to [-1,1]$ given by

$$
f(\lambda) = \frac{2\lambda}{1 + \lambda^2}.
$$
It is easy to see that $f$ a homeomorphism from $[-1, 1]$ onto $[-1, 1]$. The inverse function is
\[ g(t) := \begin{cases} 0 & t = 0 \\ \frac{1}{t} - \sqrt{\frac{1-t^2}{t^2}} & t > 0 \\ \frac{1}{t} + \sqrt{\frac{1-t^2}{t^2}} & t < 0 \end{cases} \]
for $t \in [-1, 1]$, but notice that the formula for $g$ makes sense for all $t \in \mathbb{R}$. Since both $f(0) = 0$ and $g(0) = 0$, we may apply the Abstract Spectral Theorem to define $f(A)$ and $g(A)$ for all $A \in \mathcal{A}_{s.a.}$, even when $\mathcal{A}$ is not unital. Now let $B \in \mathcal{B}_{s.a} \cap \mathcal{A}_{s.a.}$, and define $\hat{B} = g(B)$. We then choose $\hat{A} \in \mathcal{A}_{s.a.}$ to belong to an appropriate strong neighborhood of $\hat{B}$, specified below, and then define $A = f(\hat{A})$. Evidently $A \in \mathcal{A}_{s.a.} \cap \mathcal{B}_{s.a.}$. By what we have noted above, and by Corollary 2.23, $f(\hat{B}) = f(g(B)) = f g(B) = B$.

Now compute, using $\hat{B}^2 - \hat{A}^2 = \hat{A}(\hat{B} - \hat{A}) + (\hat{B} - \hat{A})\hat{B}$ in the last step,
\[
A - B = f(\hat{A}) - f(\hat{B}) = 2 \frac{1}{1 + \hat{A}^2} \left( \hat{A}(1 + \hat{B}^2) - (1 + \hat{A}^2)\hat{B} \right) \frac{1}{1 + B^2} = 2 \frac{1}{1 + \hat{A}^2} \left( \hat{A} - \hat{B} \right) \frac{1}{1 + B^2} + 2 \frac{\hat{A}}{1 + \hat{A}^2} \left( \hat{B}^2 - \hat{A}^2 \right) \frac{\hat{B}}{1 + B^2} = 2 \frac{1}{1 + \hat{A}^2} \left( \hat{A} - \hat{B} \right) \frac{1}{1 + B^2} - 2 \frac{\hat{A}^2}{1 + \hat{A}^2} \left( \hat{A} - \hat{B} \right) \frac{1}{1 + B^2} - 2 \frac{1}{1 + \hat{A}^2} \left( \hat{A} - \hat{B} \right) \frac{\hat{B}^2}{1 + B^2}.
\]

Note that $\left\| \frac{1}{1 + \hat{A}^2} \right\| \left\| \frac{\hat{A}^2}{1 + \hat{A}^2} \right\| < 1$, and therefore if $\hat{A}$ satisfies
\[
\left\| (\hat{A} - \hat{B}) \frac{1}{1 + B^2} \right\| < \frac{\epsilon}{6} \quad \text{and} \quad \left\| (\hat{A} - \hat{B}) \frac{\hat{B}^2}{1 + B^2} \right\| < \frac{\epsilon}{6},
\]
then $\|(A - B)\xi\| < \epsilon$. This proves that for every strong neighborhood $U$ of $B$, there is a strong neighborhood $V$ of $\hat{B}$ so that if $\hat{A} \in \mathcal{A}_{s.a.}$ is chosen in $V$, then $A = f(\hat{A}) \in U$. Since $A \in \mathcal{A}_{s.a.} \cap \mathcal{B}_{s.a.}$, this shows that $\mathcal{A}_{s.a.} \cap \mathcal{B}_{s.a.}$ is strongly dense in $\mathcal{M}_{s.a.} \cap \mathcal{B}_{s.a.}$.

To pass to the general case, consider $\mathcal{B}(\mathcal{H}) \otimes M_2(\mathbb{C})$, consisting of $2 \times 2$ block matrices
\[
\begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}
\]
with entries in $\mathcal{B}(\mathcal{H})$; we identify $\mathcal{B}(\mathcal{H}) \otimes M_2(\mathbb{C})$ with $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. A neighborhood basis at $0$ for the strong topology in $\mathcal{B}(\mathcal{H}) \otimes M_2(\mathbb{C})$ is given by the sets $\tilde{U}_{\epsilon, \xi_1, \cdots, \xi_n}$ consisting of operators of the form $\begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$ in $\mathcal{B}(\mathcal{H}) \otimes M_2(\mathbb{C})$ such that $B_{i,j} \in U_{\epsilon, \xi_i, \cdots, \xi_n}$ with $U_{\epsilon, \xi_1, \cdots, \xi_n}$ specified in (3.1). It is then easy to see that $\mathcal{A} \otimes M_2(\mathbb{C})$ is dense in $\mathcal{M} \otimes M_2(\mathbb{C})$. For $B \in \mathcal{M}$, define
\[
\tilde{B} = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}.
\]
Then $\|\tilde{B}\| = \|B\|$, so that for $B \in \mathcal{B}_{s.a.}$, by what we have just proved, for any neighborhood $\tilde{U}_{\epsilon, \xi_1, \cdots, \xi_n}$, there is $\tilde{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \in \mathcal{A} \otimes M_2(\mathbb{C})$ with $A_{2,1} = A_{1,2}^*$ and $A_{1,1}, A_{2,2} \in \mathcal{A}_{s.a.}$ and $\|\tilde{A}\| \leq 1$, such that $\tilde{A} - \tilde{B} \in \tilde{U}_{\epsilon, \xi_1, \cdots, \xi_n}$. Then $A_{1,2} - B \in U_{\epsilon, \xi_1, \cdots, \xi_n}$, and since $\|\tilde{A}\| \leq 1$, $\|A_{1,2}\| \leq 1$. \hfill $\square$
We now come to an important class of operator algebras singled out by von Neumann.

**3.28 Definition.** A von Neumann algebra is $C^*$ subalgebra of $\mathcal{B}(\mathcal{H})$ that contains the identity on $\mathcal{B}(\mathcal{H})$ and is closed in the weak operator topology.

Replacing the strong operator topology by the weak operator topology would not make the definition any more inclusive since the same convex sets (and hence subspaces) of $\mathcal{B}(\mathcal{H})$ are closed for the two topologies. Corollary 3.23 shows that any weakly closed $C^*$ subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is unital, and the unit is an orthogonal projection in $\mathcal{B}(\mathcal{H})$. Restricting the operators in $\mathcal{A}$ to the range of this projection, nothing essential is lost, and then the identity in the algebra is the identity operator on the smaller Hilbert space.

### 3.6 Trace class operators and the $\sigma$-weak topology

**3.29 Definition** (Trace class). Let $\mathcal{H}$ be a Hilbert space. An operator $A \in \mathcal{B}(\mathcal{H})$ is trace class in case

$$\|A\|_1 := \sup \left\{ \sum_{j=1}^{n} |\langle \eta_j, A\xi_j \rangle| : \{\eta_1, \ldots, \eta_n\}, \{\xi_1, \ldots, \xi_n\} \text{ orthonormal}, n \in \mathbb{N} \right\} < \infty.$$ \hfill (3.20)

The set of trace class operators is denoted by $\mathcal{T}(\mathcal{H})$. The function $A \mapsto \|A\|_1$ on $\mathcal{T}(\mathcal{H})$ defined by (3.20) is called the trace norm on $\mathcal{T}(\mathcal{H})$; this terminology is justified below.

**3.30 Theorem.** For any Hilbert space $\mathcal{H}$, $\mathcal{T}(\mathcal{H})$ is a (non-closed) $*$-ideal in $\mathcal{B}(\mathcal{H})$, and the function $A \mapsto \|A\|_1$ is a norm on $A$ under which the involution $*$ is isometric.

Moreover, for all $A \in \mathcal{T}(\mathcal{H})$ and all $B \in \mathcal{B}(\mathcal{H})$,

$$\|AB\|_1 \leq \|B\|\|A\|_1 \quad \text{and} \quad \|BA\|_1 \leq \|B\|\|A\|_1.$$ \hfill (3.21)

and there is equality in both inequalities in (3.21) whenever $B$ is unitary.

**Proof.** Since $|\langle \eta, (A+B)\xi \rangle| \leq |\langle \eta, A\xi \rangle| + |\langle \eta, B\xi \rangle|$, it is evident that $\mathcal{T}(\mathcal{H})$ is closed under addition, and moreover that $\|A+B\|_1 \leq \|A\|_1 + \|B\|_1$. It is also evident that $\mathcal{T}(\mathcal{H})$ is closed under scalar multiplication. For $A \in \mathcal{T}(\mathcal{H})$, $\|A\|_1 \geq \|A\| = \sup\{ |\langle \eta, A\xi \rangle| : \|\eta\| = \|\xi\| = 1 \}$. Hence $\|A\|_1 = 0$ if and only if $A = 0$. This shows that the trace norm is indeed a norm. Since $|\langle \xi, A^*\eta \rangle| = |\langle \eta, A\xi \rangle|$, $\|A^*\|_1 = \|A\|_1$ and hence $\mathcal{T}(\mathcal{H})$ is closed under the involution, and the involution is isometric in the trace norm.

Finally, let $A \in \mathcal{T}(\mathcal{H})$, and let $U$ be unitary in $\mathcal{B}(\mathcal{H})$. Then since $\{U\eta_1, \ldots, U\eta_n\}$ is orthonormal if and only if $\{\eta_1, \ldots, \eta_n\}$ is orthonormal, it follows that $AU$ and $UA$ are trace class, and that $\|UA\|_1 = \|AU\|_1 = \|A\|_1$.

Now $\epsilon > 0$, and let $B \in \mathcal{B}(\mathcal{H})$ satisfy $\|B\| < 1 - \epsilon$. By Theorem 2.42, for $m > 2/\epsilon$, $B$ has the form $B = \frac{1}{m} \sum_{j=1}^{m} U_j$, where each $j$ is unitary. Then $\|BA\|_1 \leq \frac{1}{m} \sum_{j=1}^{m} \|U_jA\|_1 = \|A\|_1$. Since $\epsilon > 0$ is arbitrary, it follows that for all $\|B\|$ in the unit ball of $\mathcal{B}(\mathcal{H})$, $\|BA\|_1 \leq \|A\|_1$, and then that for all $B \in \mathcal{B}(\mathcal{H})$, the first inequality in (3.21) is valid. The same argument (or use of the isometry property of the involution) proves the second inequality. Finally, (3.21) shows that $\mathcal{T}(\mathcal{H})$ is an ideal in $\mathcal{B}(\mathcal{H})$. 

To prove completeness, recall that the operator norm is dominated by the trace norm; i.e., \( \| \cdot \| \leq \| \cdot \|_1 \). Let \( \{ A_n \}_{n \in \mathbb{N}} \) be a Cauchy sequence in \( \mathcal{T}(\mathcal{H}) \). Then it is a Cauchy sequence in \( \mathcal{B}(\mathcal{H}) \) and there exists \( A \in \mathcal{B}(\mathcal{H}) \) such that \( \lim_{n \to \infty} \| A - A_n \| = 0 \). Pick \( \epsilon > 0 \) and \( N \in \mathbb{N} \) so that for \( m, n > N \), \( \| A_m - A_n \|_1 < \epsilon \). Then for any \( p \in \mathbb{N} \), and any orthonormal sets \( \{ \eta_1, \ldots, \eta_p \} \) and \( \{ \xi_1, \ldots, \xi_p \} \),
\[
\sum_{j=1}^p |\langle \eta_j, (A_m - A_n)\xi_j \rangle| < \epsilon .
\]
and this implies that \( \| A - A_n \|_1 \leq \epsilon \), showing at the same time that \( A \in \mathcal{T}(\mathcal{H}) \) and that \( \lim_{n \to \infty} \| A - A_n \|_1 = 0 \). \( \square \)

3.31 LEMMA. For all \( A \in \mathcal{T}(\mathcal{H}) \), and all \( \epsilon > 0 \), there is a finite rank operator \( A_\epsilon \) such that \( \| A - A_\epsilon \|_1 < \epsilon \). Moreover, ker\((A)^\perp \) and ran\((A)\) are separable subspaces of \( \mathcal{H} \), and hence there exists a separable subspace \( \mathcal{K} \) of \( \mathcal{H} \) such that \( A|_{\mathcal{K}^\perp} = 0 \) and \( A\mathcal{K} \subset \mathcal{H} \).

Proof. Let \( A \in \mathcal{T}(\mathcal{H}) \), and let \( A = U|A| \) be its polar decomposition. Then \( |A| = U^*A \in \mathcal{T}(\mathcal{H}) \), and \( \| |A| \|_1 = \| A \|_1 \). By the Cauchy-Schwarz inequality
\[
\| |A| \|_1 := \sup \left\{ \sum_{j=1}^n |\langle \xi, |A|\xi_j \rangle|, \{ \xi_1, \ldots, \xi_n \} \text{ orthonormal}, n \in \mathbb{N} \right\} < \infty .
\]

Fix \( \epsilon > 0 \), and let \( P_\epsilon \) be the spectral projection of \( |A| \) for the interval \([\epsilon, |A|]\. If \( \{ \eta_1, \ldots, \eta_m \} \) is an orthonormal set in the range of \( P_\epsilon \), then \( \sum_{j=1}^m |\langle \eta_j, |A|\eta_j \rangle| \leq cm \), and hence \( m \leq \| A \|_1/\epsilon \).

Let \( P = \lim_{\epsilon \to 0} P_\epsilon \) which exists in the strong operator topology, and is the orthogonal projection onto ker\((|A|)^\perp = \ker(A)^\perp \). Since the range of each \( P_\epsilon \) is finite dimensional, the range of \( P \) is a separable subspace of \( \mathcal{H} \). Replacing \( A \) with \( A^* \), we see that the closure of the range of \( A \) is also a separable subspace of \( A \). Let \( \mathcal{K} \) be the closed span of ker\((A)^\perp \cup \operatorname{ran}(A) \). Then \( \mathcal{K} \) is separable, invariant under \( A \), and \( A|_{\mathcal{K}^\perp} = 0 \).

Moreover, for all \( \xi, \langle \xi, |A|\xi \rangle - \langle P_\epsilon \xi, |A|P_\epsilon \xi \rangle = \langle (\xi - P_\epsilon \xi), |A|\xi \rangle + \langle P_\epsilon \xi, |A|\xi \rangle - \langle P_\epsilon \xi, |A|P_\epsilon \xi \rangle \), and then by the strong convergence of \( P_\epsilon \) to \( \epsilon \), for all \( \{ \xi_1, \ldots, \xi_m \} \) orthonormal,
\[
\lim_{\epsilon \to 0} \sum_{j=1}^m \langle \xi_j, P_\epsilon |A|P_\epsilon \xi_j \rangle = \sum_{j=1}^m \langle \xi_j, |A|\xi_j \rangle .
\]
Thus for all \( \delta > 0 \), there exists \( \epsilon > 0 \) such that \( \| P_\epsilon |A|P_\epsilon \| \geq \| |A| \|_1 - \delta \). Since \( P_\epsilon \) commutes with \( |A| \), \( |A| = P_\epsilon |A|P_\epsilon + P_\epsilon^\perp |A|P_\epsilon^\perp \). By the triangle inequality, \( \| |A| \|_1 \leq \| P_\epsilon |A|P_\epsilon \| + \| P_\epsilon^\perp |A|P_\epsilon^\perp \|_1 \). But by (3.22), restricting to sets \( \{ \xi_1, \ldots, \xi_n \} \) such that for each \( j \), \( \xi_j \) is in the range of either \( P_\epsilon \) or \( P_\epsilon^\perp \), one sees that \( \| |A| \|_1 \leq \| P_\epsilon |A|P_\epsilon \| + \| P_\epsilon^\perp |A|P_\epsilon^\perp \|_1 \), and hence \( \| |A| \|_1 = \| P_\epsilon |A|P_\epsilon \| + \| P_\epsilon^\perp |A|P_\epsilon^\perp \|_1 \). Hence \( \| A - UP_\epsilon |A|P_\epsilon \|_1 = \| UP_\epsilon^\perp |A|P_\epsilon^\perp \|_1 \leq \delta \). Since \( \delta > 0 \) is arbitrary and \( UP_\epsilon |A|P_\epsilon \) is always finite rank, this proves the density of finite rank operators. \( \square \)

3.32 LEMMA. Let \( A \in \mathcal{T}(\mathcal{H}) \) be positive, and let \( \mathcal{K} \) be any separable subspace of \( \mathcal{H} \) that is invariant under \( A \) and is such that \( A|_{\mathcal{K}^\perp} = 0 \). Then for any orthonormal basis \( \{ \eta_j \}_{j \in \mathbb{N}} \) of \( \mathcal{K} \),
\[
\| A \|_1 = \sum_{j=1}^\infty \langle \eta_j, A\eta_j \rangle .
\]
Proof. Let $A \in \mathcal{F}(\mathcal{H})$ be a positive operator on a separable Hilbert space $\mathcal{H}$, and let $\{\eta_j\}_{j \in \mathbb{N}}$ and $\{\xi_k\}_{k \in \mathbb{N}}$ be two orthonormal bases for $\mathcal{H}$. Then

$$\sum_{j=1}^{\infty} \langle \eta_j, A \eta_j \rangle = \sum_{j=1}^{\infty} \|A^{1/2} \eta_j\|_{\mathcal{H}} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left| \langle \xi_k, A^{1/2} \eta_j \rangle \right|^2.$$ 

Since infinite series of non-negative terms may be summed in any order, the right hand side is actually symmetric in $\{\eta_j\}_{j \in \mathbb{N}}$ and $\{\xi_k\}_{k \in \mathbb{N}}$. Therefore, by symmetry in the two orthonormal bases,

$$\sum_{j=1}^{\infty} \langle \eta_j, A \eta_j \rangle = \sum_{k=1}^{\infty} \langle \xi_k, A \xi_k \rangle,$$

showing that $\sum_{j=1}^{\infty} \langle \eta_j, A \eta_j \rangle$ depends only on $A$, and not the particular orthonormal basis $\{\eta_j\}_{j \in \mathbb{N}}$.

Now suppose that $\mathcal{H}$ is not separable. Let $A \in \mathcal{F}(\mathcal{H})$ be positive, and let $\mathcal{K}$ be a separable subspace of $\mathcal{H}$ that “carries” $A$, as in Lemma 3.31. Let $\epsilon > 0$, $\{\xi_1, \ldots, \xi_m\}$ be orthonormal in $\mathcal{K}$ be such that $\sum_{j=1}^{m} \langle \xi_j, A \xi_j \rangle \geq \|A\|_1 - \epsilon$. Let $\widetilde{\mathcal{H}}$ be the span of $\mathcal{H} \cap \{\xi_1, \ldots, \xi_m\}$. Let $\{\xi_j\}_{j \in \mathbb{N}}$ be an orthonormal basis of $\widetilde{\mathcal{H}}$ extending $\{\xi_1, \ldots, \xi_m\}$.

Let $\{\eta_j\}_{j \in \mathbb{N}}$ be any orthonormal basis of $\mathcal{H}$. If necessary, add finitely many vectors with indices less than 1 such that $\{\eta_j\}_{j > p}$ is an orthonormal basis for $\mathcal{K}$; all vectors with indices $p \leq j < 1$ are in ker$(A)$. By the invariance proved above

$$\|A\|_1 \geq \sum_{j=1}^{\infty} \langle \eta_j, A \eta_j \rangle = \sum_{j=p}^{\infty} \langle \eta_j, A \eta_j \rangle = \sum_{j=1}^{\infty} \langle \xi_j, A \xi_j \rangle \geq \sum_{j=1}^{m} \langle \xi_j, A \xi_j \rangle \geq \|A\|_1 - \epsilon.$$ 

Since $\epsilon > 0$ is arbitrary, this proves (3.23). \hfill \Box

For any $A \in \mathcal{F}(\mathcal{H})$, write $A = X + iY$, with $X, Y$ self adjoint. Since $X = \frac{1}{2} (A + A^*)$, $\|X\|_1 \leq \|A\|_1$ and likewise $\|Y\|_1 \leq \|A\|_1$. Then $XP = \frac{1}{2} (|X| + X)$ and $X = \frac{1}{2} (|X| - X)$, so that $\|X^+_\| \|X^-\| \leq \|A\|_1$, and likewise $\|Y^+_\| \|Y^-\| \leq \|A\|_1$. By what we have just proved, for any orthonormal basis $\{\eta_j\}_{j \in \mathbb{N}}$,

$$\sum_{j=1}^{\infty} \langle \eta_j, A \eta_j \rangle = \sum_{j=1}^{\infty} \langle \eta_j, X \eta_j \rangle + i \sum_{j=1}^{\infty} \langle \eta_j, Y \eta_j \rangle = \sum_{j=1}^{\infty} \langle \eta_j, X \eta_j \rangle + i \sum_{j=1}^{\infty} \langle \eta_j, X \eta_j \rangle = \sum_{j=1}^{\infty} \langle \eta_j, X \eta_j \rangle$$

converges absolutely and is independent of the orthonormal basis.

3.33 DEFINITION (Trace). For all $A \in \mathcal{F}(\mathcal{H})$, the trace of $A$, $\text{Tr}[A]$, is defined by

$$\text{Tr}[A] = \sum_{j=1}^{\infty} \langle \eta_j, A \eta_j \rangle$$

where $\{\eta_j\}$ is any orthonormal basis of any separable subspace $\mathcal{H}$ invariant under $A$ with $A|_{\mathcal{H}^\perp} = 0$.

3.34 THEOREM (Properties of the trace). The functional $A \mapsto \text{Tr}[A]$ is linear on $\mathcal{F}(\mathcal{H})$ and $\text{Tr}[A^*] = \text{Tr}[A]^*$ for all $A \in \mathcal{F}(\mathcal{H})$. Moreover:
(i) For all $A \in \mathcal{I}(\mathcal{H})$ and all $B \in \mathcal{B}(\mathcal{H})$
$$\text{Tr}[AB] = \text{Tr}[BA]$$ (3.25)

(ii) For all $A \in \mathcal{I}(\mathcal{H})$ and all $B \in \mathcal{B}(\mathcal{H})$
$$|\text{Tr}[AB]| \leq ||A||_1 ||B||$$ (3.26)

(iii) For all $\zeta, \xi \in \mathcal{H}$, and all $B \in \mathcal{B}(\mathcal{H})$, \(\text{Tr}[\zeta \langle \xi | A] = \langle \zeta, A\zeta \rangle_{\mathcal{H}}.\)

**Proof.** The linearity is evident, and since \(\langle \eta, A\eta \rangle_{\mathcal{H}} = \langle \eta, A^*\eta \rangle_{\mathcal{H}}\), it follows that \(\text{Tr}[A^*] = \text{Tr}[A]^*\). In view of the linearity and the fact that every \(B \in \mathcal{B}(\mathcal{H})\) is a linear combination of 4 unitaries, it suffices to prove (3.25) when \(B\) is unitary. In this case, \(AB = B^*(BA)B\) and \(\{B\eta_j\}_{j \in \mathbb{N}}\) is an orthonormal basis when \(\{\eta_j\}_{j \in \mathbb{N}}\) is, and hence
$$\text{Tr}[AB] = \sum_{j=1}^{\infty} \langle \eta_j, AB\eta_j \rangle = \sum_{j=1}^{\infty} \langle B\eta_j, BAB\eta_j \rangle = \text{Tr}[BA].$$

For part (ii), note that for all $A \in \mathcal{I}(\mathcal{H})$, $|\text{Tr}[A]| \leq ||A||_1$, and hence for all $A \in \mathcal{I}(\mathcal{H})$ and all $B \in \mathcal{B}(\mathcal{H})$, $|\text{Tr}[AB]| \leq ||AB||_1 \leq ||A||_1 ||B||$, using (3.21) in the last step.

Part (iii) is a simple computation using any orthonormal basis \(\{\eta_j\}_{j \in \mathbb{N}}\) in which $\eta_j = ||\eta||^{-1} \eta$ when $\eta \neq 0$, and is trivial otherwise. 

**3.35 THEOREM.** Every $B \in \mathcal{B}(\mathcal{H})$ defines a linear functional $\psi_B$ on $\mathcal{I}(\mathcal{H})$ by
$$\psi_B(X) = \text{Tr}[BX] \quad \text{for all } X \in \mathcal{I}(\mathcal{H}).$$ (3.27)

The mapping $B \mapsto \psi_B$ is an isometric isomorphism of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{I}(\mathcal{H})^*$. 

**Proof.** For any $\zeta, \zeta' \in \mathcal{H}$, consider the rank-one operator $|\zeta\rangle \langle \zeta'|$. Then
$$|||\zeta\rangle \langle \zeta'|| = |||\zeta\rangle \langle \zeta'||_1 = ||\zeta|| ||\zeta'||.$$ (3.28)

Let $\psi \in \mathcal{I}(\mathcal{H})^*$. Define a sesquilinear form $q_\psi$ on $\mathcal{H} \times \mathcal{H}$ by $q_\psi(\zeta, \zeta') = \psi(|\zeta\rangle \langle \zeta'|)$ for all $\zeta, \zeta' \in \mathcal{H}$. By (3.28),
$$|\psi(|\zeta\rangle \langle \zeta'|)| \leq ||\psi|| |||\zeta\rangle \langle \zeta'||_1 = ||\psi|| ||\zeta|| ||\zeta'||.$$ By the Riesz Lemma, there exists $B \in \mathcal{B}(\mathcal{H})$ with $||B|| = ||\psi||$ such that for all $\zeta, \zeta' \in \mathcal{H}$, $q_\psi(\zeta, \zeta') = \langle \zeta, B\zeta' \rangle$. Then for $\zeta, \zeta' \in \mathcal{H}$, $\psi(|\zeta\rangle \langle \zeta'|) = \langle \zeta, B\zeta' \rangle = \text{Tr}[B|\zeta\rangle \langle \zeta'|]$ and hence by linearity, $\psi(X) = \text{Tr}[BX]$ for all finite rank $X$. Since finite rank operators are dense in $\mathcal{I}(\mathcal{H})$ in the trace norm, this is valid for all $X \in \mathcal{I}(\mathcal{H})$. This shows that $B \mapsto \psi_B$ is a linear map of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{I}(\mathcal{H})^*$. By (3.26), $||\psi_B|| \leq ||B||$. Since $||B|| = \sup\{ ||\zeta, B\zeta'\rangle : ||\zeta|| \mathcal{H}, ||\zeta'|| \mathcal{H} = 1 \}$ and since
$$|\langle \zeta, B\zeta' \rangle| = |\text{Tr}[B|\zeta\rangle \langle \zeta'|]| = |\psi_B(|\zeta\rangle \langle \zeta'|)| \leq ||\psi_B|| ||\zeta|| ||\zeta'||,$$
we also have $||B|| \leq ||\psi_B||$. Hence $||\psi_B|| = ||B||_1$, and the map is isometric and hence also one-to-one. 

\[
\]

\[
\]

3.36 Definition. The \( \sigma \)-weak operator topology on \( \mathcal{B} (\mathcal{H}) \) is the weak-* topology on \( \mathcal{B} (\mathcal{H}) \), which by Theorem 3.35 is the weakest topology making all of the functions \( B \mapsto \text{Tr}[AB] \), \( A \in \mathcal{F} (\mathcal{H}) \), continuous.

By Theorem 3.8, the \( \sigma \)-weak operator topology is stronger than the weak operator topology, which is the weakest topology making all of the functions \( B \mapsto \text{Tr}[AB] \), \( A \) finite rank, continuous. However:

3.37 Lemma. The weak operator topology and the \( \sigma \)-weak topology induce the same relative topology on the unit ball \( B_{\mathcal{B}(\mathcal{H})} \), or, more generally, on bounded subsets of \( \mathcal{B}(\mathcal{H}) \).

Proof. We need only show that each relative \( \sigma \)-weak neighborhood contains a relative weak neighborhood. Let \( S \subset \mathcal{B}(\mathcal{H}) \) satisfy \( \|A\| \leq L \) for all \( A \in S \). The general \( \sigma \)-weak neighborhood of \( A_0 \in S \) has the from

\[
V_{T_1, \ldots, T_n, 1} = \{ A \in S : |\text{Tr}[T_j(A - A_0)]| < 1 \quad j = 1, \ldots, n \},
\]

where \( n \in \mathbb{N} \) and \( \{T_1, \ldots, T_n\} \subset \mathcal{F}(\mathcal{H}) \). By Lemma 3.31, we may approximate each \( T_j \) in \( \mathcal{F}(\mathcal{H}) \) by \( \tilde{T}_j \) such that \( \tilde{T}_j \) is finite rank, and \( \|T_j - \tilde{T}_j\| < \frac{1}{2^k} \). Then \( V_{T_1, \ldots, T_n, 1} \) contains

\[
W_{\tilde{T}_1, \ldots, \tilde{T}_n, 1} = \{ A \in S : |\text{Tr}[\tilde{T}_j(A - A_0)]| < \frac{1}{2} \quad j = 1, \ldots, n \},
\]

and this is a neighborhood of \( A_0 \) in the relative weak operator topology.

Now let \( \mathcal{M} \) be a von Neumann subalgebra of \( \mathcal{B}(\mathcal{H}) \). Since \( \mathcal{M} \) is weakly closed in \( \mathcal{B}(\mathcal{H}) \), it is \( \sigma \)-weakly closed. Define \( \mathcal{M}^\perp \) to be the annihilator of \( \mathcal{M} \) in the dual pairing between \( \mathcal{F}(\mathcal{H}) \) and \( \mathcal{B}(\mathcal{H}) \). That is,

\[
\mathcal{M}^\perp = \{ A \in \mathcal{F}(\mathcal{H}) : \text{Tr}[AB] = 0 \text{ for all } A \in \mathcal{M} \}.
\]

Evidently, \( \mathcal{M}^\perp \) is norm closed in \( \mathcal{F}(\mathcal{H}) \). We define \( \mathcal{M}_* \) to be the Banach space \( \mathcal{F}(\mathcal{H})/\mathcal{M}^\perp \) consisting of equivalence classes in \( \mathcal{F}(\mathcal{H}) \) under the equivalence relation \( A \sim C \) if and only if \( A - C \in \mathcal{M}^\perp \), and we equip \( \mathcal{M}_* \) with the quotient norm. Then, by a standard result in the theory of Banach spaces, the dual of \( \mathcal{M}_* \) is isometrically isomorphic to \( \mathcal{M} \).

To see this, first note that \( \mathcal{M} \) is the annihilator of \( \mathcal{M}^\perp \) in the dual pairing between \( \mathcal{F}(\mathcal{H}) \) and \( \mathcal{B}(\mathcal{H}) \). That is:

\[
\mathcal{M} = \{ B \in \mathcal{B}(\mathcal{H}) : \text{Tr}[AB] = 0 \text{ for all } A \in \mathcal{M} \} =: \mathcal{M}^{\perp \perp}.
\]

(3.29)

This is a consequence of the Hahn-Banach Theorem: Clearly \( \mathcal{M} \subset \mathcal{M}^{\perp \perp} \). If the containment were proper, there would exist \( X \in \mathcal{M}^{\perp \perp} \) not belonging to the \( \sigma \)-weakly closed subspace \( \mathcal{M} \), and then by the Hahn-Banach Theorem, there would exist a \( \sigma \)-weakly continuous linear functional \( \phi \) on \( \mathcal{B}(\mathcal{H}) \) with \( \phi(\mathcal{M}) = 0 \) and \( \psi(X) = 1 \). and every \( \sigma \)-weakly continuous linear functional on \( \mathcal{B}(\mathcal{H}) \) is of the form \( B \mapsto \text{Tr}[AB] \) for some \( A \in \mathcal{F}(\mathcal{H}) \). This is impossible since then \( \phi \in \mathcal{M}^\perp \) and \( X \in \mathcal{M}^{\perp \perp} \) which is incompatible with \( \phi(X) = 1 \).

Let \( \psi \in (\mathcal{M}_*)^* \). Let \( Q : \mathcal{F}(\mathcal{H}) \to \mathcal{M}_* \) be the quotient map, which, by the definition of the quotient norm, is a contraction. Consequently, \( ||\psi \circ Q|| (\mathcal{F}(\mathcal{H}))^* \leq ||\psi|| (\mathcal{M}_*)^* \). Thus \( \psi \circ Q \in (\mathcal{F}(\mathcal{H}))^* \), and since \( Q \) vanishes on \( \mathcal{M}^{\perp \perp} \), \( \psi \circ Q \in \mathcal{M}^{\perp \perp} = \mathcal{M} \).
Conversely, suppose that \( \phi \in \mathcal{M} = (\mathcal{M}^\perp)^\perp \), Define the linear functional \( \hat{\phi} \) on \( \mathcal{M} \) by \( \hat{\phi}(\{A\}) = \phi(A) \), which is well-defined since \( \phi \) vanishes on \( \mathcal{M}^\perp \). Moreover, for all \( C \sim A \), \( |\hat{\phi}(\{A\})| = |\phi(C)| \leq \|\phi\|\|C\| \). Taking the infimum over \( C \sim A \), \( |\hat{\phi}(\{A\})| \leq \|\phi\|\|\{A\}\| \). That is, \( \|\hat{\phi}\|_{(\mathcal{M}^\star)^\star} \leq \|\phi\| \). By construction, for all \( A \in \mathcal{T}(\mathcal{H}) \), \( \hat{\phi} \circ Q(A) = \hat{\phi}(\{A\}) = \phi(A) \); That is, \( \hat{\phi} \circ Q = \phi \).

Therefore, the map \( \psi \mapsto \psi \circ Q \) is a linear map from \( (\mathcal{M}^\star)^\star \) onto \( \mathcal{M} = \mathcal{M}^{\perp \perp} \), and its inverse on \( \mathcal{M} \) is the map \( \phi \mapsto \hat{\phi} \) defined in the previous paragraph. Since both maps are contractions, they are both isometries; that is \( \psi \mapsto \psi \circ Q \) is a linear isometry of \( (\mathcal{M}^\star)^\star \) onto \( \mathcal{M} \). We have proved:

**3.38 THEOREM.** Every von Neumann algebra \( \mathcal{M} \) is isomorphic as a Banach space to the dual of a Banach space. More precisely, \( \mathcal{M} \) is isomorphic to the dual of \( \mathcal{M}^\star \), where \( \mathcal{M}^\star \) is the quotient space \( \mathcal{T}(\mathcal{H})/\mathcal{M}^\perp \), where \( \mathcal{M}^\perp \) is the subspace of \( \mathcal{T}(\mathcal{H}) \) annihilated by \( \mathcal{M} \) in the dual pairing of \( \mathcal{T}(\mathcal{H}) \) and \( \mathcal{B}(\mathcal{H}) \).

## 4 \( C^* \) algebras as operator algebras

### 4.1 Representations of \( C^* \) algebras

**4.1 DEFINITION.** A representation of a \( C^* \)-algebra \( \mathcal{A} \) is a \(*\)-homomorphism \( \pi \) from \( \mathcal{A} \) into \( \mathcal{B}(\mathcal{H}) \) for some Hilbert space \( \mathcal{H} \). For any subspace \( \mathcal{K} \) of \( \mathcal{H} \), we define

\[
\pi(\mathcal{A})\mathcal{K} = \{ \pi(A)\eta : a \in \mathcal{A}, \eta \in \mathcal{K} \}.
\]

A subspace \( \mathcal{K} \) of \( \mathcal{H} \) is invariant under \( \pi \) in case \( \pi(\mathcal{A})\mathcal{K} \subset \mathcal{K} \). The representation \( \pi \) is irreducible in case no non-trivial subspace \( \mathcal{K} \) of \( \mathcal{H} \) is invariant under \( \pi \). The representation \( \pi \) is non-degenerate in case \( \pi(\mathcal{A})\mathcal{H} = \mathcal{H} \). Let \( \pi_1 \) and \( \pi_2 \) be two representations of \( \mathcal{A} \) on Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively. Then \( \pi_1 \) and \( \pi_2 \) are equivalent representations of \( \mathcal{A} \) in case there exists a unitary transformation from \( U \) from \( \mathcal{H}_1 \) onto \( \mathcal{H}_2 \) such that for all \( A \in \mathcal{A} \),

\[
\pi_2(A)U = U\pi_1(A).
\]

**4.2 LEMMA.** Let \( \mathcal{A} \) be a \( C^* \) algebra, and let \( \pi \) be a non-zero representation of it as an algebra of operators on some Hilbert space \( \mathcal{H} \). Then a closed subspace \( \mathcal{K} \) of \( \mathcal{H} \) is invariant under \( \pi(\mathcal{A}) \) if and only if the orthogonal projection of \( \mathcal{H} \) onto \( \mathcal{K} \) belongs to \( (\pi(\mathcal{A}))' \).

**Proof.** First, \( \mathcal{K} \) is invariant under \( \pi(\mathcal{A}) \) if and only if \( \mathcal{K} \) is invariant under \( \pi(\mathcal{A}) \). To see this, let \( \zeta \in \mathcal{K} \) and \( \xi \in \mathcal{K} \), and \( A \in \mathcal{A} \). If \( \mathcal{K} \) is in variant, \( \pi(A^*)\xi \in \mathcal{K} \), and hence

\[
\langle \pi(A)\zeta, \xi \rangle_{\mathcal{K}} = \langle \zeta, \pi(A^*)\xi \rangle_{\mathcal{K}} = 0.
\]

Thus the invariance of \( \mathcal{K} \) implies the invariance of \( \mathcal{K} \), and then by symmetry, the reverse implication is valid as well.

Now let \( P \) be the orthogonal projection onto \( \mathcal{K} \). Then when \( \mathcal{K} \) is invariant, for all \( A \in \mathcal{A} \),

\[
0 = P\pi(A)(1 - P) = P\pi(A) - P\pi(A)P = P\pi(A) - \pi(A)P
\]

where the last equality is true since the range of \( \pi(A)P \) lies in \( \mathcal{K} \). Therefore, \( P \in (\pi(\mathcal{A}))' \).

Conversely, if \( P \in (\pi(\mathcal{A}))' \) and \( \xi \in \mathcal{K} \), then for all \( A \in \mathcal{A} \),

\[
\pi(A)\xi = \pi(A)P\xi = P\pi(A)\xi \in \mathcal{K},
\]

which shows the invariance of \( \mathcal{K} \). \( \square \)
Lemma 4.2 permits us to make the following definition:

4.3 DEFINITION. For a representation \( \pi \) of a \( \mathbb{C}^* \) algebra \( \mathcal{A} \) on a Hilbert space \( \mathcal{H} \), and a non-zero projector \( P \in (\pi(\mathcal{A}))' \), \( \pi_P \) is the subrepresentation obtained by restricting \( \pi \) to \( \text{ran}(P) \).

4.4 THEOREM. Let \( \mathcal{A} \) be a \( \mathbb{C}^* \) algebra, and let \( \pi \) be a non-zero representation of it as an algebra of operators on some Hilbert space \( \mathcal{H} \). Then \( \pi \) is irreducible if and only if \((\pi(\mathcal{A}))' \) consists of scalar multiples of the identity.

Proof. If \((\pi(\mathcal{A}))' \) consists of scalar multiples of the identity, then \((\pi(\mathcal{A}))' \) contains no non-trivial orthogonal projections, and hence by Lemma 4.2, \( \pi \) is irreducible. On the other hand, if \((\pi(\mathcal{A}))' \) contains some operator that is not a multiple of the identity, then it contains a self adjoint operator \( A \) that is not a multiple of the identity. Any such \( A \in (\pi(\mathcal{A}))' \) has a non-trivial spectral projection that is also in \((\pi(\mathcal{A}))' \) since \((\pi(\mathcal{A}))' \) is a strongly closed \( \ast \)-algebra containing \( A \).

4.2 States on a \( \mathbb{C}^* \) algebra

4.5 DEFINITION. Let \( \mathcal{A} \) be a \( \mathbb{C}^* \) algebra. A linear functional \( \varphi \) on \( \mathcal{A} \) regarded as a Banach space, is positive in case \( \varphi(A) \geq 0 \) for all \( A \in \mathcal{A}^+ \). A positive linear functional \( \varphi \) is faithful in case

\[
\varphi(A^*A) = 0 \quad \Rightarrow \quad A = 0.
\] (4.1)

Every self-adjoint \( A \in \mathcal{A} \) is the difference of two elements of \( \mathcal{A}^+ \): For \( f(t) := \max\{0,t\} \), define \( A_1 = f(A) \) and \( A_2 = A - f(A) \). Then \( A = A_1 - A_2 \), \( A_1, A_2 \in \mathcal{A}^+ \). Moreover, with this choice of \( A_1 \) and \( A_2 \), \( \|A_1\|, \|A_2\| \leq \|A\| \) and \( A_1 A_2 = A_2 A_1 = 0 \). In this case we say that \( A_1 \) is the positive part of \( A \), written \( A_+ \), and \( A_2 \) is the negative part of \( A \), written \( A_- \). Consequently, for a positive linear functional \( \varphi \), \( \varphi(A) = \varphi(A_+) - \varphi(A_-) \), and hence \( \varphi \) is real on the self-adjoint elements of \( \mathcal{A} \).

Every \( A \in \mathcal{A} \) has a canonical decomposition

\[
A = X_1 - X_2 + i(Y_1 - Y_2)
\] (4.2)

where \( X_1, X_2, Y_1, Y_2 \in \mathcal{A}^+ \), and \( \|X_1\|, \|X_2\|, \|Y_1\|, \|Y_2\| \leq \|A\| \). To see this, define

\[
X := \frac{A + A^*}{2} \quad \text{and} \quad Y := \frac{A - A^*}{2i},
\]

and then decompose \( X \) and \( Y \) into their positive and negative parts.

For all positive linear functionals \( \varphi \) on a \( \mathbb{C}^* \) algebra \( A \), the map

\[
(A, B) \mapsto \varphi(A^*B) =: \langle A, B \rangle_\varphi
\]

defines a (possibly degenerate) inner product on \( \mathcal{A} \); this inner product is non-degenerate if and only if \( \varphi \) is faithful. In any case, the fact that \( \langle A, A \rangle_\varphi \geq 0 \) for all \( A \in \mathcal{A} \) yields the Cauchy-Schwarz inequality:

\[
|\langle A, B \rangle_\varphi| \leq \langle A, A \rangle_\varphi^{1/2} \langle B, B \rangle_\varphi^{1/2}.
\] (4.3)

4.6 LEMMA (Positivity and continuity). Let \( \mathcal{A} \) be a \( \mathbb{C}^* \) algebra. Every positive linear functional is bounded, and

\[
\|\varphi\| = \sup_{A \in \mathcal{A} \cap B_\mathcal{A}} \{\|\varphi(A)\|\}.
\] (4.4)
If furthermore $\mathcal{A}$ is unital,

(1) For all $\varphi \in \mathcal{A}_+^*$, $\|\varphi\| = \varphi(1)$.

(2) Every bounded linear functional $\varphi$ such that $\|\varphi\| = \varphi(1)$ is positive.

Proof. As noted above, each $A \in B_{\mathcal{A}}$ has decomposition $A = X_1 - X_2 + i(Y_1 - Y_2)$ with $X_1, X_2, Y_1, Y_2$ all in $\mathcal{A}^+ \cap B_{\mathcal{A}}$. Therefore, for any positive linear functional $\varphi$,

$$\sup_{\|A\| \leq 1} \{\|\varphi(A)\|\} \leq 4 \sup_{A \in \mathcal{A}^+ \cap B_{\mathcal{A}}} \{\|\varphi(A)\|\} .$$

Suppose $\varphi$ is a positive linear functional, and that $\varphi \in \mathcal{A}^*$ is not bounded. Then by the remarks just above, there is a sequence $\{X_j\}_{j \in \mathbb{N}}$ in $\mathcal{A}^+$ with $\|X_j\| = 1$ and $|\varphi(X_j)| \geq 4^j$ for all $j$. Define

$$X := \sum_{j=1}^{\infty} 2^{-j}X_j .$$

Since $\mathcal{A}^+$ is closed, $X \in \mathcal{A}^+$, Moreover for all $n > m$, $\sum_{j=m+1}^{n} 2^{-j}X_j \in \mathcal{A}^+$, and taking $n \to \infty$, and using the closure of $\mathcal{A}^+$ once more, for all $m \in \mathbb{N}$, $X - \sum_{j=1}^{m} 2^{-j}X_j \in \mathcal{A}^+$. Therefore,

$$\varphi(X) \geq \sum_{j=1}^{m} 2^{-j}\varphi(X_j) \geq \sum_{j=1}^{m} 2^j ,$$

and this evidently yields a contradiction for $m$ large enough. Next, let $\epsilon > 0$, and pick $A \in B_{\mathcal{A}}$ such that $|\varphi(A)| + \epsilon \geq \|\varphi\|$. By Theorem 2.32, there exists $E \in \mathcal{A}^+ \cap B_{\mathcal{A}}$ such that $\|AE - A\| \leq \epsilon$. Then $|\varphi(AE)| \geq |\varphi(A)| - \|\varphi\|\epsilon$, and by the Cauchy-Schwarz inequality,

$$|\varphi(AE)| \leq (\varphi(A^*A)(\varphi(E^2))^{1/2} \leq \sup_{X \in \mathcal{A}^+ \cap B_{\mathcal{A}}} \{\|\varphi(X)\|\}$$

since both $A^*A$ and $E^2$ belong to $\mathcal{A}^+ \cap B_{\mathcal{A}}$. Since $\epsilon > 0$ is arbitrary, this proves (4.4).

For the second part, suppose that $\mathcal{A}$ is unital. Since $\|\varphi\| \geq \varphi(1) \geq \varphi(A)$ for all $A \in \mathcal{A}^+ \cap B_{\mathcal{A}}$, (1) is an immediate consequence of (4.4).

To prove (2), suppose that $\varphi \in \mathcal{A}^*$ and $\varphi(1) = \|\varphi\|$. If $\varphi = 0$, it is positive. If $\varphi \neq 0$, we may divide by $\|\varphi\|$ and thus may suppose that $\|\varphi\| = \varphi(1) = 1$.

We claim that for all $\varphi \in \mathcal{A}^*$ such that $\varphi(1) = \|\varphi\|$, $\varphi(A)$ belongs to the convex hull of $\sigma_{\mathcal{A}}(A)$ for all $A \geq 0$ in $\mathcal{A}$. To see this suppose that the closed disc of radius $r$ centered on $\lambda$ contains $\sigma_{\mathcal{A}}(A)$. Then $\lambda 1 - A$ is normal, and its spectrum is continued in $\{\lambda - t : t \in \sigma_{\mathcal{A}}(A)\}$, and hence the spectral radius of $\lambda 1 - A$ is at most $r$. Since $\lambda 1 - A$ is normal, $\|\lambda 1 - A\| \leq r$. Therefore,

$$|\lambda - \varphi(A)| = |\varphi(\lambda 1 - A)| \leq \|\lambda 1 - A\| \leq r .$$

Thus for all $r > 0$ and $\lambda \in \mathbb{C}$, $\varphi(A)$ is contained in the closed disc of radius $r$ centered on $\lambda$ contains $\sigma_{\mathcal{A}}(A)$. The intersection over all such discs is the convex hull of $\sigma_{\mathcal{A}}(A)$.

By the first part of Lemma 4.6, for every $C^*$ algebra $\mathcal{A}$, every positive linear functional on $\mathcal{A}$ belongs to $A^*$, the Banach space dual to $\mathcal{A}$. We write $A_+^*$ to denote the set of positive linear functionals.
4.7 DEFINITION (State and quasi-state). Let $\mathcal{A}$ be a $C^*$ algebra. A state on $\mathcal{A}$ is an element $\varphi$ of $\mathcal{A}^*_+$ with $\|\varphi\| = 1$. A quasi-state on $\mathcal{A}$ is an element $\varphi$ of $\mathcal{A}^*_+$ with $\|\varphi\| \leq 1$. We write $S_{\mathcal{A}}$ to denote the set of state on $\mathcal{A}$, and $Q_{\mathcal{A}}$ to denote the set of quasi-states. We equip both $S_{\mathcal{A}}$ and $Q_{\mathcal{A}}$ with the relative weak-$*$ topology.

Notice that $Q_{\mathcal{A}} = \bigcap_{A \in \mathcal{A}_+} \{ \varphi \in \mathcal{A}^*_+ : \|\varphi\| \leq 1 \text{ and } \varphi(A) \geq 0 \}$, which is evidently a weak-$*$ closed subset of the unit ball in $\mathcal{A}^*$. Then by the Banach-Alaoglu Theorem, $Q_{\mathcal{A}}$ is weak-$*$ compact.

Suppose that $\mathcal{A}$ is not unital, and let $\mathcal{A}_1$ be the $C^*$ algebra obtained by adjoining a unit in the canonical manner. Let $\varphi \in Q_{\mathcal{A}}$, and define $\tilde{\varphi}$ on $\mathcal{A}_1$ by

$$\tilde{\varphi}(\lambda, A) = \|\varphi\| \lambda + \varphi(A) .$$

Let $(\lambda, A) \in \mathcal{A}_1^+$. Then $\lambda \geq 0$ and $\sigma_{\mathcal{A}}(A) \subset [-\lambda, \infty)$. Hence

$$\tilde{\varphi}(\lambda, A) = \|\varphi\| \lambda + \varphi(A_+ - \varphi(A_-) \geq \|\varphi\| \lambda - \varphi(A_-) .$$

Since $\sigma_{\mathcal{A}}(A) \subset [-\lambda, 0], \|A_-\| \leq \lambda$. Therefore $\varphi(A_-) \leq \|\varphi\| \|A_-\| \leq \|\varphi\| \lambda$. Thus, $\tilde{\varphi}(\lambda, A) \geq 0$, showing that $\tilde{\varphi}$ is positive. By part (1) of Lemma 4.6, $\|\tilde{\varphi}\| = \|\varphi\|$, so that $\tilde{\varphi} \in Q_{\mathcal{A}_1}$, and if $\varphi \in S_{\mathcal{A}}$, then $\tilde{\varphi} \in S_{\mathcal{A}_1}$.

Let $\psi \in S_{\mathcal{A}_1}$. Define a positive linear function $\psi|_{\mathcal{A}}$ on $\mathcal{A}$ by

$$\psi|_{\mathcal{A}}(A) = \psi((0, A)) .$$

Evidently $\|\psi|_{\mathcal{A}}\| \leq \|\psi\| = 1$, so that $\psi|_{\mathcal{A}} \in Q_{\mathcal{A}}$. Let $\psi_0$ denote the state on $\mathcal{A}_1$ given by $\psi_0(\lambda, A) = \lambda$. For $\psi \in S_{\mathcal{A}_1}$, $\psi \neq \psi_0, \|\psi|_{\mathcal{A}}\| > 0$. Then with $c := \|\psi|_{\mathcal{A}}\|, c^{-1}\psi|_{\mathcal{A}} \in S_{\mathcal{A}}$, and

$$\psi((\lambda, A)) = \lambda + \psi((0, A)) = \lambda + \psi|_{\mathcal{A}}(A) = (1 - c)\lambda + c (\lambda + c^{-1}\psi|_{\mathcal{A}}(A)) .$$

That is,

$$\psi = c\psi_0 + (1 - c)c^{-1}\psi|_{\mathcal{A}} ,$$

which shows that every state $\psi \in S_{\mathcal{A}_1}$ is a convex combination of $\psi_0$ and a state of the form $\tilde{\varphi}, \varphi \in S_{\mathcal{A}}$.

By Lemma 4.6, when $\mathcal{A}$ is unital, when $\varphi \in \mathcal{A}^*_+, \|\varphi\| = \varphi(1)$, and hence in this case, reasoning as above,

$$S_{\mathcal{A}} = \bigcap_{A \in \mathcal{A}_+} \{ \varphi \in \mathcal{A}^*_+ : \varphi(A) \geq 0 \text{ and } \varphi(1) = 1 \}$$

which displays $S_{\mathcal{A}}$ as the intersection over a family of weak-$*$ closed subsets of the unit ball of $\mathcal{A}^*$. Hence in the unital case, the state space is compact. When $\mathcal{A}$ is not unital, we still have that

$$Q_{\mathcal{A}} = \bigcap_{A \in \mathcal{A}_+} \{ \varphi \in \mathcal{A}^*_+ : \varphi(A) \geq 0 \} \cap B_{\mathcal{A}^*} ,$$

which displays $Q_{\mathcal{A}}$ as the intersection over a family of weak-$*$ closed subsets of the unit ball of $\mathcal{A}^*$. Hence $Q_{\mathcal{A}}$ is compact in the weak-$*$ topology.

4.8 LEMMA. Let $\mathcal{A}$ be a $C^*$ algebra. For all self adjoint $A \in \mathcal{A}$, there exists a state $\varphi$ such that $|\varphi(A)| = \|A\|$. 
Proof. We may assume \( A \neq 0 \). First suppose that \( \mathcal{A} \) is unital. Consider the \( C^* \) algebra \( C\{1, A\} \) generated by 1 and \( A \). This is a commutative \( C^* \) algebra, and by Corollary 1.34, there exists a character \( \varphi_0 \) of \( C\{1, A\} \) such that \(|\varphi_0(A)| = \nu(A)\), which equals \(|A|\). Since \( \varphi_0 \) is a character \( \varphi(1) = 1 \). Then by Lemma 4.6, \( \varphi_0 \in \mathcal{A}^*_+ \), and so \( \varphi \) is a state on \( C\{1, A\} \).

By the Hahn-Banach Theorem, there is a norm preserving extension \( \varphi \) of \( \varphi_0 \) (as a linear functional) to \( \mathcal{A} \). Then \( \varphi(1) = \varphi_0(1) = 1 \), and hence by Lemma 4.6, \( \varphi \) is a state, and since \( \varphi \) extends \( \varphi_0 \), \( \varphi(A) = ||A|| \).

When \( \mathcal{A} \) is not unital, consider \( \mathcal{A}_1 \), the \( C^* \) algebra \( \mathcal{A}_1 \) obtained by adjoining a unit in the canonical manner. Then \( (0, A) \) is self adjoint in \( \mathcal{A}_1 \), and by what we have just proved, there is a character \( \psi \) on \( \mathcal{A}_1 \) such that \( \psi((0, A)) = ||(0, A)|| = ||A|| \). Define \( \varphi \) to be the quasi-state \( \varphi = \psi|_{\mathcal{A}} \); i.e., for all \( B \in \mathcal{A} \), \( \varphi(B) = \psi((0, B)) \). Since \( \varphi(A) = ||A|| \), \( ||\varphi|| = 1 \), and \( \varphi \) is a state on \( \mathcal{A} \).

We have seen that for a unital \( C^* \) algebra \( \mathcal{A} \), \( S_\mathcal{A} \) is compact in the weak-* topology, and it is evidently convex. The Krein-Milman Theorem says that every non-empty convex set in \( \mathcal{A}^*_+ \) that is compact in the weak-* topology is the convex hull of its extreme points. Hence there exist extreme points in \( \mathcal{A}^*_+ \).

4.9 DEFINITION (Pure state). Let \( \mathcal{A} \) be a unital \( C^* \) algebra. A pure state is an extreme point of \( \mathcal{A}^*_+ \).

4.10 THEOREM. Let \( \mathcal{A} \) be a unital \( C^* \) algebra. For all self adjoint \( A \in \mathcal{A} \), there exists a pure state \( \varphi \) such that \( |\varphi(A)| = ||A|| \).

Proof. By Lemma 4.8, the set \( S \) of states \( \varphi \) such that \( \varphi(A) = ||A|| \) is non-empty, and evidently it is convex and closed in the weak-* topology. By the Krein-Milman Theorem, \( S \) has at least one extreme point \( \psi \). We now show that \( \psi \) is extreme in \( S_\mathcal{A} \) as well as in \( S \).

Suppose that \( \psi_1, \psi_2 \in S_\mathcal{A} \) and that \( \psi = t\psi_1 + (1 - t)\psi_2 \) for some \( t \in (0, 1) \). Evaluating both sides at \( A \),

\[ ||A|| = \psi(A) = t\psi_1(A) + (1 - t)\psi_2(A) \leq t||A|| + (1 - t)||A|| = ||A||. \]

Hence \( \psi_1, \psi_2 \in S \), and so \( \psi_1 = \psi_2 = \psi \).

4.11 DEFINITION. Let \( \pi \) be a representation of a \( C^* \) algebra \( \mathcal{A} \) on a Hilbert space \( \mathcal{H} \). A vector \( \eta \in \mathcal{H} \) is cyclic for \( \pi \) in case \( A \mapsto \pi(A)\eta \) has dense range, and is a separating vector for \( \pi \) in case \( A \mapsto \pi(A)\eta \) is injective. If a cyclic vector exists, then \( \pi \) is a cyclic representation.

For any representation \( \pi \) of \( \mathcal{A} \) on \( \mathcal{H} \), and any unit vector \( \eta \in \mathcal{H} \), the functional \( \varphi_\eta \in \mathcal{A}^* \) defined by

\[ \varphi_\eta(A) = \langle \eta, \pi(A)\eta \rangle_{\mathcal{H}} \tag{4.6} \]

is a state. Evidently, \( \varphi_\eta(A^*A) = \langle \eta, \pi(A^*A)\eta \rangle_{\mathcal{H}} = ||\pi(A)\eta||^2_{\mathcal{H}} \), and hence \( \eta \) is separating for \( \pi \) if and only if \( \varphi_\eta \) is faithful. The next theorem links cyclicity and irreducibility.

4.12 THEOREM. Let \( \pi \) be a representation of a \( C^* \) algebra \( \mathcal{A} \) on a Hilbert space \( \mathcal{H} \), and let \( \eta \) be a cyclic unit vector for \( \pi \). Then with \( \varphi_\eta \) denoting the state defined in (4.6). Then \( \pi \) is irreducible if and only if \( \varphi_\eta \) is pure.

The heart of the matter is the following lemma:
4.13 **Lemma.** Let $\pi$ be a representation of a unital $C^*$ algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and let $\eta$ be a cyclic unit vector for $\pi$. Then with $\varphi_\eta$ denoting the state defined in (4.6), suppose that $\psi \in S_{\mathcal{A}}$, and that for some $r \in (0, \infty)$,

$$\psi(A) \leq r\varphi_\eta(A) \quad \text{for all} \quad A \in \mathcal{A}^+.$$ 

Then there is a positive operator $X \in (\pi(\mathcal{A}))'$ such that $\|X\| \leq r$ and for all $A, B \in \mathcal{A}$,

$$\psi(A^*B) = \langle \eta, \pi(A), X\pi(B)\eta \rangle_\mathcal{H}.$$

(4.7)

**Proof.** Define a positive sesquilinear form $q$ on $\pi(\mathcal{A})\eta$ by $q(\pi(A)\eta, \pi(B)\eta) = \psi(A^*B)$. We have

$$|q(\pi(A)\eta, \pi(A)\eta)| \leq r|\langle \pi(A)\eta, \pi(A)\eta \rangle_\mathcal{H}| \leq r\|\pi(A)\|^2_\mathcal{H}.$$

Since $\eta$ is cyclic, $q$ is densely defined on $\mathcal{H}$ and extends to a positive sesquilinear form on all of $\mathcal{H}$, still denoted by $q$, that, by the Cauchy-Schwarz inequality, satisfies $|q(\zeta, \xi)| \leq \|\zeta\| \|\xi\| \|\mathcal{H}\| \|\xi\| \|\mathcal{H}\|$ for all $\zeta, \xi \in \mathcal{H}$. By Reisz’s Lemma, there exists a self adjoint operator $X \in \mathcal{B}(\mathcal{H})$ such that $q(\zeta, \xi) = \langle \zeta, X\xi \rangle_\mathcal{H}$ for all $\zeta, \xi \in \mathcal{H}$, and $\|X\| \leq r$. Since $q(\zeta, \zeta) \geq 0$ for $\zeta$ in the dense set $\pi(\mathcal{A})\eta$, $X$ is positive.

Finally, note that for all $A, B, C \in \mathcal{A}$, $A^*(BC) = (B^*A)^*C$, and hence $\psi(A^*(BC)) = \psi((B^*A)^*C)$. This means that $q(\pi(A)\eta, \pi(B)\pi(C)\eta) = q(\pi(B^*)\pi(A)\eta, \pi(C)\eta)$ which is the same as

$$\langle \pi(A)\eta, X\pi(B)\pi(C)\eta \rangle_\mathcal{H} = \langle \pi(A)\eta, \pi(B)X\pi(C)\eta \rangle_\mathcal{H}.$$

Thus for all $\zeta, \xi$ in a dense subset of $\mathcal{H}$, $\langle \zeta, X\pi(B)\xi \rangle_\mathcal{H} = \langle \zeta, \pi(B)X\xi \rangle_\mathcal{H}$ and this shows that $X$ commutes with $\pi(B)$ for arbitrary $B \in \mathcal{A}$.

Proof of Theorem 4.12. Suppose that $\pi$ is irreducible. Let $\psi_1, \psi_2$ be two states such that $\varphi_\eta = t\psi_1 + (1-t)\psi_2$ for some $t \in (0, 1)$. By Lemma 4.13, applied to $\psi_1$, which satisfies $\psi_1 \leq t^{-1}\varphi_\eta$, there is a positive $X \in (\pi(\mathcal{A}))'$

$$\psi_1(A^*B) = \langle \eta, \pi(A), X\pi(B)\eta \rangle_\mathcal{H} \quad \text{for all} \quad A, B \in \mathcal{A}.$$

(4.8)

Since $\pi$ is irreducible, $X$ must be a scalar multiple of the identity. Since $\psi_1$ is a state, taking $A = B = 1$ in (4.8), $1 = \psi(1) = \langle \eta, X\eta \rangle_\mathcal{H}$, which shows that $X = 1$. Then taking $A = 1$ in (4.8) shows that $\psi(B) = \varphi_\eta(B)$ so that $\psi_1 = \varphi_\eta$. By symmetry, $\psi_2 = \varphi_\eta$ as well, and this proves $\varphi_\eta$ is extreme.

For the converse, suppose that $\pi$ is not irreducible. Then there exists a projection $P \in (\pi(\mathcal{A})')'$ such that neither $P$ nor $P^\perp$ is zero. Suppose that $P\eta = 0$. Then for all $A \in \mathcal{A}$, $P(\pi(A)\eta) = \pi(A)P\eta = 0$ and this would mean that $P$ vanishes on a sense subspace, which is not the case. Hence $\|P\eta\|_\mathcal{H} > 0$, and the same reasoning shows that $\|P^\perp\eta\|_\mathcal{H} > 0$. Define $\eta_1 = \|P\eta\|_\mathcal{H}^{-1}P\eta$ and $\eta_2 = \|P^\perp\eta\|_\mathcal{H}^{-1}P^\perp\eta$. For all $A \in \mathcal{A}$,

$$\langle \eta_1, \pi(A)\eta_2 \rangle_\mathcal{H} = \langle P\eta_1, \pi(A)P^\perp\eta_2 \rangle_\mathcal{H} = \langle \eta_1, PP^\perp\pi(A)\eta_2 \rangle_\mathcal{H} = 0.$$

Define $t \in (0, 1)$ by $t = \|P\eta\|_\mathcal{H}^2$. Since $\|P\eta\|_\mathcal{H}^2 + \|P^\perp\eta\|_\mathcal{H}^2 = 1$, $\|P^\perp\eta\|_\mathcal{H}^2 = 1 - t$. Then by the orthogonality proved just above, for all $A \in \mathcal{A}$,

$$\varphi_\eta(A) = \langle \sqrt{t}\eta_1 + \sqrt{1-t}\eta_2, \pi(A)(\sqrt{t}\eta_1 + \sqrt{1-t}\eta_2) \rangle_\mathcal{H}$$

$$= t\langle \eta_1, \pi(A)\eta_1 \rangle_\mathcal{H} + (1-t)\langle \eta_2, \pi(A)\eta_2 \rangle_\mathcal{H},$$

and this displays $\varphi_\eta$ as a non-trivial convex combination of states. Hence $\varphi_\eta$ is not extreme. □
4.3 Grothendieck’s Decomposition Theorem

4.14 DEFINITION. Let \( \mathcal{A} \) be a C* algebra. A linear functional \( \phi \in \mathcal{A}^* \) is Hermitian in case \( \phi(A^*) = \overline{\phi(A)} \) for all \( A \in \mathcal{A} \). The real vector space of all Hermitian elements of \( \mathcal{A}^* \) is denoted \( (\mathcal{A}^*)_{\text{s.a.}} \). Let \( (\mathcal{A}_{\text{s.a.}})^* \) denote the real Batch space dual to the real Banach space \( \mathcal{A}_{\text{s.a.}} \). Note that the elements of \( (\mathcal{A}^*)_{\text{s.a.}} \) are bounded complex linear functionals on \( \mathcal{A} \), while the elements of \( (\mathcal{A}_{\text{s.a.}})^* \) are bounded real linear functionals on \( \mathcal{A}_{\text{s.a.}} \).

4.15 LEMMA. A linear functional \( \phi \in \mathcal{A}^* \) is Hermitian if and only if \( \phi : \mathcal{A}_{\text{s.a.}} \to \mathbb{R} \). For \( \phi \in (\mathcal{A}^*)_{\text{s.a.}} \), define \( \phi_R \) to be the restriction of \( \phi \) to \( \mathcal{A}_{\text{s.a.}} \). For \( \varphi \in (\mathcal{A}_{\text{s.a.}})^* \), define \( \varphi_C : \mathcal{A} \to \mathbb{C} \) by
\[
\varphi_C(X + iY) = \varphi(X) + i\varphi(Y) \tag{4.9}
\]
for \( X, Y \in \mathcal{A}_{\text{s.a.}} \). Then \( \varphi_C \in (\mathcal{A}^*)_{\text{s.a.}} \), and the map \( \varphi \mapsto \varphi_C \) is an isometric isomorphism of \( (\mathcal{A}_{\text{s.a.}})^* \) onto \( (\mathcal{A}^*)_{\text{s.a.}} \).

Proof. Evidently if \( \phi \) is Hermitian and \( A \in \mathcal{A}_{\text{s.a.}} \), then \( \phi(A) \in \mathbb{R} \). Conversely, if \( \phi \in \mathcal{A}^* : \mathcal{A}_{\text{s.a.}} \to \mathbb{R} \), then \( \phi \) is Hermitian, since any \( A \in \mathcal{A} \) can be written as \( A = X + iY \), \( X, Y \in \mathcal{A}_{\text{s.a.}} \), and then \( \phi(A^*) = \phi(X) - i\phi(Y) = \overline{\phi(X + iY)} = \overline{\phi(A)} \).

If \( \phi \in \mathcal{A}^* \) is Hermitian, then the restriction \( \phi_R \) of \( \phi \) to the real Banach space \( \mathcal{A}_{\text{s.a.}} \) is a bounded real linear functional on \( \mathcal{A}_{\text{s.a.}} \), and since \( \phi \) is determined by its action on \( \mathcal{A}_{\text{s.a.}} \), \( \phi \mapsto \phi_R \) is one-to-one.

For any \( \varphi \in (\mathcal{A}_{\text{s.a.}})^* \), define an extension \( \varphi_C \) to \( \mathcal{A} \) by (4.9). Then \( \varphi_C(i(X + iY)) = i\varphi(X) - i\varphi(Y) = i\varphi_C(X + iY) \), and from here one readily sees that \( \varphi_C \) is complex linear functional on \( \mathcal{A} \), and moreover, \( \varphi_C \) is Hermitian, and the restriction of \( \varphi_C \) to \( \mathcal{A}_{\text{s.a.}} \) is simply \( \varphi \) itself.

Furthermore, for \( A = X + iY \), \( X, Y \in \mathcal{A}_{\text{s.a.}} \), and if \( \mathcal{A} \in B_{\mathcal{A}} \), then \( X, Y \in B_{\mathcal{A}_{\text{s.a.}}} \). It follows immediately that for all \( A \in B_{\mathcal{A}} \), \( |\varphi_C(A)| \leq 2\|\varphi\| \), so that so that \( \varphi_C \) is a Hermitian element of \( \mathcal{A}^* \). In fact, \( \|\varphi_C\| = \|\varphi\| \). To see this, fix \( \epsilon > 0 \), and choose \( A \in B_{\mathcal{A}} \) such that \( |\varphi(A)| + \epsilon > \|\varphi_C\| \). Choose \( \theta \in [0, 2\pi) \) so that \( e^{i\theta} \varphi(A) > 0 \). The replace \( A \) by \( e^{i\theta} A \), and write \( A = X + iY \), \( X, Y \in \mathcal{A}_{\text{s.a.}} \). Then \( \varphi_C(A) = \varphi(X) + i\varphi(Y) \) is real, so that \( \varphi(Y) = 0 \) and
\[
\|\varphi_C\| - \epsilon \leq \varphi_C(A) = \varphi(X) \leq \|\varphi\|\|X\| \leq \|\varphi\|
\]
since \( \|X\| \leq \|A\| \leq 1 \). Hence \( \|\varphi_C\| \leq \|\varphi\| \), and since \( \varphi \) is the restriction of \( \varphi_C \) to \( \mathcal{A}_{\text{s.a.}} \), the opposite inequality is trivially true.

4.16 LEMMA. The unit ball in \( (\mathcal{A}^*)_{\text{s.a.}} \) is the convex hull of \( S_{\mathcal{A}} \) and \( -S_{\mathcal{A}} \).

Proof. Let \( K \) be the convex hull of \( S_{\mathcal{A}} \) and \( -S_{\mathcal{A}} \). This is the image of \( [0, 1] \times S_{\mathcal{A}} \times S_{\mathcal{A}} \) under the map \( (t, \varphi, \psi) \mapsto (1 - t)\varphi - t\psi \), which is continuous with the natural product topology on the domain. As the continuous image of a compact set, \( K \) is compact. Let \( K_R \) be set of restrictions of elements of \( K \) to \( \mathcal{A}_{\text{s.a.}} \). Then \( K_R \) is weak *-compact and convex in \( (\mathcal{A}_{\text{s.a.}})^* \).

Suppose that there exists some \( \psi \) in the unit ball of \( (\mathcal{A}_{\text{s.a.}})^* \) that does not belong to \( K_R \). Then by the Hahn-Banach Separation Theorem, there is an \( A \in \mathcal{A}_{\text{s.a.}} \) and a \( t \in \mathbb{R} \) such that \( \psi(A) > t \), but \( \varphi(A) \leq t \) for all \( \varphi \in K \). Since \( -\varphi \in \mathcal{A}_{\text{s.a.}} \) whenever \( \varphi \in \mathcal{A}_{\text{s.a.}} \), \( |\varphi(A)| \leq t \). By Theorem 4.8, \( \|A\| \leq t \). But then \( \psi(A) > t \) is impossible for \( \psi \) in the unit ball of \( (\mathcal{A}_{\text{s.a.}})^* \). Hence no such \( A \) exists. By Lemma 4.9, \( K \) is then all of \( (\mathcal{A}^*)_{\text{s.a.}} \).
4.17 Lemma. Let $\mathcal{A}$ be a $C^*$ algebra. For every two positive linear functionals $\varphi$ and $\psi$ on $\mathcal{A}$, the following are equivalent:

1. $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|
2. For all $\epsilon > 0$, and all $A \in \mathcal{A}^+$, there exists $X,Y \in \mathcal{A}^+$ with $X + Y \in B_{\mathcal{A}}$ and $X - Y \in B_{\mathcal{A}}$ such that

   $$\varphi(X) < \epsilon, \quad \psi(Y) < \epsilon, \quad \varphi(Y) + \epsilon > \|\varphi\| \quad \text{and} \quad \psi(X) + \epsilon > \|\psi\|. \quad (4.10)$$

   and

   $$\|A(X + Y) - A\| = \|(X + Y)A - A\| < \epsilon. \quad (4.11)$$

Furthermore, when $\mathcal{A}$ is unital, one may take $X$ and $Y$ to satisfy $X + Y = 1$, so that (4.11) is trivially true for all $A$.

Proof. Suppose that (1) is valid. Pick $B,C \in \mathcal{A}^+ \cap B_{\mathcal{A}}$ such that $\varphi(B) + \epsilon > \|\varphi\|$ and $\psi(C) + \epsilon > \|\psi\|$. Since $\varphi - \psi \in (\mathcal{A}_{\text{s.a.}})^*$, for all $\epsilon > 0$, there exists $D$ in the unit ball of $\mathcal{A}_{\text{s.a.}}$ such that $\varphi(D) - \psi(D) + \epsilon \geq \|\varphi - \psi\|$.

Suppose $\mathcal{A}$ is not unital. By Theorem 2.32 applied to $\{A, B, C, D\}$, there exists $E \in \mathcal{A}^+ \cap B_{\mathcal{A}}$ such that $\|Z\| < \epsilon$ and $\|EZ - Z\| < \epsilon$ when $Z$ is any element of $\{A, B, C, D\}$. Then $EZE - Z = (EZE - Z) + EZ - Z$ so that $\|EZE - Z\| < 2\epsilon$ when $Z$ is any element of $\{A, B, C, D\}$.

Let $\mathcal{A}_1$ be the $C^*$ algebra obtained by adjoining an identity to $\mathcal{A}$ in the canonical manner. Then for self adjoint $Z$ in $B_{\mathcal{A}}$, the spectrum of $1 \pm Z$ lies in $[0, 2]$ and hence it follows that $E(1 \pm Z)E = E^2 \pm EZE$ have spectrum in $[0, 2]$. Thus, $E^2 \geq EZE$ and $\frac{1}{2}(E^2 \pm EZE)$ both belong to $\mathcal{A}^* \cap B_{\mathcal{A}}$.

If $\mathcal{A}$ is unital, things are much simpler: We may simply take $E = 1$. In either case, with $E = 1$ when $\mathcal{A}$ is unital, by (1),

$$\varphi(EAE) - \psi(EAE) + (1 + 2\|\varphi\| + 2\|\psi\|)\epsilon \geq \|\varphi\| + \|\psi\| \geq \varphi(E^2) + \psi(E^2). \quad (4.12)$$

Define

$$X := \frac{1}{2}(E^2 - EAE) \quad \text{and} \quad Y := \frac{1}{2}(E^2 + EAE).$$

Then as noted above, $X, Y \in \mathcal{A}^+$, and $X + Y = E^2 \in B_{\mathcal{A}}$ while $X - Y = -EAE \in B_{\mathcal{A}}$. Hence, with these definitions, (4.12) becomes

$$\psi(Y) + \varphi(X) \leq \frac{1}{2}(1 + 2\|\varphi\| + 2\|\psi\|)\epsilon. \quad (4.13)$$

Also, $\varphi(X + Y) = \varphi(E^2) \geq \varphi(EBE) \geq \|\varphi\| - (1 + 2\|\varphi\|)\epsilon$. Then by (4.13),

$$\varphi(Y) \geq \|\varphi\| - (2 + 3\|\varphi\| + \|\psi\|)\epsilon,$$

In the same way, we obtain a similar lower bound for $\psi(X)$. Finally, since $X + Y = E^2$ and $E^2A - A = E(EA - E) + (EA - a)\|E^2A - A\| \leq 2\epsilon$ then since $\epsilon > 0$ is arbitrary, this proves that (1) implies (2). Note that when $\mathcal{A}$ is unital $E^2 = 1$, and hence $X + Y = 1$.

Suppose that (2) is valid for any $A \in \mathcal{A}^+$, e.g., $A = 0$. Pick $\epsilon > 0$, and suppose that $X$ and $Y$ satisfy the conditions in (2). Then $X - Y \in B_{\mathcal{A}}$ and $(\varphi - \psi)(Y - X) \geq \|\varphi\| - \|\psi\| - 4\epsilon$. Hence $\|\varphi - \psi\| \geq \|\varphi\| + \|\psi\|\epsilon$, and the opposite inequality is trivial. \qed
4.18 DEFINITION. Let $\mathcal{A}$ be a $C^*$ algebra. Two positive linear functionals on $\mathcal{A}$ are mutually singular in case $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|$, in which case we write $\varphi \perp \psi$.

4.19 THEOREM (Grothendieck’s Decomposition Theorem). Let $\mathcal{A}$ be a $C^*$ algebra and let $\phi \in (\mathcal{A}^*)_{\text{s.a.}}$. Then $\phi$ has a unique decomposition $\phi = \varphi - \psi$ where $\varphi$ and $\psi$ are mutually singular positive linear functionals on $\mathcal{A}$.

Proof. We may suppose without loss of generality that $\|\phi\| = 1$. By Lemma 4.16, there are $\phi_1, \phi_2 \in \mathcal{A}^+$ and $t \in [0,1]$ such that $\phi = (1 - t)\phi_1 - t\phi_2$. Define $\varphi = (1 - t)\phi_1$ and $\psi = t\phi_2$. Then

$$1 = \|\phi\| = \|\varphi - \psi\| \leq \|\varphi\| + \|\psi\| = (1 - t)\|\phi_1\| + t\|\phi_2\| \leq 1.$$ 

Hence equality must hold throughout, and hence $\varphi$ and $\psi$ are mutually singular. This proves the existence of such a decomposition.

Now let $\phi = \varphi - \psi$ be one such decomposition, and suppose that $\phi = \tilde{\varphi} - \tilde{\psi}$ is another. Pick $\epsilon > 0$ and $A \in \mathcal{A}^+$, and pick $X,Y$ satisfying the conditions of (2) of Lemma 4.17 for the first decomposition $\phi = \varphi - \psi$. As we have observed above, $\phi(Y - X) \geq \|\phi\| - 4\epsilon$. Hence $\tilde{\varphi}(Y) + \tilde{\psi}(X) - \tilde{\varphi}(X) - \tilde{\psi}(Y) \geq \|\tilde{\varphi}\| + \|\tilde{\psi}\| - 4\epsilon$. It follows that $\tilde{\varphi}(X), \tilde{\psi}(Y) \leq 4\epsilon$.

Next, $\varphi(Y) = \psi(Y) + \tilde{\psi}(Y) = \psi(Y) + \psi(Y)$ and hence $\varphi(Y) \leq \|\varphi\| - 5\epsilon$. It follows that $\|\tilde{\varphi}\| \geq \|\varphi\|$. By symmetry, $\|\tilde{\psi}\| = \|\varphi\|$, and hence $\tilde{\varphi}(Y) \geq \|\varphi\| - 5\epsilon$. A similar argument applies to $X$ and $\psi$. Thus, replacing $\epsilon$ by $5\epsilon$, we have that (4.11) is of satisfied with the same pair $X,Y$ for both decompositions, and (4.11) is independent of the decomposition. Hence for each $A \in \mathcal{A}^+$, and each $\epsilon > 0$, there is a single pair $X,Y$ satisfying the conditions in (2) of Lemma 4.17. Using this, we show the two decompositions coincide.

By the Cauchy-Schwarz inequality, $|\varphi(AX)| \leq (\varphi(X^{1/2}AX^{1/2})\varphi(X))^{1/2} \leq \epsilon$, and likewise for $\tilde{\varphi}(AX), \psi(AY)$ and $\tilde{\psi}(AY)$. Hence

$$|\varphi(AX)|, |\tilde{\varphi}(AX)|, |\psi(AY)|, |\tilde{\psi}(AY)| < \epsilon. \quad (4.14)$$

Now for $\epsilon > 0$, let $X,Y$ satisfy the conditions (2) of Lemma 4.17 for both decompositions. Then

$$|\varphi(AY) - \tilde{\varphi}(AY)| \leq |\varphi(A(X + Y)) - \tilde{\phi}(A(X + Y))| + |\varphi(AX) - \tilde{\phi}(AX)| \leq |\varphi(A(X + Y)) - \tilde{\phi}(A(X + Y))| + 2\epsilon.$$ 

Since $\|A(X + Y)\| < \epsilon$, $|(|\varphi(A(X + Y)) - \tilde{\phi}(A(X + Y))| - (\varphi(A) - \tilde{\phi}(A))| \leq 2\|\varphi\|\epsilon$. Therefore, since $\varphi - \tilde{\phi} = \psi - \tilde{\psi}$,

$$|\varphi(A) - \tilde{\phi}(A)| \leq 2(1 + \|\varphi\|)\epsilon + |\varphi(AY) - \tilde{\phi}(AY)| = 2(1 + \|\varphi\|)\epsilon + |\tilde{\psi}(AY) - \psi(AY)| \leq 2(1 + \|\varphi\|)\epsilon$$

Since $\epsilon > 0$ is arbitrary, and since Hermitian linear functional are determined by their values on $\mathcal{A}^+$, $\tilde{\varphi} = \varphi$.

For $\varphi \in \mathcal{A}^*$, define $\varphi^*$ by

$$\varphi^*(A) = \overline{\varphi(A^*)} \quad \text{for all} \quad A \in \mathcal{A}. \quad (4.15)$$

Note that $\varphi$ is Hermitian if and only if $\varphi^* = \varphi$. In any case, $\frac{1}{2}(\varphi + \varphi^*)$ and $\frac{1}{2i}(\varphi - \varphi^*)$ are Hermitian, and $\varphi = \frac{1}{2}(\varphi + \varphi^*) + \frac{1}{2i}(\varphi + \varphi^*)$, so that every $\varphi \in \mathcal{A}^*$, is a linear combination of two elements of $(\mathcal{A}^*)_{\text{s.a.}}$.
4.4 Extreme points on the unit ball in a von Neumann algebra

Let \( \mathcal{A} \) be a C* algebra. Then \( B_{\mathcal{A}} \) may contain no extreme points at all. For example, let \( \mathcal{A} = \mathcal{C}_0(X) \), where \( X \) is a locally compact Hausdorff space that is not compact, so that \( \mathcal{A} \) is not unital. Let \( f \in B_{\mathcal{A}} \), and let \( K \) be a compact set outside of which \( |f(x)| \leq 1/2 \). Let \( g \in B_{\mathcal{A}} \) have support in \( K^c \), and note that \( f = \frac{1}{2}(f + fg) + \frac{1}{2}(f - fg) \) and \( f \pm fg \in B_{\mathcal{A}} \).

4.20 LEMMA. In any unital C* algebra, every unitary \( U \) is extreme in \( B_{\mathcal{A}} \).

Proof. We first show that the identity \( 1 \) is extreme. Suppose that for some \( t \in (0, 1) \) and \( X, Y \in \mathcal{A} \),

\[
1 = (1 - t)X + tY. \quad \text{if} \quad t < \frac{1}{2}, \quad 1 = \frac{1}{2}X + \frac{1}{2}W \quad \text{where} \quad W = (1 - 2t)X + 2tY \in B_{\mathcal{A}}, \quad \text{and} \quad X = Y \quad \text{if and only if} \quad X = W. \quad \text{A similar argument applies when} \quad t > 1/2. \quad \text{Hence it suffices to show that when} \quad 1 = \frac{1}{2}(X + Y), \quad X, Y \in B_{\mathcal{A}}, \quad \text{then} \quad X = Y = 1.
\]

Suppose \( X, Y \in B_{\mathcal{A}} \) and \( 1 = \frac{1}{2}(X + Y) \). Let \( A = \frac{1}{2}(X + X^*) \) and \( B = \frac{1}{2}(X + X^*) \). Then \( 1 = \frac{1}{2}(X + Y) \), and \( A, B \in B_{\mathcal{A}} \) are self adjoint. Since \( B = 2I - A \) and since \( \sigma(A) \subset [-1, 1] \), \( \sigma(B) \subset [1, 3] \) But since \( B \in B_{\mathcal{A}} \), \( \sigma(B) \subset [-1, 1] \). Hence \( \sigma(B) = \{1\} \), and hence \( B = 1 \), from which it follows that \( A = 1 \). Then \( X \) has the form \( 1 + iC \), where \( C \) is self adjoint. Therefore \( X \) is normal and its norm is its spectral radius. Hence if \( C \neq 0 \), \( \|X\| > 1 \), which is not the case. Hence \( X = 1 \), and in the same way, we conclude \( Y = 1 \).

Now let \( U \) be unitary, and suppose that for some \( U = \frac{1}{2}(X + Y) \) for \( X, Y \in B_{\mathcal{A}} \). Then \( 1 = \frac{1}{2}(U^*X + U^*Y) \), and \( U^*X, U^*Y \in B_{\mathcal{A}} \). Hence \( U^*X = U^*Y \), and then \( X = Y \). \( \square \)

A theorem of Kadison [11] specifies the extreme points of the unit ball of any unital C* algebra \( \mathcal{A} \). There turn out to be closely related to unitarities: The extreme points are a special sort of partial isometry.

4.21 DEFINITION. An element \( U \in \mathcal{A} \) is a partial isometry in case \( P = U^*U \) is an idempotent; i.e., \( P^2 = P \), and since \( P \) is self adjoint, this is the case exactly when \( \sigma(P) \subset \{0, 1\} \).

Since \( UU^* \) has the same spectrum as \( U^*U \), note that \( U^* \) is a partial isometry if and only if \( U \) is a partial isometry. Of course, every unitary is a partial isometry, but not every partial isometry is unitary.

At this point we only prove Kadison’s Theorem in the von Neumann algebra setting, in which its proof is facilitated by the use of the polar decomposition. However, the theorem is true as stated for all unital C* algebras. It is the von Neumann algebra result that we need in the next sections.

4.22 THEOREM (Kadison’s Extreme Point Theorem). Let \( \mathcal{A} \) be a von Neumann algebra. Then the extreme points of \( B_{\mathcal{A}} \) are precisely the elements \( U \in \mathcal{A} \) such that

\[
(1 - UU^*)\mathcal{A}(1 - U^*U) = 0, \tag{4.16}
\]

and all of these are isometries.

Proof. If \( U \) satisfies (4.16), then \( U^*((1 - UU^*)(1 - U^*U)) = 0 \), but the left hand side is \( U^*U(1 - U^*U)^2 \), and so \( \sigma(U^*U) \subset \{0, 1\} \), and hence \( U \) is a partial isometry.

We now show that (4.16) is necessary, starting with a proof of the consequent fact that every extreme point of \( B_{\mathcal{A}} \) must be an isometry. For \( X \in B_{\mathcal{A}} \), we have from the from the polar decomposition

\[
X = U|X| = \frac{1}{2}(U(2|X| - |X|^2) + U|X|^2). \]
Since $|X|$ has spectrum in $[0, 1]$, $U(2|X| - |X|^2), U|X|^2 \in B_{\mathcal{A}}$. Moreover, $U(2|X| - |X|^2) = U|X|^2$ if and only if $|X| = |X|^2$. This is the case if and only if $|X|$, and hence $|X|^2 = X^*X$ has spectrum in $\{0, 1\}$. That is, unless $X$ is a partial isometry, $X$ cannot be an extreme point.

Let $U$ be an idempotent in $\mathcal{A}$, and define the projections $P = U^*U$ and $Q = UU^*$. Simple computations show that $(U - UP)^*(U - UP) = 0v$ and $(U - QU)^*(U - QU) = 0$. Hence $U = UP = QU$.

Let $A \in B_{\mathcal{A}}$, and define $B = Q^\perp AP^\perp$. Then $U + B = QUP + Q^\perp AP^\perp$. Then
\[
\|U + B\|^2 = \|(U + B)^*(U + B)\| = \|PU^*QUP + P^\perp A^*Q^\perp AP^\perp\| = \max\{\|U^*U\|, \|A\|^2\} \leq 1,
\]
where in the last line we have used the fact that $U^*U = PU^*QUP$ and $P^\perp A^*Q^\perp AP^\perp$ are commuting and self adjoint, and their product is zero. Thus, if $B \neq 0$, the decomposition $U = \frac{1}{2}(U + B) + \frac{1}{2}(U - B)$, the decomposition shows that $U$ is not extreme. Thus, (4.16) is a necessary condition for $U$ to be extreme.

Now suppose that $U$ satisfies (4.16). As we have noted at the beginning, $U$ is an isometry. As above, define $P = U^*U$ and $Q = UU^*$. Suppose $X,Y \in B_{\mathcal{A}}$ and $U = \frac{1}{2}(X + Y)$. Since $U = UP = QU$, we also have $U = \frac{1}{2}(XP + YP)$, and then
\[
P = \frac{1}{4}(PX^*XP + PX^*YP + PY^*XP + PY^*YP).
\]
Since $P$ is the identity in the $C^*$ algebra $P_{\mathcal{A}}P$, $P$ is extreme, and hence of the terms on the right in (4.17) are equal to $P$. But then $P(X - Y)^*(X - Y)P = 0$, and so $XP = YP$. The same sort of argument shows that $QX = QY$, and then $X - Y = (1 - Q)(X - Y)(1 - P)$, and this is 0 by (4.16). Hence $X = Y$, and $U$ is extreme.

**4.23 REMARK.** Only in the second paragraph of the proof, where we used the polar decomposition, did we make any arguments that are not valid in a general unital $C^*$ algebra. But we really need in this paragraph was a that each $X \in B_{\mathcal{A}}$ should have a decomposition $X = VB$ where $B$ is positive with spectrum in $[0, 1]$ and $\|U\| \in B_{\mathcal{A}}$. This can be done by other means.

**4.5 Normal functionals**

**4.24 DEFINITION.** Let $\mathcal{M}$ be a von Neumann algebra, and let $\mathcal{M}_*$ be its predual, as described in Theorem 3.38. The second dual of $\mathcal{M}_*, (\mathcal{M}_*)^{**}$ is $\mathcal{M}^*$, and we may identify $\mathcal{M}_*$ with a subspace of $\mathcal{M}^*$ using the canonical embedding of $\mathcal{M}_*$ into $(\mathcal{M}_*)^{**}$. A linear functional $\phi \in \mathcal{M}^*$ is a normal functional if and only if it is in the image of $\mathcal{M}_*$ under this canonical embedding. We use $\mathcal{M}_*$ to denote the set of normal functionals on $\mathcal{M}$.

By Theorem 3.38, every $\phi \in \mathcal{M}_*$ is of the form $\phi(B) = \text{Tr}[AB]$ for some $A \in \mathcal{T}(\mathcal{H})$. Therefore, a neighborhood base at 0 for the weak-$*$ topology on $\mathcal{M}$ is given by the sets
\[
V_{A_1, \ldots, A_m, \epsilon} = \{B \in \mathcal{M} : |\text{Tr}[A_jB]| < \epsilon, j = 1, \ldots, m \}.
\]
Since $\mathcal{B}(\mathcal{H})$ is the dual of $\mathcal{T}(\mathcal{H})$, there same sets are a neighborhood base at 0 for the relative $\sigma$-weak topology on $\mathcal{M}$ considered as a subset of $\mathcal{B}(\mathcal{H})$.

For any Banach space $X$, a linear functional $\phi \in X^{**}$ is continuous with respect to the weak-$*$ topology on $X^*$ if and only if it is in the image of the canonical embedding of $X$ into $X^{**}$. Hence
φ ∈ ℳ* is normal if and only if it is continuous in the weak-* topology on ℳ, and by what we have just observed, this is the case if and only if φ is continuous on ℳ equipped with the relative σ-weak topology.

4.25 THEOREM. Let ℳ be a von Neumann algebra, and let φ ∈ ℳ*. Then the following are equivalent:

(i) φ normal.
(ii) φ is continuous on ℳ when ℳ is equipped with the σ-weak topology.
(iii) There exists A ∈ ℰ(ℋ) such that φ(B) = Tr[AB] for all B ∈ ℳ.

Proof. The discussion just before the statement of the theorem proves the equivalence of (i) and (ii). The equivalence of (i) and (iii) is an immediate consequence of Theorem 3.38.

Later in this section, we shall sharpen (iii) considerably: It turns out that there is an A ∈ ℰ(ℋ) such that φ(B) = Tr[AB] for all B ∈ ℳ with ||A||₁ = ||φ||, and that we may choose A ≥ 0 if φ ≥ 0. Thus every element of ℳ* has a norm-preserving extension to ℬ(ℋ), and if φ is positive, there is a norm-preserving positive extension to ℬ(ℋ). Before we are able to prove this, we must draw some more immediate consequences from Theorem 4.25.

4.26 COROLLARY. Let ℳ be a von Neumann algebra. Then ℳ* is a norm closed subspace of ℳ. Furthermore, let φ be a normal functional on a von Neumann algebra ℳ. Then:

(i) φ* is also normal, and hence the Hermitian and skew-Hermitian parts of φ are normal.
(ii) For all B ∈ ℳ, define Bφ to be the linear functional Bφ(X) = φ(BX) for all X ∈ ℳ, and define φB be the linear functional φB(X) = φ(XB) for all X ∈ ℳ. Then Bφ and φB are also normal.
(iii) Let φ be a Hermitian normal functional on ℳ, and let φ = φ⁺ − φ⁻ be its Grothendieck decomposition. then φ⁺ and φ⁻ are normal.

Proof. Let {φₙ}ₙ∈ℕ be a sequence in ℳ* converging in norm to φ ∈ ℳ*. Since the canonical embedding is isometric, {φₙ}ₙ∈ℕ is a Cauchy sequence sequence in ℳ*, and ℳ* is complete. Hence φ ∈ ℳ*.

Let φ be normal. By Theorem 4.25, there is A ∈ ℰ(ℋ) such that φ(X) = Tr[AX] for all X ∈ ℳ, and conversely, any linear functional of this form is normal.

To prove (i), note that φ*(X) = Tr[AX*] = Tr[XA*] = Tr[A*X]. Since A* ∈ ℰ(ℬ), φ* ∈ ℳ*, and then of course 1/2(φ + φ*) and 1/2(φ − φ*) belong to ℳ*.

To prove (ii), note that Bφ*(X) = Tr[BAX] and BA ∈ ℰ(ℋ). The same reasoning applies to φB.

To prove (iii), pick ε > 0, and then by the final part of Lemma 4.17, since ℳ is unital, there exist Y ≥ 0 with Y ∈ ℬₙ such that φ⁺(1 − Y), φ⁻(Y) < ε. Then for any X ∈ ℳ,

\[ \|φ⁺(X) − Yφ(X)\| = \|φ⁺((1 − Y)X) − φ⁻(YX)\| \leq (φ⁺(1 − Y)φ⁺(X*(1 − Y)X))^{1/2} + (φ⁻(Y)φ⁺(X*YX))^{1/2} \leq ε(\|φ⁺\| + \|φ⁻\|)||X|| = ε\|φ\||||X||. \]

since Yφ is normal by part (ii), and since X and ε > 0 are arbitrary, φ⁺ is in the norm closure of ℳ*, which is ℳ* itself. □
4.27 THEOREM (Sakai’s Polar Factorization Theorem). Let $\mathcal{M}$ be a von Neumann algebra, and let $\phi \in \mathcal{M}_*$. Then there exists a unique positive normal functional $|\phi|$ such that $|||\phi|| = \|\phi\|$ and such that

$$|\phi(X)|^2 \leq ||\phi|||\phi|(X^*X)$$

(4.18)

for all $X \in \mathcal{M}$. Furthermore, there is a partial isometry $U \in \mathcal{M}$ such that $\phi = U|\phi|$ and $|\phi| = U^*\phi$.

**Proof.** Let $||\phi|| = 1$. The set $\{X \in B_{\mathcal{M}} : \phi(X) = 1\}$ is a non-empty, convex, $\sigma$-weakly compact subset of $B_{\mathcal{M}}$. Hence this set contains an extreme point $U^*$, which is also evidently an extreme point of $B_{\mathcal{M}}$. By Theorem 4.22, $U^*$ is a partial isometry such that with $P := U^*U$ and $Q := UU^*$, $P^\perp M Q^\perp = 0$. Evidently $U$ is also a partial isometry.

Now define two linear functionals $\psi_\ell$ and $\psi_r$ by $\psi_\ell := U^*\phi$ and $\psi_r := \phi U^*$. Evidently, $||\psi_\ell||, ||\psi_r|| \leq ||\phi|| = 1$, and both are normal by Corollary 4.26. Since

$$||\psi_\ell|| \geq \psi_\ell(1) = \phi(U^*) = 1 \geq ||\psi_\ell||,$$

$\psi_\ell$ is positive and $||\psi_\ell|| = ||\phi|| = 1$. Likewise, $\psi_r$ is positive and $||\psi_r|| = ||\phi|| = 1$.

Since $(U - UU^*U^*)(U - UU^*) = 0$, $U = UP = QU$. Taking adjoints, $U^* = PU^*$ and $U^* = U^*Q$. Then $\psi_\ell(Q) = \phi(U^*Q) = \phi(U^*) = 1$, and hence $\psi_\ell(Q^\perp) = 0$. The same sort of argument shows that $\psi_r(P^\perp) = 0$. Now for all $X \in \mathcal{M}$, using $P^\perp M Q^\perp = 0$,

$$\phi(P^\perp X) = \phi(P^\perp XQ) + \phi(P^\perp XQ^\perp) = \phi(P^\perp XQ) \leq \phi(P^\perp (U^* X^* U)X) \leq (\psi_r(P^\perp))^{1/2} = 0.$$

Since $X$ is arbitrary, $P^\perp \phi = 0$, and $\phi = P\phi$. But then for all $X$, $U\psi_\ell(X) = \psi_\ell(UX) = \phi(PX) = \phi(X)$, and hence $U\psi_\ell = \phi$. (The same sort of argument shows that $\phi = Q\phi$, and that $\psi_rU = \phi$.)

By the Cauchy-Schwarz inequality, for all $X \in \mathcal{M}$, $|\phi(X)|^2 = |\psi_\ell(UX)|^2 \leq \psi_\ell(U^*U)\psi_\ell(X^*X) \leq \||\psi_\ell||\psi_\ell(X^*X)$. Likewise, $|\phi(X)|^2 = |\psi_r(XU)|^2 \leq \||\psi_r||\psi_r(X^*X)$ We now show that there is exactly one positive functional such $\psi$ with $||\psi|| = 1$ such that for all $X$,

$$|\phi(X)|^2 \leq ||\psi||\psi(X^*X).$$

(4.19)

This will prove that $\psi_\ell = \psi_r$, and that (4.19) characterizes $\psi_\ell = \psi_r$.

Suppose $\psi$ is any positive functional with $||\psi|| = 1$ that satisfies (4.19). For $X$ self adjoint,

$$|\psi_\ell(X)|^2 = |\phi(U^*X)|^2 \leq \psi(XUU^*X) \leq \psi(X^2).$$

Replacing $X$ by $1 + \epsilon X$, this yields $|\psi_\ell(1 + \epsilon X)|^2 \leq \psi((1 + \epsilon X)^2)$, and since $\psi_\ell(1) = \psi(1) = 1$, and since $\epsilon > 0$ is arbitrary, this yields. $\psi_\ell(X) \leq \psi(X)$ for all self adjoint $X$. Replacing $X$ with $-X$, we get that $\psi_\ell = \psi$. We now define $|\phi| = \psi_\ell$. We note that the same reasoning would show that $\psi_\ell$ is the unique positive functional such $\psi$ with $||\psi|| = 1$ such that for all $X$, $|\phi(X)|^2 \leq ||\psi||\psi(X^*X)$; note the difference with (4.19).

Let $\psi$ be a positive normal functional on a von Neumann algebra $\mathcal{M}$ on the Hilbert space $\mathcal{H}$. Then there exists $A \in \mathcal{S}(\mathcal{M})$ such that $\phi(X) = \text{Tr}[AX]$ for all $X \in \mathcal{M}$. Since $\phi$ is Hermitian, for all self adjoint $X \in \mathcal{M}$, $\phi(X) = \text{Tr}[AX] = \text{Tr}[(AX)^*] = \text{Tr}[XA^*] = \text{Tr}[A^*X]$. Since $\phi$ is determined by its action $\mathcal{M}_{sa}$, it follows that we may replace $A$ with $\frac{1}{2}(A + A^*)$ without changing $\phi$. That is, we may assume that $A$ is self adjoint. We next show that we may take $A$ to be positive with $\text{Tr}[A] = ||\phi||$. 


4.28 THEOREM (Dixmier’s Extension Theorem). Let $\psi$ be a positive normal functional on a von Neumann algebra $\mathcal{M}$ on the Hilbert space $\mathcal{H}$. Then there exists a positive $A \in \mathcal{F}(\mathcal{H})$ with $\text{Tr}[A] = \|\phi\|$. In particular, $\phi$ has a norm-preserving normal extension to $\mathcal{B}(\mathcal{H})$ that is also positive.

Proof. We have seen that we may write $\phi$ in the form $\phi(X) = \text{Tr}[AX]$ for some self-adjoint $A \in \mathcal{F}(\mathcal{H})$. Then $A$ has the spectral expansion $A = \sum_{j=1}^{\infty} \lambda_j |\zeta_j\rangle \langle \zeta_j|$, where $\{\zeta_j\}_{j \in \mathcal{N}}$ is orthonormal, $|\lambda_{j+1}| \geq |\lambda_j|$, and $\sum_{j=1}^{\infty} |\lambda_j| = \|A\|_1$.

Let $\mathcal{H} = \bigoplus_{j=1}^{\infty} \mathcal{H}_j$ where each $\mathcal{H}_j$ is a copy of $\mathcal{H}$. For $(\eta_j)_{j \in \mathbb{N}} \in \mathcal{H}$ and $x \in \mathcal{M}$, define $\pi(x)(\eta_j)_{j \in \mathbb{N}} = (\lambda_j \eta_j)_{j \in \mathbb{N}}$. Let $\hat{\xi}_0 := \|A\|^{-1/2} |\lambda_j|^{1/2} \zeta_j \rangle \langle \zeta_j|_{j \in \mathbb{N}}$, where $\{\zeta_j\}_{j \in \mathcal{N}}$ is the orthonormal sequence in the spectral expansion of $A$, and $A\zeta_j = \lambda_j \zeta_j$ for each $j$. Define $\mathcal{K}$ to be the closure of $\pi(\mathcal{M}) \hat{\xi}_0$. The $\mathcal{K}$ is invariant under $\pi$, and $\hat{\xi}_0$ is a cyclic unit vector for the resulting representation on $\mathcal{B}(\mathcal{K})$.

Now observe that for all $X \in \mathcal{M}^+$,

$$\langle \hat{\xi}_0, \pi(X)\hat{\xi}_0 \rangle = \frac{1}{\|A\|} \sum_{j=1}^{\infty} |\lambda_j| \langle \zeta_j, X \zeta_j \rangle \geq \frac{1}{\|\phi\|} \sum_{j=1}^{\infty} \lambda_j \langle \zeta_j, X \zeta_j \rangle = \frac{1}{\|\phi\|} \phi(X) .$$

By Lemma 4.13, there exists a positive operator $\hat{B}$ in $(\pi(\mathcal{M}))'$, so that

$$\frac{1}{\|\phi\|} \phi(X) = \langle \hat{\xi}_0, \hat{B} \pi(X) \hat{\xi}_0 \rangle = \langle \hat{B}^{1/2} \hat{\xi}_0, \pi(X)\hat{B}^{1/2} \hat{\xi}_0 \rangle .$$

Taking $X = 1$, we see that $\hat{B}^{1/2} \hat{\xi}_0$ is a unit vector. Define $\{\eta_j\}_{j \in \mathbb{N}}$ and $B \in \mathcal{F}(\mathcal{H})$ by

$$\hat{B}^{1/2} \hat{\xi}_0 = (\eta_j)_{j \in \mathbb{N}} \quad \text{and} \quad B := \sum_{j=1}^{\infty} |\eta_j| \eta_j| ,$$

it being evident from $\|\hat{B}^{1/2} \hat{\xi}_0\|_1^2 = \sum_{j=1}^{\infty} \|\eta_j\|^2 = 1$, that the series defining $B$ converges in trace norm. With these definitions, $\phi(X) = \|\phi\| \text{Tr}[BX]$, and $\text{Tr}[B] = 1$. Hence $C := \|\phi\|B$ is a positive trace class operator on $\mathcal{H}$ such that $\phi(X) = \text{Tr}[CX]$ for all $X \in \mathcal{M}$, $C \geq 0$, and $\|C\|_1 = \|\phi\|$. Clearly $X \mapsto \text{Tr}[CX]$ is a normal functional on all of $\mathcal{B}(\mathcal{H})$, and its norm is $\|C\|_1 = \|\phi\|$. □

4.29 COROLLARY (Sakai). Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$. Every normal functional $\phi$ on $\mathcal{M}$ has an extension $\hat{\phi}$ to $\mathcal{B}(\mathcal{H})$ such $\|\hat{\phi}\| = \|\phi\|$. □

Proof. This is an immediate consequence of Dixmier’s Extension Theorem and Sakai’s Polar Decomposition Theorem. □

4.6 Order and normality

Let $\mathcal{S}$ be a bounded upward directed sets of operators in a von Neumann algebra $\mathcal{M}$. Then its least upper bound, $\bigvee_{A \in \mathcal{S}} A$, exists and is in the strong closure of $\mathcal{S}$. It is therefore also in the weak
closure, and since the weak and the σ-weak topologies coincide on bounded sets, \( \bigvee_{A \in S} A \) belongs to the σ-weak closure of \( S \).

It is then evident that if \( \phi \) is any positive normal functional on \( \mathcal{M} \),

\[
\phi \left( \bigvee_{A \in S} A \right) = \bigvee_{A \in S} \phi(A) .
\]

(4.20)

In fact the property (4.20) is characteristic of normal positive functionals:

**4.30 THEOREM.** A positive functional \( \phi \) on a von Neumann algebra \( \mathcal{M} \) is normal if and only if (4.20) is valid for every bounded upward directed set \( S \) in \( \mathcal{M} \).

**Proof.** It remains to show that for any positive \( \phi \), if (4.20) is valid for every bounded upward directed set \( S \) in \( \mathcal{M} \), then \( \phi \) is normal. We make two temporary definitions to be used in the proof: A positive linear functional \( \phi \) is order-continuous in case (4.20) is valid for every bounded upward directed set \( S \) in \( \mathcal{M} \), and we say that an element \( B \in \mathcal{M} \) is regular for the order continuous positive functional \( \phi \) in case \( \phi B \) is normal.

**Step 1:** Let \( S \) be any bounded upward directed set of regular elements of \( \mathcal{M} \). Let \( B = \bigvee_{A \in S} A \) which belongs to \( \mathcal{M} \) by Vigier’s Theorem. Then \( B \) is regular.

To prove this, it suffices to show that \( \phi B \) is the norm limit of a sequence in \( \{ \phi A_n \} \), \( \{ A_n \}_{n \in \mathbb{N}} \subset S \) since \( \mathcal{M} \) is norm closed. By the order continuity of \( \phi \), for each \( n \in \mathbb{N} \), there exists \( A_n \in S \) such that \( \phi(B - A_n) < 1/n \). Then for any \( X \in B \mathcal{M} \),

\[
|\phi(B - \phi A_n)(X)| = |\phi(X(B - A_n))| = |\phi(X((B - A_n)^{1/2}(B - A_n)^{1/2})| \\
= \phi(B - A_n)^{1/2} \phi(B - A_n)^{1/2} X(B - A_n)^{1/2} \leq \frac{1}{n} \|B\| \|\phi\| .
\]

Therefore \( \|\phi B - \phi A_n\| \leq \frac{1}{n} c \) where \( c \) is the least upper bound on the norms of elements of \( S \).

**Step 2:** Let \( M \in \mathcal{M} \) be non-zero and positive. Then there exists a non zero \( B \in \mathcal{M} \) such that \( 0 \leq B \leq M \), ands such that

\[
0 \leq X \leq cB \quad \text{for some } c > 0 \quad \Rightarrow \quad \phi(X) \leq \langle \xi, X \xi \rangle .
\]

(4.21)

To prove this, choose a vector \( \xi \in \mathcal{H} \) such that \( \langle \xi, M \xi \rangle > \phi(M) \), and define

\[
\mathcal{R} := \{ A \in \mathcal{M} : 0 \leq A \leq M , \langle \xi, A \xi \rangle \leq \phi(A) \} .
\]

The linear functional \( X \mapsto \langle \xi, A \xi \rangle \) is evidently normal, and hence order continuous, as is \( \phi \). Hence if \( S \) is any totally ordered set in \( \mathcal{R} \), it has a least upper bound that is in \( \mathcal{M} \) by Vigier’s Theorem, and which belongs to \( \mathcal{R} \) by order continuity. Then by Zorn’s Lemma, \( \mathcal{R} \) contains a maximal element \( C \). Define \( B = M - C \). Then \( 0 \leq B \leq M \) and \( \langle \xi, B \xi \rangle = \langle \xi, M \xi \rangle - \langle \xi, C \xi \rangle > \phi(M - C) \geq 0 \). Hence \( B \neq 0 \). Note that

\[
\phi(B) = \phi(M) - \phi(C) < \langle \xi, M \xi \rangle - \langle \xi, C \xi \rangle = \langle \xi, B \xi \rangle ,
\]

Suppose for some \( 0 \leq X \leq B \), \( \phi(X) \geq \langle \xi, X \xi \rangle \). Then

\[
\phi(C + X) = \phi(C) + \phi(X) \geq \langle \xi, C \xi \rangle + \langle \xi, X \xi \rangle = \langle \xi, (C + X) \xi \rangle ,
\]
and \(0 \leq C + X \leq C + B = M\). For \(X \neq 0\), this contradicts the maximality of \(C\). Hence for all \(0 \leq X \leq B\), \(\phi(X) = \langle \xi, X\xi \rangle\). Then (4.21) follows by homogeneity in \(X\).

**Step 3:** For \(B\) satisfying (4.21), \(\phi B\) is normal. Consequently, for every non-zero positive \(M \in \mathcal{M}\), there exists a non-zero regular \(\phi B \in \mathcal{M}\) such that \(0 \leq \phi B \leq M\).

To see this, note that for all \(A \in \mathcal{M}\), by the Cauchy-Schwarz inequality, \(|\phi B(A)|^2 = |\phi(AB)|^2 \leq \phi(1)\phi(B^*A^*AB) \leq \|\phi\|\phi(B^*A^*AB)\). Then since \(B^*A^*AB \leq \|A\|^2B^2 \leq \|A\|^2\|B\|\|B\|\), (4.21) implies that \(\phi(B^*A^*AB) \leq \langle \xi, B^*A^*AB\xi \rangle = \|AB\xi\|^2\). Altogether, \(|\phi B(A)| \leq \|\phi\|^{1/2}\|AB\xi\|\) and \(\phi B\) is continuous in the strong operator topology. But the strong and weak operator topologies have the same set of continuous linear functionals, and the \(\sigma\)-weak topology is stronger than the weak operator topology. Hence \(\phi B\) is normal.

**Step 4:** By Zorn’s Lemma and the result of the first step, we may choose a maximal regular \(B\) such that \(0 \leq B \leq 1\). By what we proved in the third step, the maximal element must be 1. Hence 1 is regular, and \(\phi\) is normal. \(\square\)

### 4.7 Normal morphisms

**4.31 Definition** (Normal maps). Let \(\mathcal{M}\) and \(\mathcal{N}\) be von Neumann algebras. A bounded linear transformation \(\Phi : \mathcal{M} \to \mathcal{M}\) is **normal** in case \(\Phi\) is continuous with respect to the \(\sigma\)-weak topology on \(\mathcal{M}\) and \(\mathcal{N}\).

Let \(\mathcal{L}(\mathcal{M}, \mathcal{N})\) denote the set of bounded linear transformations from \(\mathcal{M}\) to \(\mathcal{N}\). It is evident that \(\Phi \in \mathcal{L}(\mathcal{M}, \mathcal{N})\) is normal if and only if \(\varphi \circ \Phi \in \mathcal{M}_s\) for all \(\varphi \in \mathcal{N}_s\), and then decomposing \(\varphi\) into a linear combination of positive components, it suffices that \(\varphi \circ \Phi \in \mathcal{M}_s\) for all \(\varphi \in \mathcal{N}_s^+\). \(\mathcal{L}_N(\mathcal{M}, \mathcal{N})\) denote the subspace of \(\mathcal{L}(\mathcal{M}, \mathcal{N})\) consisting of normal linear transformations. Then \(\mathcal{L}_N(\mathcal{M}, \mathcal{N})\) is norm closed in \(\mathcal{L}(\mathcal{M}, \mathcal{N})\): If \(\{\Phi_n\}_{n \in \mathbb{N}}\) is a sequence in \(\mathcal{L}_N(\mathcal{M}, \mathcal{N})\) converging to \(\Phi \in \mathcal{L}(\mathcal{M}, \mathcal{N})\), for each \(\varphi \in \mathcal{N}_s\), \(\{\varphi \circ \Phi_n\}_{n \in \mathbb{N}}\) converges to \(\varphi \circ \Phi\) in \(\mathcal{N}_s\), a closed subspace of \(\mathcal{M}_s^*\).

Let \(\Phi\) be a linear transformation on \(\mathcal{M}\) such that \(\Phi(A) \in \mathcal{N}_s^+\) for all \(A \in \mathcal{M}_s^+\). Then for all \(\phi \in \mathcal{M}_s^*, \phi \circ \Phi\) is a positive linear functional on \(\mathcal{M}\), and is therefore bounded. By the Uniform Boundeness Principle, there exists a finite constant \(c\) such that \(\|\phi \circ \Phi\| \leq c\) for all \(\phi \in \mathcal{N}_s^*\) with \(\|\phi\| \leq 1\). But then \(\|\Phi\| \leq c\). Hence positive linear transformations, like positive linear functionals, are always bounded.

**4.32 Lemma.** Let \(\mathcal{M}\) and \(\mathcal{N}\) be von Neumann algebras, and let \(\Phi : \mathcal{M} \to \mathcal{N}\) be a positive map. Then \(\Phi\) is normal if and only if whenever \(S\) be a bounded directed set in \(\mathcal{M}_{s,a}\),

\[
\Phi \left( \bigvee_{A \in S} A \right) = \bigvee_{A \in S} \Phi(A) .
\]

**Proof.** It is evident that \(\Phi\) is normal if and only if \(\varphi \circ \Phi \in \mathcal{M}_s\) for all \(\varphi \in \mathcal{N}_s^+\). Let \(S\) be a bounded directed set in \(\mathcal{M}_{s,a}\). By Theorem 4.30, for every morphism \(\Phi\) and every positive linear functional \(\varphi\), \(\varphi \circ \Phi \in \mathcal{M}_s\) if and only if

\[
\varphi \circ \Phi \left( \bigvee_{A \in S} A \right) = \bigvee_{A \in S} \varphi \circ \Phi(A) .
\]

Since \(\Phi(A) \in \mathcal{N}_s^+\) when \(A \in \mathcal{M}_s^+\), \(\Phi(S)\) is a bounded upward directed set in \(\mathcal{M}\), Suppose that whenever \(S\) is a bounded directed set in \(\mathcal{M}_{s,a}\),
Then by Theorem 4.30 again, for every \( \varphi \in \mathcal{N} \), (4.23) is valid, and then \( \varphi \circ \Phi \in \mathcal{M} \). Therefore, \( \Phi \) is normal. Conversely, let \( B := \bigvee_{A \in S} A \) and \( C := \bigvee_{A \in S} \Phi(A) \). Then \( \Phi(B) \geq C \). If (4.22) is false, since normal linear functionals separate, there exists \( \varphi \in \mathcal{N} \) such that \( \varphi(\Phi(B)) > \varphi(C) \). Then (4.23) fails for this \( \varphi \).

Among positive maps, the von Neumann algebra morphisms are particularly important: A linear transformation \( \Phi : \mathcal{M} \to \mathcal{N} \), where \( \mathcal{M} \) and \( \mathcal{N} \) are von Neumann algebras, is morphism in case it is a \( \ast \)-homomorphism. Then for all \( A \in \mathcal{M}^+ \), \( A = B^*B \), \( B \in \mathcal{M} \), and then \( \Phi(A) = \Phi(B)^*\Phi(B) \in \mathcal{N}^+ \). That is, every morphism is positive.

4.33 Lemma. Every isomorphism \( \Phi \) between von Neumann algebras is normal.

**Proof.** Let \( S \) be any bounded directed set in \( \mathcal{M} \). Then \( \Phi(S) = \Phi(A) \) \( : A \in S \) is a bounded directed set in \( \mathcal{N} \). Let \( B := \bigvee_{A \in S} A \) and \( C := \bigvee_{A \in S} \Phi(A) \). Then \( B \geq C \), and since \( S = \Phi^{-1}(\Phi(S)) \), \( C \geq B \). Thus \( C = B \), and the claim follows from Lemma 4.32.

4.34 Theorem (Kernels and ranges of normal morphisms). \( \mathcal{M} \) and \( \mathcal{N} \) be von Neumann algebras, and let \( \Phi : \mathcal{M} \to \mathcal{N} \) be a normal morphism. Then \( \ker(\Phi) \) is a strongly closed ideal in \( \mathcal{M} \), and \( \Phi(\mathcal{M}) \) is a von Neumann subagebra of \( \mathcal{N} \).

**Proof.** Since \( \Phi \) is normal, \( \mathcal{J} := \ker(\Phi) \) is \( \sigma \)-weakly closed, and hence is norm closed. Hence \( \mathcal{J} \) is a norm closed ideal in \( \mathcal{M} \), and hence a \( \mathcal{C}^* \) subalgebra of \( \mathcal{M} \). We must show that is is strongly closed, or, what is the same since \( \mathcal{J} \) is convex, that it is weakly closed.

The unit ball in \( \mathcal{M} \), \( B_\mathcal{M} \) is \( \sigma \)-weakly closed, and even compact. Hence \( B_\mathcal{M} \cap \mathcal{J} \) is sigma-weakly closed. Since the weak and \( \sigma \)-weak topologies coincide on bounded sets, \( B_\mathcal{M} \) is weakly closed, and then since \( B_\mathcal{M} \) is convex, it is strongly closed.

Let \( B \) belong to the norm closure of \( \mathcal{J} \), and suppose that \( \|B\| \leq 1 \). By Kaplansky’s Density Theorem, \( B \) belongs to the strong closure of \( B_\mathcal{M} \), and since \( B_\mathcal{M} \) is closed, to \( B_\mathcal{M} \). The assumption that \( \|B\| \leq 1 \) is then eliminated by scaling.

By Theorem 2.50, \( \Phi \) induces an isometric \( \ast \)-isomorphism of \( \mathcal{M} / \mathcal{J} \) onto \( \Phi(\mathcal{M}) \). We must show that \( \Phi(\mathcal{M}) \) is strongly closed in \( \mathcal{N} \). Let \( C \) belong to the strong closure of \( \Phi(\mathcal{M}) \), and suppose \( \|C\| \leq 1 \). Again by Kaplansky’s Density Theorem, every weak neighborhood of \( C \) contains points in \( \Phi(\mathcal{M}) \) of norm no greater than 1. Since the weak and the \( \sigma \)-weak topology coincide on bounded sets, the same is true for every \( \sigma \)-weak neighborhood of \( C \).

Since \( \Phi \) induced an isometry of \( \mathcal{M} / \mathcal{J} \) onto \( \Phi(\mathcal{M}) \), every weak neighborhood of \( C \) contains points of the form \( \Phi(A) \), \( A \in B_\mathcal{M} \). Since \( B_\mathcal{M} \) is \( \sigma \)-weakly compact, and since continuous images of compact sets are compact, the image of of this ball under \( \Phi \) is compact, and hence closed. By what we have noted above, it contains \( C \). The assumption that \( \|C\| \leq 1 \) is then eliminated by scaling.

4.8 The GNS construction, Sherman’s Theorem, and some applications
A construction due to Gelfand, Neumark and Segal, known as the GNS construction, associates to every state \( \varphi \) on an \( \mathcal{C}^* \) algebra \( \mathcal{A} \) a representation \( \pi \) of \( \mathcal{A} \) on a Hilbert space built out of \( \mathcal{A} \) itself and the state \( \varphi \).
4.35 THEOREM (The GNS construction). Let \( \mathcal{A} \) be a unital \( C^* \) algebra with identity \( 1 \), and let \( \varphi \) be a state on \( \mathcal{A} \). Then there exists a Hilbert space \( \mathcal{H} \) and a cyclic representation \( \pi \) of \( \mathcal{A} \) on \( \mathcal{H} \) with a distinguished cyclic unit vector \( \eta \) such that for all \( A \in \mathcal{A} \),

\[
\varphi(A) = \langle \eta, \pi(A)\eta \rangle_{\mathcal{H}} .
\]  

(4.24)

The representation \( \pi \) is irreducible if and only if \( \varphi \) is a pure state.

Proof. Let \( \langle A, B \rangle_\varphi \) be the possibly degenerate inner product on \( \mathcal{A} \) defined by \( \langle A, B \rangle_\varphi = \varphi(A^*B) \). Define

\[
\mathcal{N} := \{ a \in \mathcal{A} : \langle A, A \rangle_\varphi = 0 \} .
\]

Since \( \varphi \) is continuous, \( \mathcal{N} \) is closed. In fact, \( \mathcal{N} \) is a closed left ideal. To see this, consider \( B \in \mathcal{A} \) and \( A \in \mathcal{N} \). Then

\[
\langle BA, BA \rangle_\varphi = \varphi(A^*B^*BA) = \langle A, B^*BA \rangle_\varphi \leq \langle A, A \rangle_\varphi^{1/2} \langle B^*BA, B^*BA \rangle_\varphi^{1/2} = 0 .
\]

A similar but simpler argument shows that \( \mathcal{N} \) is a subspace.

Now consider the vector space \( \mathcal{A} / \mathcal{N} \). With \( \sim \) denoting equivalence mod \( \mathcal{N} \), we have

\[
A \sim A' \quad \text{and} \quad B \sim B' \quad \Rightarrow \quad \langle A, B \rangle_\varphi = \langle A', B' \rangle_\varphi ,
\]

and hence we may define a non-degenerate inner product on \( \mathcal{A} / \mathcal{N} \) by \( \langle \{ A \}, \{ B \} \rangle = \langle A, B \rangle_\varphi \). Let \( \mathcal{H} \) be the completion of \( \mathcal{A} / \mathcal{N} \) in the corresponding Hilbert norm, and let \( \langle , , \rangle_{\mathcal{H}} \) denote the resulting inner product on \( \mathcal{H} \).

For \( A \in \mathcal{A} \), let \( \pi(A) \) denote the linear operator on \( \mathcal{A} / \mathcal{N} \) defined by \( \pi(A)\{ B \} = \{ AB \} \) which is well-defined since \( \mathcal{N} \) is a left ideal. Next note that since \( B^*A^*AB = \|A\|^2B^*B + B^*(\|A^*A\|1 - A^*A)B \), and \( B^*(\|A^*A\|1 - A^*A)B \) is positive,

\[
\|\pi(A)\{ B \}\|_{\mathcal{H}}^2 = \varphi(B^*A^*AB) \leq \|A\|^2\varphi(B^*B) = \|A\|^2\|\{ B \}\|_{\mathcal{H}}^2 .
\]

Since \( \mathcal{A} / \mathcal{N} \) is dense in \( \mathcal{H} \), \( \pi(A) \) extends to a bounded operator on \( \mathcal{H} \) with \( \|\pi(A)\| \leq \|A\| \). It is evident that \( \pi \) is a homomorphism of \( \mathcal{A} \) into \( \mathcal{B}(\mathcal{H}) \), and note that for all \( X, Y \in \mathcal{A} \),

\[
\langle \{ X \}, \pi(A)\{ Y \} \rangle_{\mathcal{H}} = \varphi(X^*AY) = \varphi((A^*X)^*Y = \langle \pi(A)\{ X \}, \{ Y \} \rangle_{\mathcal{H}} ,
\]

showing that \( \pi(A^*) = \pi(A)^* \), and thus \( \pi \) is a *-homomorphism.

The representation \( \pi \) is cyclic since for all \( A \in \mathcal{A} \), \( \{ A \} = \{ A1 \} = \pi(A)\{ 1 \} \), showing that \( \eta := \{ 1 \} \) is a cyclic vector for \( \pi \). Finally, note that \( \langle \eta, \pi(A)\eta \rangle_{\mathcal{H}} = \varphi(1^*A1) = \varphi(A) \), and this proves (4.24). The final statement now follows from Theorem 4.12. \( \square \)

4.36 COROLLARY. Let \( \mathcal{A} \) be a unital \( C^* \) algebra. For every non-zero \( A \in \mathcal{A} \), there is a representation \( \pi \) of \( \mathcal{A} \) such that \( \|\pi(A)\| = \|A\| \).

Proof. By Lemma 4.8, there exists \( \varphi \in S_\mathcal{A} \) such that \( |\varphi(A^*A)| = \|A\|^2 \). Let \( \pi \) be the GNS representation of \( \mathcal{A} \) associated to \( \varphi \), and \( \eta \) the associated distinguished cyclic unit vector. Then

\[
\|\pi(a)\eta\|_{\mathcal{H}}^2 = \langle \eta\pi(A^*A)\eta \rangle_{\mathcal{H}} = \varphi(A^*A) = \|A\|^2 ,
\]

showing that \( \|\pi(A)\| \geq \|A\| \), and since it is automatic that \( \|\pi(A)\| \leq \|A\| \), \( \|\pi(A)\| = \|A\| \). \( \square \)
We now arrive at the Non-Commutative Gelfand-Neumark Theorem:

4.37 THEOREM (Non-Commutative Gelfand-Neumark Theorem). Every $C^*$ algebra $\mathcal{A}$ with an identity is isometrically $^*$-isomorphic to a $C^*$ algebra of operators.

Proof. For each $\varphi \in S_{\mathcal{A}^*}$, let $\pi_{\varphi}$ be the representation of $A$ on $\mathcal{B}(\mathcal{H}_{\varphi})$ provided by the GNS construction. Define $\mathcal{H} = \bigoplus_{\varphi \in S_{\mathcal{A}^*}} \mathcal{H}_{\varphi}$, and define $\pi = \bigoplus_{\varphi \in S_{\mathcal{A}^*}} \pi_{\varphi}$. For each $A \in \mathcal{A}$, there exists $\varphi \in S_{\mathcal{A}}$ such that $\|\pi_{\varphi}(A)\| = \|A\|$, and hence $\|A\| = \|\pi(A)\|$. □

With $\pi = \bigoplus_{\varphi \in S_{\mathcal{A}^*}} \pi_{\varphi}$, by the von Neumann Double Commutant Theorem, $(\pi(\mathcal{A}))''$ is a von Neumann algebra in which $\pi(\mathcal{A})$ is strongly dense. This von Neumann algebra, known as the enveloping von Neumann algebra, turns out to be the bidual of $\mathcal{A}$ considered as a Banach space. This fact is extremely useful for studying linear maps $\Phi$ from one $C^*$ algebra $\mathcal{A}$ to another, $\mathcal{B}$. The bidual $\mathcal{A}'' : \mathcal{A}'' \to \mathcal{B}''$ has the same norm as $\Phi$. Moreover, we can identify $\mathcal{A}$ with a subspace of $\mathcal{A}''$ using the canonical embedding, which identifies $A \in \mathcal{A}$ with $L_A$, the evaluation functional $L_A(\psi) = \psi(A)$ on $\mathcal{A}^*$. Then for all $\psi \in \mathcal{B}$,

$$\Phi''(L_A)(\psi) = L_A(\Phi^* \psi) = (\Phi^* \psi)(A) = \psi(\Phi(A)) = L_{\Phi(A)}(\psi).$$

That is, identifying $\mathcal{A}$ with its image under the canonical embedding, $\Phi''|\mathcal{A} = \Phi$. Once we know that the bidual of a $C^*$ algebra is isometrically isomorphic to its enveloping von Neumann algebra, we have a norm preserving extension of $\Phi : \mathcal{A} \to \mathcal{B}$ to a map, that we still denote by $\Phi''$, from $(\pi(\mathcal{A}))''$ to $(\pi(cB))''$. In the investigation of $\Phi''$, and hence $\Phi$, we may then use the rich structure that come along with von Neumann algebras; e.g., the measurable functional calculus and the consequent fact that von Neumann algebras are generated by the projections that they contain. Hence the following theorem is fundamentally important.

4.38 THEOREM (Sherman’s Theorem). Let $\mathcal{A}$ be a $C^*$ algebra, and let $\mathcal{H} = \bigoplus_{\varphi \in S_{\mathcal{A}^*}} \mathcal{H}_{\varphi}$, and $\pi = \bigoplus_{\varphi \in S_{\mathcal{A}^*}} \pi_{\varphi}$ be given as in Theorem 4.37. Let $\mathcal{M} := (\pi(\mathcal{A}))''$. Let $\mathcal{A}''$ be the bidual of $\mathcal{A}$ considered as a Banach space. Then $\mathcal{A}''$ is isometrically isomorphic to $\mathcal{M}$.

Proof. We have seen that every element of $\mathcal{A}''$ is a linear combination of at most 4 states on $\mathcal{A}$. For $\varphi \in S_{\mathcal{A}^*}$, let $\xi_{\varphi}$ be the unit vector in $\mathcal{H}_{\varphi}$ such that for all $A \in \mathcal{A}$, $\varphi(A) = \langle \xi_{\varphi}, \pi(\varphi) \xi_{\varphi} \rangle$; we identify $\xi_{\varphi}$ with a vector in $\mathcal{H}$ in the obvious manner. Note that $\|\phi\| = \sup_{A \in B_{\mathcal{A}}} \{|\phi(A)\|} = \sup_{A \in B_{\mathcal{A}}} \{|\hat{\phi}(\pi(A))\|}$, and by Kaplansky’s Density Theorem, $\pi(B_{\mathcal{A}}) = B_{\mathcal{B}_{\mathcal{A}''}}$ is dense in $B_{\mathcal{B}_{\mathcal{A}''}}$ in the strong, and hence weak, topology on $B_{\mathcal{B}_{\mathcal{A}''}}$, which is the same as the $\sigma$-weak topology on $B_{\mathcal{B}_{\mathcal{A}''}}$. Hence $\|\phi\| = \sup_{A \in B_{\mathcal{A}''}} \{|\hat{\phi}(\pi(A))\|} = \hat{\phi}$. Therefore, $\phi \mapsto \hat{\phi}$ is an isometric isomorphism from $\mathcal{A}^*$ into $\mathcal{M}$, and it is surjective since for any $\varphi \in \mathcal{M}$, $\varphi|_{\mathcal{A}} \in \mathcal{A}^*$, and the extension of $\varphi|_{\mathcal{A}} \in \mathcal{A}^*$ to $\mathcal{M}$ is $\varphi$, again using the weak density of $\pi(\mathcal{A})$ in $\mathcal{M}$, and the weak continuity of $\varphi|_{\mathcal{A}}$. Then $\mathcal{A}''$ is isometrically isomorphic to $(\mathcal{M})^*$ = $\mathcal{M}$. □

Going forward, it will be useful to identify $\mathcal{A}$ with $\pi(\mathcal{A})$, and to write $\mathcal{A}''$ to denote the universal enveloping von Neumann algebra of $\mathcal{A}$. **
4.39 COROLLARY. Let $\mathcal{A}$ and $\mathcal{B}$ be two $C^*$ algebras, and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a bounded linear transformation. Then there is a norm preserving extension of $\Phi$, denoted by $\Phi^{**}$, that maps $\mathcal{A}''$ to $\mathcal{B}''$.

Proof. The proof has already been given in the remarks preceding the statement of Sherman’s Theorem. \hfill $\Box$

We now come to an important application of this corollary, namely Tomiyama’s Theorem. We first define some classes of maps between $C^*$ algebras with which we shall be concerned in what follows.

4.40 DEFINITION. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras. A linear transformation $\Phi : \mathcal{A} \to \mathcal{B}$ is positive in case $\Phi(A) \geq 0$ whenever $A \geq 0$. When $\mathcal{A}$ and $\mathcal{B}$ are unital then $\Phi$ is unital in case $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. $\Phi$ is Hermitian in case $\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{A}$. Finally, suppose that $\mathcal{B}$ is a $C^*$ subalgebra of $\mathcal{A}$. Then $\Phi$ is a projection in case $\Phi(B) = B$ for all $B \in \mathcal{B}$.

Before turning to Tomiyama’s Theorem, we first prove a Lemma of Kadison [11]. In the rest of this section, it is more the proof of Kadison’s Lemma than the statement that will be of use to us. The proof is a variant of the “Ahren’s Trick”; see [13, p. 24] for more discussion.

4.41 LEMMA (Kadison’s Lemma). Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$ algebras. Let $\Psi : \mathcal{A} \to \mathcal{B}$ be a unital linear transformation such that $\|\Psi(X)\| \leq \|X\|$ when $X$ is normal. Then $\Psi$ is Hermitian.

Proof. Let $A \in \mathcal{A}$, $A = A^*$, $\|A\| = 1$. Write $\Psi(A) = B + iC$, $B, C \in \mathcal{B}_{sa}$. If $C \neq 0$, there is some $\lambda \in \mathbb{R}\setminus\{0\}$ in $\sigma(C)$. Replacing $A$ by $-a$ if needed, we may suppose that $\lambda > 0$. For all $t > 0$,

$$\|A + it1\| \leq (1 + t^2)^{1/2} = t(1 + t^{-2})^{1/2} \leq t + (2t)^{-1},$$

and hence $\|A + it1\| \leq t + \lambda$ for $t \in (0, (2\lambda)^{-1})$. Since $\lambda \in \sigma(C)$, $t + \lambda \leq \|C + t\| \leq \|B + iC + it\|$, where we used Lemma ?? in the last step. But $\|B + iC + it\| = \|\Psi(A + it1)\| = \|A + it\|$ since $A + it1$ is normal. Altogether we have $\|A + it1\| < \|A + it1\|$, and this contradiction proves the claim. \hfill $\Box$

4.42 THEOREM (Tomiyama’s Theorem). Let $\mathcal{A}$ be a unital $C^*$ algebra, and let $\mathcal{B}$ be a unital $C^*$ subalgebra of $\mathcal{A}$. Let $\Phi$ be a norm 1 projection from $\mathcal{A}$ to $\mathcal{B}$. Then $\Phi$ is unital and positive, and

$$\Phi(B_1AB_2) = B_1\Phi(A)B_2 \quad \text{for all } B_1, B_2 \in \mathcal{B}, \; A \in \mathcal{A} \quad (4.25)$$

and for all $A \in \mathcal{A}$,

$$\Phi(A)^*\Phi(A) \leq \Phi(A^*A) \quad (4.26)$$

Proof of Tomiyama’s Theorem. By Sherman’s Theorem and its Corollary, $\Phi^{**}$ may be regarded as a norm one linear map from $\mathcal{A}''$ to $\mathcal{B}''$, whose restriction to $\mathcal{B}$ is the identity, since $\Phi^{**}|_{\mathcal{A}} = \Phi$, and $\Phi$ is a projection onto $\mathcal{B}$. More is true: $\Phi^{**}(B) = B$ for all $B \in \mathcal{B}''$. To see this, note that if $\Lambda \in \mathcal{B}^{**} \subset \mathcal{A}^{**}$, and $\psi \in \mathcal{B}^*$, then $(\Phi^{**}\Lambda)(\psi) = \Lambda(\Phi^*(\psi)) = \Lambda(\psi \circ \Phi) = \Lambda(\psi)$. Hence $\Phi^{**}$ is the identity on $\mathcal{B}^{**}$, and thus $\Phi^{**}$ is a norm one projection of $\mathcal{A}''$ onto $\mathcal{B}''$. Therefore, it suffices to prove the theorem when $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras; we now assume this is the case.

First, since $\Phi$ maps $\mathcal{A}$ onto $\mathcal{B}$ and $1 \in \mathcal{B}$, there is some $A \in \mathcal{A}$ with $1 = \Phi(A)$. Hence $\Phi(1) = \Phi^2(A) = \Phi(A) = 1$. Hence $\Phi$ is unital.
We next show that $\Phi$ is Hermitian. Arguing as in Kadison’s Lemma, let $A \in \mathcal{A}_{a.a.}$ and write $\Phi(A) = B + iC$ where $A$ and $B$ are self adjoint, and suppose that some $\lambda > 0$ belongs to the spectrum of $C$. Then for $t > 0$, since $\Phi(1) = 1$,

$$t + \lambda \leq \|C + t1\| \leq \|B + iC + it1\| = \|\Phi(A + it1)\| \leq \|A + it1\| \leq (t^2 + \|A\|^2)^{1/2} \leq t + \frac{\|A\|^2}{2t}.$$ 

This is impossible for large $t$. Hence $C$ has no positive spectrum. Repeating the argument with $-A$ in place of $A$, we conclude that $C$ has no negative spectrum, and hence $C = 0$.

The next step is to show that $\Phi$ is positive; here Tomiyama cites the argument of Kadison [11]. Suppose that $A \in \mathcal{A}^+$, but that $\Phi(A)$ has spectrum in $(\infty, 0)$. We may assume without loss of generality that $0 \leq A \leq 1$. Then $\|1 - A\| \leq 1$, but $\Phi(1 - A) = 1 - \Phi(A)$ is self adjoint and has spectrum in $(1, \infty)$, so that $\|\Phi(1 - A)\| > 1$. This contradiction proves that $\Phi$ is positive.

Now let $0 \leq A \leq 1$, $A \in \mathcal{A}$, and let $P$ be a projection in $\mathcal{B}$, Then $P \geq PAP$ and hence $P \geq \Phi(PAP)$. It follows that $P\Phi(PAP)P = \Phi(PAP)$. Therefore, for all $X \in \mathcal{A}$,

$$P\Phi(PXP)P = \Phi(PXP). \tag{4.27}$$

Now pick $X \in \mathcal{A}$, $\|X\| \leq 1$, and define $Y := PX(1 - P)$. Our immediate goal is to show that

$$P\Phi(Y)P = 0. \tag{4.28}$$

First note that for all $t \in \mathbb{R}$,

$$\|Y + tP\| = \|(Y^* + tP)(Y + tP)\|^{1/2} = \|(1 - P)X^*PX(1 - P) + t^2P\|^{1/2} \leq (1 + t^2)^{1/2}.$$

Therefore, $\|\Phi Y + tP\| \leq (1 + t^2)^{1/2}$ for all $t \in \mathbb{R}$.

However,

$$\|\Phi Y + tP\| \geq \|P(\Phi Y)P + tP\| \geq \left\| \frac{P(\Phi Y)P + P(\Phi Y)^*P}{2} + tP \right\|.$$

Let $\lambda$ belong to the spectrum of $\frac{1}{2}(P(\Phi Y)P + P(\Phi Y)^*P)$. If $\lambda > 0$, then for $t > 0$, $\|\Phi Y + tP\| \geq \lambda + t$. If $\lambda < 0$, and $t < 0$, $\|\Phi Y + tP\| \geq -\lambda - t$. Either way, there is a contradiction for suitable values of $t$. Hence $\frac{1}{2}(P(\Phi Y)P + P(\Phi Y)^*P) = 0$. The same reasoning shows that $\frac{1}{2}(P(\Phi Y)P - P(\Phi Y)^*P) = 0$, and thus (4.28) is proved.

A similar argument applied to $\|Y + t(1 - P)\|$ proves that

$$(1 - P)(\Phi Y)(1 - P) = 0. \tag{4.29}$$

At this point we have $\Phi Y = (1 - P)\Phi Y P + P\Phi Y (1 - P)$. Therefore, for $t > 0$,

$$\Phi Y + t(1 - P)\Phi Y P = P\Phi Y (1 - P) + (t + 1)(1 - P)\Phi Y P.$$

Since the norm of block matrix

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$$

is given by $\max\{\|A\|, \|B\|\}$, for all sufficiently large $t$,

$$\|\Phi Y + t(1 - P)\Phi Y P\| = (t + 1)\|(1 - P)\Phi Y P\|.$$
However, since $\Phi$ is a contractive projection,
\[
\|\Phi Y + t(1 - P)\Phi Y P\| \leq \|Y + t(1 - P)\Phi Y P\| = \|PX(1 - P) + t(1 - P)\Phi Y P\| = \max\{\|PX(1 - P)\|, t\|(1 - P)\Phi Y P\|\}.
\]

For large $t$, $(t + 1)\|(1 - P)\Phi Y P\| \leq t\|(1 - P)\Phi Y P\|$, and hence $\|(1 - P)\Phi Y P\| = 0$.

At this point we have $(1 - P)\Phi Y P = 0$, $(1 - P)(\Phi Y)(1 - P) = 0$ and $P(\Phi Y)P = 0$. It follows that
\[
\Phi Y = P\Phi Y(1 - P).
\]

Now define $Z := (1 - P)XP$, which is just like the definition of $Y$, except with $P$ and $1 - P$ interchanged. By Symmetry, we have
\[
\Phi Z = (1 - P)\Phi ZP.
\]

Now for any $X \in \mathcal{A}$, we have, using (4.27) once with $P$, and once with $(1 - P)$ in place of $P$,
\[
\Phi X = \Phi(PXP + PX(1 - P) + (1 - P)XP + (1 - P)X(1 - P)) = P\Phi(PXP)P + \Phi Y + \Phi Z + (1 - P)\Phi((1 - P)X(1 - P))(1 - P)
\]
Therefore, by (4.30) and (4.31) $(1 - P)\Phi XP = (1 - P)\Phi ZP = \Phi Z = \Phi((1 - P)XP)$ and then, also using (4.27), $P\Phi XP = P\Phi(PXP)P = \Phi(PXP)$. Summing, $(\Phi X)P = \Phi(XP)$. Since a von Neumann algebra is generated by the projection it contains, for all $A \in \mathcal{B}$, $\Phi(X)A = \Phi(XA)$, and taking adjoints, since $\Phi$ preserves self-adjointness, $A\Phi X = \Phi(AX)$. This proves (4.25).

Next, since $(X - \Phi X)^\ast(X - \Phi X) \geq 0$ and since $\Phi$ preserves positivity, using (4.25) twice,
\[
0 \leq \Phi(X^\ast X - X^\ast \Phi X - \Phi X^\ast X + (\Phi X)^\ast \Phi X) = \Phi(X^\ast X) - (\Phi X)^\ast \Phi X,
\]
and this proves (4.26)

\section{5 Completely positive maps}

\subsection{5.1 Hilbert-Schmidt operators}

Let $A \in \mathcal{B}(\mathcal{H})$ and let $A = U|A|$ be the polar decomposition of $A$. Then $AA^* = UA^*AU^*$ so that whenever $A^*A \in \mathcal{T}(\mathcal{H})$, $AA^* \in \mathcal{T}(\mathcal{H})$. By symmetry, $A^*A \in \mathcal{T}(\mathcal{H})$ if and only if $AA^* \in \mathcal{T}(\mathcal{H})$.

**5.1 DEFINITION** (Hilbert-Schmidt operators). An operator $A \in \mathcal{B}(\mathcal{H})$ is Hilbert-Schmidt in case $A^*A \in \mathcal{T}(\mathcal{H})$, or equivalently, in case $AA^* \in \mathcal{T}(\mathcal{H})$. The set of all Hilbert-Schmidt operators on $\mathcal{H}$ is denoted $\mathcal{C}_2(\mathcal{H})$.

It is evident that $\mathcal{C}_2(\mathcal{H})$ is self adjoint; i.e., $A \in \mathcal{C}_2(\mathcal{H})$ if and only if $A^* \in \mathcal{C}_2(\mathcal{H})$. We now show that $\mathcal{C}_2(\mathcal{H})$ is a two-sided $*$-ideal in $\mathcal{B}(\mathcal{H})$. Suppose that $A \in \mathcal{C}_2(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$. Then
\[
(BA)^* (BA) = A^* (B^* B) A \leq \| B \|^2 A^* A \in \mathcal{T}(\mathcal{H}),
\]
\[(5.1)\]
and hence $BA \in \mathcal{C}_2(\mathcal{H})$ for all $A \in \mathcal{C}_2(\mathcal{H})$ and all $B \in \mathcal{B}(\mathcal{H})$. Since $\mathcal{C}_2(\mathcal{H})$ is self adjoint, $AB \in \mathcal{C}_2(\mathcal{H})$ for all $A \in \mathcal{C}_2(\mathcal{H})$ and all $B \in \mathcal{B}(\mathcal{H})$. To see that $\mathcal{C}_2$ is closed under addition, note that

$$(A + B)^*(A + B) + (A - B)^*(A - B) = 2A^*A + 2B^*B.$$  

This proves that $\mathcal{C}_2(\mathcal{H})$ is a two-sided $*$-ideal in $\mathcal{B}(\mathcal{H})$.

For any two orthonormal sets $\{\eta_1, \ldots, \eta_n\}$ and $\{\xi_1, \ldots, \xi_n\}$,

$$\sum_{j=1}^{n} \langle \eta_j, A^*B\xi_j \rangle \leq \left( \sum_{j=1}^{n} \langle \eta_j, A^*A\eta_j \rangle \right)^{1/2} \left( \sum_{j=1}^{n} \langle \xi_j, B^*B\xi_j \rangle \right)^{1/2} \leq (\text{Tr}[A^*A])^{1/2} (\text{Tr}[B^*B])^{1/2}.$$  

It follows that for all $A, B \in \mathcal{C}_2(\mathcal{H})$, $A^*B \in \mathcal{F}(\mathcal{H})$, and $|\text{Tr}[A^*B]| \leq (\text{Tr}[A^*A])^{1/2} (\text{Tr}[B^*B])^{1/2}$.

**5.2 DEFINITION** (Hilbert-Schmidt inner product and norm). For all $A, B \in \mathcal{C}_2(\mathcal{H})$, define

$$\langle A, B \rangle_{\text{HS}} = \text{Tr}[A^*B].$$  

By what we have noted above, this is a positive definite sesquilinear form on $\mathcal{C}_2(\mathcal{H})$. Let $\| \cdot \|_{\text{HS}}$ denote the corresponding norm. That is, $\|A\|_{\text{HS}} = (\text{Tr}[A^*A])^{1/2}$.

Note that for any $\eta, \xi \in \mathcal{H}$,

$$\|A\|_{\text{HS}} = \text{Tr}[(\langle \eta, \xi \rangle^* \langle \eta, \xi \rangle) = \|\eta\|^2 \text{Tr}[|\xi\rangle \langle \xi|] = \|\eta\|^2 \|\xi\|^2. \quad (5.2)$$

Let $A \in \mathcal{B}(\mathcal{H})$ and $\epsilon > 0$. There exist unit vectors $\eta, \xi$ such that $|\langle \eta, A\xi \rangle| \geq (1 - \epsilon)\|A\|$. By the Cauchy-Schwarz inequality for the Hilbert-Schmidt inner product and (5.2),

$$(1 - \epsilon)\|A\| \leq |\langle \eta, A\xi \rangle| = |\text{Tr}[\langle \eta, \langle \xi, A \rangle] \|A\| \leq \|\eta\| \|\xi\| \|A\|_{\text{HS}} = \|A\|_{\text{HS}}.$$  

Since $\epsilon > 0$ is arbitrary, $\|A\| \leq \|A\|_{\text{HS}}$ for all $A \in \mathcal{C}_2(\mathcal{H})$.

**5.3 THEOREM.** $\mathcal{C}_2(\mathcal{H})$ is complete in the norm $\| \cdot \|_{\text{HS}}$, and hence is a Hilbert space for the inner product $\langle \cdot, \cdot \rangle_{\text{HS}}$.

**Proof.** Let $\{A_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}_2(\mathcal{H})$ for the Hilbert-Schmidt norm. By what has been noted above, $\{A_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the operator norm, and hence there exists $A \in \mathcal{B}(\mathcal{H})$ such that $\lim_{n \to \infty} \|A_n - A\| = 0$. We now show that, in fact, $\lim_{n \to \infty} \|A_n - A\|_{\text{HS}} = 0$.

Let $\{\xi_1, \ldots, \xi_n\}$ be any orthonormal set in $\mathcal{H}$. Then

$$\limsup_{m \to \infty} \|A_n - A_m\|^2_{\text{HS}} \geq \limsup_{m \to \infty} \sum_{j=1}^{n} \langle \xi_j, (A_n - A_m)^*(A_n - A_m)\xi_j \rangle$$

$$= \lim_{m \to \infty} \sum_{j=1}^{n} \|A_n\xi_j - A_m\xi_j\|^2 = \sum_{j=1}^{n} \langle \xi_j, (A_n - A)^*(A_n - A)\xi_j \rangle.$$  

However, for all $\epsilon > 0$, since $\{A_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}_2(\mathcal{H})$, there is an $N$ such that for $m, n \geq N$, $\|A_n - A_m\|_{\text{HS}} < \epsilon$. It then follows that for $n \geq N$, $\|A_n - A\|_{\text{HS}} < \epsilon$, in particular $A = A_n - (A_n - A) \in \mathcal{C}_2(\mathcal{H})$, and $\lim_{n \to \infty} \|A_n - A\|_{\text{HS}} = 0$. This proves the completeness. \qed
Let \( A \in \mathcal{C}_2(\mathcal{H}) \). Then \( A^*A \in \mathcal{F}(\mathcal{H}) \), and as we have observed earlier, \( 1_{[\epsilon^2,\|A\|^2]}(|A|^2) \) is a finite rank projection. Let \( A = U|A| = U(A^*A)^{1/2} \) be the polar decomposition of \( A \). Then

\[
|A| = |A|1_{[\epsilon^2,\|A\|^2]}(|A|^2) + |A|1_{[0,\epsilon^2]}(|A|^2) \quad \text{and} \quad A - A1_{[\epsilon^2,\|A\|^2]}(|A|^2) = A1_{[0,\epsilon^2]}(|A|^2).
\]

Since \( \lim_{\epsilon \to 0} \|A1_{[\epsilon^2,\|A\|^2]}(|A|^2)\|_{HS} = 0 \), \( A = \lim_{\epsilon \to 0} A1_{[\epsilon^2,\|A\|^2]}(|A|^2) \).

Let \( \{\xi_1, \ldots, \xi_n\} \) be an orthonormal basis for \( \text{ran}(1_{[\epsilon^2,\|A\|^2]}(|A|^2)) \) consisting of eigenvectors of \( A \) with \( |A|\xi_j = \lambda_j \xi_j \), and we may suppose that the labeling is such that \( \lambda_{j+1} \leq \lambda_j \) for each \( j \). Then

\[
A1_{[\epsilon^2,\|A\|^2]}(|A|^2) = \sum_{j=1}^{\infty} \lambda_j |U\xi_j\rangle\langle \xi_j|.
\]

As \( \epsilon \) decreases, the range of \( 1_{[\epsilon^2,\|A\|^2]} \) increases, and we can add more vectors to our orthonormal set, always with eigenvalues no larger than the ones we already have. In the limit, we obtain the sum

\[
A = \sum_{j=1}^{\infty} \lambda_j |U\xi_j\rangle\langle \xi_j|
\]

which converges in the Hilbert-Schmidt norm, and where \( \{\xi_j\}_{j \in \mathbb{N}} \) is orthonormal in \( \mathcal{H} \). It is not excluded that only finitely many of the \( \lambda_j \) are non-zero, and in any case, we see that finite rank operators are dense in \( \mathcal{C}_2(\mathcal{H}) \).

The is a simple generalization to the case of linear maps from one Hilbert space \( \mathcal{H}_1 \) to another, \( \mathcal{H}_2 \). \( \mathcal{C}_2(\mathcal{H}_1, \mathcal{H}_2) \) is the set of all \( A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) such that \( A^*A \in \mathcal{F}(\mathcal{H}_2) \). Using the polar factorization, it is easily shown that \( A^*A \in \mathcal{F}(\mathcal{H}_2) \) if and only if \( AA^* \in \mathcal{F}(\mathcal{H}_1) \). The Hilbert-Schmidt inner product on \( \mathcal{C}_2(\mathcal{H}_1, \mathcal{H}_2) \) is given, as before, by \( \langle A, B \rangle_{HS} = \text{Tr}[A^*B] \).

The algebraic tensor product of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), is the span of the vectors \( \eta \otimes \xi, \eta \in \mathcal{H}_1 \) and \( \xi \in \mathcal{H}_2 \). We equip \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) with the inner product

\[
\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \langle \eta_1, \eta_2 \rangle_{\mathcal{H}_1} \langle \xi_1, \xi_2 \rangle_{\mathcal{H}_2}.
\]

In particular,

\[
\|\eta \otimes \xi\|_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \|\eta\|_{\mathcal{H}_1} \|\xi\|_{\mathcal{H}_2}.
\] (5.3)

Let \( \overline{\mathcal{H}} \) denote the Hilbert space that, as a set, is the same as \( \mathcal{H} \), with the same vector addition, but with the scalar multiplication \( \lambda \eta = (\lambda) \in \mathcal{H} \), and the inner product

\[
\langle \eta_1, \eta_2 \rangle_{\overline{\mathcal{H}}} = \langle \eta_2, \eta_1 \rangle_{\mathcal{H}}.
\]

It is easily checked that this inner product is sesquilinear in the usual manner, linear on the right, and that \( \mathcal{H} \) is a Hilbert space. The trivial embedding of \( \mathcal{H} \) into \( \overline{\mathcal{H}} \) is a conjugate linear isometry of \( \mathcal{H} \) onto \( \overline{\mathcal{H}} \).

Given \( \eta \otimes \xi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \), define a rank-one operator from \( K_{\eta \otimes \xi} : \overline{\mathcal{H}} \) to \( \mathcal{H} \) by

\[
K_{\eta \otimes \xi} \zeta = (\langle \eta, \zeta \rangle_{\overline{\mathcal{H}}}) \xi = (\langle \zeta, \eta \rangle_{\overline{\mathcal{H}}}) \xi.
\]

It is then evident that

\[
(K_{\eta \otimes \xi})^* K_{\eta \otimes \xi} = \|\xi\|^2 |\eta| \quad \text{and} \quad \text{Tr}[(K_{\eta \otimes \xi})^* K_{\eta \otimes \xi}] = \|\xi\|^2 \|\eta\|^2.
\]
It then follows from (5.3) and polarization, or a simple direct calculation, that the map
\[ \eta \otimes \xi \mapsto K_{\eta \otimes \xi} \]
extends by linearity to an isometry of \( \mathcal{H} \otimes \mathcal{K} \) into \( \mathcal{C}_2(\mathcal{H}, \mathcal{K}) \), with the image including all finite rank operators in \( \mathcal{C}_2(\mathcal{H}, \mathcal{K}) \). Hence the range of this embedding is dense. Therefore, we may identify the completion of the algebraic tensor product in the tensor product norm with the Hilbert space \( \mathcal{C}_2(\mathcal{H}, \mathcal{K}) \).

5.2 Some useful isomorphisms

Let \( \mathcal{H} \) and \( \mathcal{K} \) be two Hilbert spaces. The general element \( \psi \) of \( \mathcal{H} \otimes \mathcal{K} \) may be identified with an element \( K \) of \( \mathcal{C}(\mathcal{H}, \mathcal{K}) \), and then there are orthonormal sets \( \{\eta_j\}_{j \in \mathbb{N}}, \{\xi_j\}_{j \in \mathbb{N}} \), and a non-increasing sequence \( \{\lambda_j\}_{j \in \mathbb{N}} \) with \( \sum_{j=1}^{\infty} \lambda_j^2 = \|\psi\|^2_{\mathcal{H} \otimes \mathcal{K}} \) such that
\[ \psi = \sum_{j=1}^{\infty} \lambda_j \eta_j \otimes \xi_j \] (5.4)
This is true without any assumption of separability on the part of either \( \mathcal{H} \) or \( \mathcal{K} \). However, we shall be particularly interested in the case in which \( \mathcal{K} \) is finite dimensional. If \( \dim(\mathcal{K}) = n \), then evidently \( \lambda_j = 0 \) for all \( j > n \), and the sum in (5.4) is finite, no matter what the dimension of \( \mathcal{H} \) may be.

Suppose that \( \dim(\mathcal{K}) = n < \infty \), and let \( \{\xi_1, \ldots, \xi_n\} \) be an orthonormal basis for \( \mathcal{K} \). Then the general element \( \psi \) of the Hilbert space \( \mathcal{H} \otimes \mathcal{K} \) has the form
\[ \psi = \sum_{j=1}^{n} \zeta_j \otimes \xi_j \] (5.5)
where, in terms of (5.4), \( \zeta_j = \lambda_j U \xi_j \). For \( m \in \mathbb{N} \), let \( \mathcal{H}_m \) denote the direct sum of \( m \) copies of \( \mathcal{K} \),
\[ \mathcal{H}_m := \mathcal{H} \oplus \cdots \oplus \mathcal{H} \], \( m \) times (5.6)
Let \( \mathcal{H}_{\mathbb{N}} \) denote the direct sum of countably infinitely many copies of \( \mathcal{K} \). We write \( \{\zeta_j\}_{1 \leq j \leq m} \) to denote the general element of \( \mathcal{H}_m \), and we write \( \{\zeta_j\}_{j \in \mathbb{N}} \) to denote the general element of \( \mathcal{H}_{\mathbb{N}} \).

For \( \dim(\mathcal{K}) = n \), we define an isomorphism from \( \mathcal{H}_n \) onto \( \mathcal{H} \otimes \mathcal{K} \) by choosing an orthonormal basis \( \{\eta_1, \ldots, \eta_n\} \) of \( \mathcal{K} \) and then define the map
\[ \{\zeta_j\}_{j \in \mathbb{N}} \mapsto \sum_{j=1}^{n} \zeta_j \otimes \eta_j . \] (5.7)
This gives a unitary map from \( \mathcal{H}_n \) onto \( \mathcal{H} \otimes \mathcal{K} \). When \( \mathcal{K} \) is infinite dimensional but separable, and \( \{\eta_j\} \) is an orthonormal basis of \( \mathcal{K} \), the map given in (5.7) is unitary from \( \mathcal{H}_{\mathbb{N}} \) onto \( \mathcal{H} \otimes \mathcal{K} \).

5.4 DEFINITION. Let \( \mathcal{H} \) be any Hilbert space and let \( \mathcal{K} \) be a separable Hilbert space. Let \( \{\eta_j\} \) be an orthonormal basis of \( \mathcal{K} \). Define \( V_j : \mathcal{K} \rightarrow \mathcal{H} \otimes \mathcal{K} \) by
\[ V_j \zeta = \zeta \otimes \eta_j . \] (5.8)
Note that \( V_j \) is an isometry from \( \mathcal{H} \) into \( \mathcal{H} \otimes \mathcal{H} \), and that
\[
\sum_{j=1}^{\dim(\mathcal{H})} V_j^* V_j = 1_{\mathcal{H}} .
\] (5.9)

Now we specialize to a case that will be important in what follows. Let \( \mathcal{H} \) be finite dimensional, and identify it with \( \mathbb{C}^n \) for \( n = \dim(\mathcal{H}) \). We may then identify \( \mathcal{B}(\mathcal{H}) \) with \( M_n(\mathbb{C}) \). Let \( \{ \eta_1, \ldots, \eta_n \} \) be any orthonormal basis of \( \mathbb{C}^n \). Then \( \{ |\eta_i\rangle \langle \eta_j| : 1 \leq i, j \leq n \} \) is a basis for for \( M_n(\mathbb{C}) \). It is easy to check that
\[
|\eta_i\rangle \langle \eta_j| |\eta_k\rangle \langle \eta_\ell| = \delta_{j,k} |\eta_i\rangle \langle \eta_\ell| .
\] (5.10)

For any \( A_1 \otimes M_1, A_2 \otimes M_2 \in \mathcal{B}(\mathcal{H}) \times M_n(\mathbb{C}) \), we define the product \( (A_1 \otimes M_1)(A_2 \otimes A_2) \) by
\[
(A_1 \otimes M_1)(A_2 \otimes M_2) = A_1 A_2 \otimes M_1 M_2,
\] and extend this by linearity. Likewise, for any \( A \in \mathcal{B}(\mathcal{H}) \) and any \( M \in M_n(\mathbb{C}) \), the involution \( (A \otimes M)^* = A^* \otimes M^* \), and also extend this by linearity.

Since \( \{ |\eta_i\rangle \langle \eta_j| : 1 \leq i, j \leq n \} \) is a basis of \( M_n(\mathbb{C}) \), the general element \( \tilde{A} \) of \( \mathcal{B}(\mathcal{H}) \otimes M_n(\mathbb{C}) \) can be written as \( \tilde{A} = \sum_{i,j=1}^n A_{i,j} \otimes |\eta_i\rangle \langle \eta_j| \). By (5.10),
\[
\left( \sum_{i,j=1}^n A_{i,j} \otimes |\eta_i\rangle \langle \eta_j| \right) \left( \sum_{k,\ell=1}^n B_{k,\ell} \otimes |\eta_k\rangle \langle \eta_\ell| \right) = \sum_{i,\ell=1}^n \left( \sum_{j=1}^n A_{i,j} B_{j,\ell} \right) \otimes |\eta_i\rangle \langle \eta_\ell| .
\] (5.11)

Define \( M_n(\mathcal{B}(\mathcal{H})) \) to be the set of \( n \times n \) matrices with entries in \( \mathcal{B}(\mathcal{H}) \). Let \( [A_{i,j}] \) denote the element of \( M_n(\mathcal{B}(\mathcal{H})) \) with \( i, j \) entry is \( A_{i,j} \). Define a linear transformation from \( \mathcal{B}(\mathcal{H}) \otimes M_n(\mathbb{C}) \) onto \( M_n(\mathcal{B}(\mathcal{H})) \) by
\[
\tilde{A} = \sum_{i,j=1}^n A_{i,j} \otimes |\eta_i\rangle \langle \eta_j| \mapsto [A_{i,j}] .
\] (5.12)

The transformation in (5.12) is evidently injective, and hence is a vector space isomorphism. In fact, the inverse map is simply given by
\[
[A_{i,j}] \mapsto \sum_{i,j=1}^n A_{i,j} \otimes |\eta_i\rangle \langle \eta_j| .
\] (5.13)

By (5.11), this vector space isomorphism is also a algebra isomorphism where \( M_n(\mathcal{B}(\mathcal{H})) \) is given the natural product, and one easily checks that \( [A_{i,j}]^* = [A^*_{j,i}] \), so that it is a *-isomorphism.

Another identification will be useful in what follows: There is a natural *-isomorphism of \( \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n) \) with \( M_n(\mathcal{B}(\mathcal{H})) \). Let \( \{ \eta_1, \ldots, \eta_n \} \) be the standard orthonormal basis of \( \mathbb{C}^n \), so that \( |\eta_i\rangle \langle \eta_j| =: E_{i,j} \), the \( n \times n \) matrix with 1 in the \( i, j \) entry, and 0 in all other entries. For \( j = 1, \ldots, n \), let \( V_j \) be the isometry from \( \mathcal{H} \) into \( \mathcal{H} \otimes \mathbb{C}^n \) given by (5.8). Define a linear transformation from \( \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n) \) to \( M_n(\mathcal{B}(\mathcal{H})) \) by
\[
\tilde{A} \mapsto [V_{i}^* \tilde{A} V_j] .
\] (5.14)

for all \( \tilde{A} \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n) \). Another way to think about this isomorphism is in terms of the identification of \( \mathcal{H} \times \mathcal{H} \) with \( \mathbb{C}^n \), as in (5.6). This clearly identifies \( \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n) \) with the \( n \times n \)}
block matrices with entries in \( \mathcal{B}(\mathcal{H}) \), and the map \( \hat{A} \mapsto V_i^* \hat{A} V_j \) picks out the \( i, j \) block. The inverse reansformation is simply

\[
[A_{i,j}] \mapsto \sum_{i,j=1}^n A_{i,j} \otimes E_{i,j} .
\]

(5.15)

It is clear that this is a \(*\) algebra isomorphism, and the operator norm on \( \mathcal{B}(\mathcal{H}) \) is a \( \mathcal{C}^*\)- algebra norm, so this norm makes \( \mathcal{B}(\mathcal{H}) \otimes M_n(\mathbb{C}) \) into a \( \mathcal{C}^*\)-algebra, and by the uniqueness of \( \mathcal{C}^*\) algebra norms, it is the only norm with this property.

We have proved:

5.5 THEOREM. For any Hilbert space \( \mathcal{H} \) and any \( n \in \mathbb{N} \), \( \mathcal{B}(\mathcal{H}) \otimes M_n(\mathbb{C}) \) and \( \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n) \), equipped with their natural \(*\)-algebra structures, are all \(*\)-algebra isomorphic. Moreover, for any orthonormal basis \( \{\eta_1, \ldots, \eta_n\} \) of \( \mathbb{C}^n \), and:

1. The map in (5.12) is a \(*\)-isomorphism of \( \mathcal{B}(\mathcal{H}) \otimes M_n(\mathbb{C}) \) onto \( M_n(\mathcal{B}(\mathcal{H})) \), and the map in (5.13) is its inverse.

2. The map in (5.14) is a \(*\)-isomorphism of \( \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n) \) onto \( M_n(\mathcal{B}(\mathcal{H})) \), and the map in (5.15) is its inverse.

The operator norm on \( \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n) \) is the unique \( \mathcal{C}^*\) algebra norm on these \(*\)-algebras.

We now return to \( \mathcal{C}^*\) algebras. There is a natural family of \( \mathcal{C}^*\) algebras associated to every \( \mathcal{C}^*\) algebra \( \mathcal{A} \), namely the \( \mathcal{C}^*\) algebras \( \mathcal{A} \otimes M_n(\mathbb{C}) \) for each \( n \in \mathbb{N} \), that we now define. As a vector space, \( \mathcal{A} \otimes M_n(\mathbb{C}) \) is the algebraic tensor product of the vector spaces \( \mathcal{A} \) and \( M_n(\mathbb{C}) \). Let \( \{\eta_1, \ldots, \eta_n\} \) be any orthonormal basis for \( \mathbb{C}^n \). Since the \( \{|\eta_i\rangle\langle\eta_j| : 1 \leq i, j \leq n\} \) is a basis for \( M_n(\mathbb{C}) \), the general element \( \tilde{A} \) of \( \mathcal{A} \otimes M_n(\mathbb{C}) \) has the form

\[
\tilde{A} = \sum_{i,j=1}^n A_{i,j} \otimes |\eta_i\rangle\langle\eta_j| ,
\]

(5.16)

where for each \( i, j, A_{i,j} \in \mathcal{A} \).

By the Gelfand-Neumark Theorem, \( \mathcal{A} \) may be \(*\)-isomorphically and isometrically embedded in \( \mathcal{B}(\mathcal{H}) \) for some \( \mathcal{H} \), and we may regard \( \mathcal{A} \) as \( \mathcal{C}^*\)-subalgebra of \( \mathcal{B}(\mathcal{H}) \), and then, evidently, may regard \( \mathcal{A} \otimes M_n(\mathbb{C}) \) as a \( \mathcal{C}^*\)-subalgebra of \( \mathcal{B}(\mathcal{H}) \otimes M_n(\mathbb{C}) \), using the \( \mathcal{C}^*\) algebra norm on \( \mathcal{B}(\mathcal{H}) \otimes M_n(\mathbb{C}) \). This gives us the unique \( \mathcal{C}^*\)-algebra norm on \( \mathcal{A} \otimes M_n(\mathbb{C}) \).

We also obtain, as an immediate corollary of Theorem 5.5, \(*\)-isomorphisms between \( \mathcal{A} \otimes M_n(\mathbb{C}) \) and \( M_n(\mathcal{A}) \), the \(*\)-algebra of \( n \times n \) matrices with entries in \( \mathcal{A} \), given by the formulas provided there.

In what follows, we will need a convenient characterization of the elements of \( M_n(\mathcal{A})^+ \).

5.6 THEOREM. Let \( \mathcal{A} \) be a \( \mathcal{C}^*\) algebra. An element \( \hat{X} \) of \( M_n(\mathcal{A}) \) is positive if and only if it is a sum of at most \( n \) elements of the form \([A_i^* A_j]\), \( \{A_1, \ldots, A_n\} \subset \mathcal{A} \).

Proof. Let \( X_k \) be the elements of \( M_n(\mathcal{A}) \) whose \( k \)th row is \((A_1, \ldots, A_n)\), and whose other rows are zero. Then \( X_k^* X_k = [A_i^* A_j] \), and this proves that \([A_i^* A_j]\) is positive. Now if \( \hat{X} \in M_n(\mathcal{A}) \) is positive, then for some \( \hat{A} \in M_n(\mathcal{A}) \), \( \hat{X} = \hat{A}^* \hat{A} \). Let \( \hat{A}_k \) be the element of \( M_n(\mathcal{A}) \) whose \( k \)th row is the same as that of \( \hat{A} \), but whose other rows are all zero. Then \( \hat{X} = \sum_{j=1}^n \hat{A}_k^* \hat{A}_k \). \( \square \)
Now let $\rho$ be a positive linear functional on $\mathcal{A}$. Let $\tilde{X} \in M_n(A)$ have the form $[A_i^*A_j]$ for $(A_1, \ldots, A_n) \subset \mathcal{A}$. Then $\rho \otimes 1(\tilde{X}) = [\rho(A_i^*A_j)]$. For any $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, 
\[
\sum_{i,j=1}^{n} \overline{\lambda_i} \rho(A_i^*A_j) \lambda_j = \rho \left( \left( \sum_{i=1}^{n} \lambda_i A_i \right)^* \left( \sum_{j=1}^{n} \lambda_j A_j \right) \right) \geq 0.
\]
Therefore, $[\rho(A_i^*A_j)]$ is a positive element of $M_n(\mathbb{C})$, and hence $\rho \otimes 1$ is positive. Then by Theorem 5.6, for every $\tilde{X} \in M_n(\mathcal{A})^+$, $(\rho \otimes 1)(\tilde{A}) \in M_n(\mathcal{C})^+$, and therefore, for any positive linear functional $\sigma$ on $M_n(\mathcal{C})$, 
\[
\rho \otimes \sigma(\tilde{X}) = \sigma(\rho \otimes 1)(\tilde{X}) \geq 0
\]
and hence $\rho \otimes \sigma$ is positive. This proves:

**5.7 COROLLARY.** Let $\rho$ be a positive linear functional on a $C^*$ algebra $\mathcal{A}$. Let $\sigma$ be a positive linear functional on $M_n(\mathcal{C})$. Then $\rho \otimes \sigma$ is a positive linear functional on $M_n(\mathcal{A})$.

**5.3 Positive and completely positive maps**

**5.8 DEFINITION.** Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$ algebras. A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is **positive** in case $\Phi(A) \in \mathcal{B}^+$ for all $A \in \mathcal{A}^+$. If $\mathcal{A}$ and $\mathcal{B}$ have identities $1_\mathcal{A}$ and $1_\mathcal{B}$ respectively, then $\Phi$ is **unital** in case $\Phi(1_\mathcal{A}) = 1_\mathcal{B}$.

**5.9 LEMMA.** $\mathcal{A}$ and $\mathcal{B}$ be $C^*$ algebras, and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a positive linear map. Then $\Phi$ is bounded, and for all $A \in \mathcal{A}$, $\Phi(A^*) = \Phi(A)^*$.

**Proof.** In any $C^*$ algebra $\mathcal{A}$, we can write the general element $A$ of $B_{\mathcal{A}}$ has a decomposition $A = \sum_{j=3}^{3} i^j B_j$ with $\{B_0, B_1, B_2, B_3\} \subset B_{\mathcal{A}} \cap \mathcal{A}^+$. 
\[
\sup\{\|\Phi(A)\| : A \in B_{\mathcal{A}}\} \leq 4 \sup\{\|\Phi(A)\| : A \in B_{\mathcal{A}} \cap \mathcal{A}^+\}.
\]
If $\mathcal{A}$ is unital, it follows that $\|\Phi\| \leq 4\|\Phi(1)\|$. Later in this chapter we shall see the the factor of 4 is superfluous. What matters now is that positive maps are bounded. To see this when $\mathcal{A}$ is not unital, we adapt the aegument used to prove that positive linear functionals are bounded: Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is positive, but unbounded. Then for each $n \in \mathbb{N}$, there exists $A_n \in B_{\mathcal{A}}$ such that $\|A_n\| = 8^n$. Define $S_n = \sum_{j=1}^{n} 2^{-j} A_j$. Then $S_n \in B_{\mathcal{A}}$ and $\|\Phi(S_{n-1})\| \leq \sum_{j=1}^{n-1} 4^j \leq \frac{1}{4} 4^n$ and so $\|\Phi(S_n)\| \geq \frac{2}{3} 4^n$. The sequence $\{S_n\}_{n \in \mathbb{N}}$ converges in norm to $S \in B_{\mathcal{A}}$, and $S \geq S_n$ for all $n$. Hence $\Phi(S) \geq \Phi(S_n)$, with the consequence that $\|\Phi(S)\| \geq \|\Phi(S_n)\| \geq \frac{2}{3} 4^n$ for all $n$, and this is a contradiction for large $n$.

The decomposition $A = \sum_{j=0}^{3} i^j B_j$ with $\{B_0, B_1, B_2, B_3\} \subset B_{\mathcal{A}} \cap \mathcal{A}^+$ shows that $\Phi(A^*) = \Phi(A)^*$ holds for all $A \in B_{\mathcal{A}}$, and then for all $A$ by homogeneity. 

If $\Phi$ is any $*$-homomorphism of $\mathcal{A}$ into $\mathcal{B}$, then for all $A \in \mathcal{A}$, $\Phi(A^*A) = \Phi(A)^*\Phi(A) \geq 0$, and since every element of $\mathcal{A}^+$ if is the form $A^*A$, it follows that every $*$-homomorphism is positive.

Here is another important example: Let $\mathcal{A} = \mathcal{B} = M_n(\mathbb{C})$, and for $A \in \mathcal{A}$, let $\Phi(a) = AT$, the transpose of $A$. Then evidently $\Phi : A \to AT$ is positive. Since for all $A_1, A_2 \in \mathcal{A}$, $(A_1A_2)^T = A_2^T A_1^T$, $\Phi$ is not a $*$-homomorphism.
5.10 DEFINITION. Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( C^* \)-algebras, and let \( \Phi \) be a bounded linear map from \( \mathcal{A} \) to \( \mathcal{B} \). Then for all \( n \in \mathbb{N} \), define \( \Phi_n := \Phi \otimes 1_{\mathbb{C}^n} \), so that

\[
\Phi_n(A \otimes M) = \Phi(A) \otimes M
\]

for all \( A \in \mathcal{A} \) and all \( M \in M_n(\mathbb{C}) \). In particular, for any orthonormal basis \( \{\eta_1, \ldots, \eta_n\} \) of \( \mathbb{C}^n \),

\[
\Phi_n \left( \sum_{i,j=1}^{n} A_{i,j} \otimes |\eta_i\rangle\langle\eta_j| \right) = \sum_{i,j=1}^{n} \Phi(A_{i,j}) \otimes |\eta_i\rangle\langle\eta_j|.
\]

Equivalently, for \( \hat{A} \) consider as \( [A_{i,j}] \in M_n(\mathcal{A}) \), \( [\Phi_n(\hat{A})]_{i,j} = [\Phi(A_{i,j})] \). That is, the action of \( \Phi_n \) on \( [A_{i,j}] \) is given by the action of \( \Phi \) on each entry of \( [A_{i,j}] \).

5.11 DEFINITION (\( n \)-positive and completely positive). Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( C^* \) algebras. A linear map \( \Phi : \mathcal{A} \to \mathcal{B} \) is \( n \)-positive in case \( \Phi_n : \mathcal{A} \otimes \mathbb{C}^n \to \mathcal{B} \otimes \mathbb{C}^n \) is positive. A linear map \( \Phi : \mathcal{A} \to \mathcal{B} \) is completely positive in case \( \Phi_n \) is positive for all \( n \in \mathbb{N} \).

5.12 THEOREM. Let \( \mathcal{A} \) be a \( C^* \)-subalgebra of \( \mathcal{B}(\mathcal{H}) \) for some Hilbert space \( \mathcal{H} \). Let \( \mathcal{K} \) be a second Hilbert space, and let \( \Phi : \mathcal{A} \to \mathcal{B}(\mathcal{K}) \) be given by

\[
\Phi(A) = \sum_{j=1}^{m} W_j^* A W_j
\]

where for each \( j = 1, \ldots, m \), \( W_j \) is a bounded linear transformation from \( \mathcal{K} \) to \( \mathcal{H} \). Then \( \Phi \) is completely positive.

Proof. Since a sum of completely positive maps is evidently completely positive, it suffices to consider the case \( \Phi(A) = W^* A W \) for a bounded linear transformation \( W \) from \( \mathcal{K} \) to \( \mathcal{H} \). But for any \( n \in \mathbb{N} \), if \( \hat{A} = [A_{i,j}] \) is any element of \( M_n(\mathcal{A}) \),

\[
\Phi_n(\hat{A}) = [W^* A_{i,j} W] = \left( \sum_{i=1}^{n} W^* \otimes |\eta_i\rangle\langle\eta_i| \right) \left( \sum_{k,\ell} A_{k,\ell} \otimes |\eta_k\rangle\langle\eta_\ell| \right) \left( \sum_{j=1}^{n} W \otimes |\eta_j\rangle\langle\eta_j| \right)^* \]

which is clearly positive.

\[\Box\]

5.13 THEOREM. Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( C^* \)-algebras, and let \( \pi : \mathcal{A} \to \mathcal{B} \) be a *-homomorphism. The \( \pi \) is completely positive.

Proof. Every \( \hat{A} \in M_n(\mathcal{A})^+ \) has the form \( \hat{A} = \hat{X}^* \hat{X} \) for some \( \hat{X} = [X_{i,j}] \in M_n(\mathcal{A}) \). That is,

\[
\hat{A} = \sum_{k=1}^{n} X_{k,i}^* X_{k,j}.
\]

Therefore \( \pi_n(\hat{A}) = \left[ \sum_{k=1}^{n} \pi(X_{k,i})^* \pi(X_{k,j}) \right] = \pi_n(\hat{X})^* \pi_n(\hat{X}) \in M_n(\mathcal{B})^+ \).

\[\Box\]
5.14 EXAMPLE (The partial transpose). Not every positive map is completely positive: Let $\mathcal{A} = M_2(\mathbb{C})$, and let $\Psi$ be the transpose map $\Psi(A) = A^T$. Then $\Psi$ is positive, but $\Psi_2$ is not. Indeed, identifying $\mathcal{A} \otimes M_2(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ with $M_2(M_2(\mathbb{C}))$ as above, for any $A, B, C, D \in M_2(\mathbb{C})$,

\[
\Psi_2 \left(\begin{array}{cc}
A & B \\
C & D
\end{array}\right) = \begin{bmatrix}
A^T & B^T \\
C^T & D^T
\end{bmatrix}
\]

need not be positive. To see this, let $A = E^{(1,1)}, B = E^{(1,2)}, C = E^{(2,1)}$, and $D = E^{(2,2)}$. Then

\[
\begin{bmatrix}
E^{(1,1)} & E^{(1,2)} \\
E^{(2,1)} & E^{(2,2)}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix} \geq 0, \text{ but } \Psi_2 \left(\begin{array}{cc}
E^{(1,1)} & E^{(1,2)} \\
E^{(2,1)} & E^{(2,2)}
\end{array}\right) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

is not positive.

It turns out the when $\mathcal{A}$ and $\mathcal{B}$ are $C^*$ algebras, and either $\mathcal{A}$ or $\mathcal{B}$ is commutative, then any unital positive linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is completely positive.

5.15 THEOREM (Størmer’s Commutative Range Theorem). Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$ algebras with $\mathcal{B}$ commutative, and let $\Phi : \mathcal{A} \to \mathcal{B}$ be unital positive. Then $\Phi$ is completely positive.

Proof. Let $\psi$ be a pure state on $\mathcal{B} \otimes M_n(\mathbb{C})$. We claim that $\psi$ is a product state. That is, there exists states $\rho$ and $\sigma$ on $\mathcal{B}$ and $M_n(\mathbb{C})$ such that $\psi = \rho \otimes \sigma$.

Suppose that this is proved. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be positive. Then

\[
\psi \circ \Phi_n = (\rho \otimes \sigma) \circ (\Phi \otimes 1) = \rho \circ \Phi \otimes \sigma.
\]

By Corollary 5.7, $\psi$, being the tensor product of positive linear functionals, is positive. Hence $\psi \circ \Phi_n$ is positive for every state $\psi$ on $\mathcal{B} \otimes M_n(\mathbb{C})$. It follows that $\Phi_n$ is positive.

To complete the proof, we validate the claim that every pure state on $\mathcal{B} \otimes M_n(\mathbb{C})$ is a product state. Since $\mathcal{B} \otimes 1$ is the center of $\mathcal{B} \otimes M_n(\mathbb{C})$, if $B \in \mathcal{B}$ and $0 \leq B \leq 1$, then $0 \leq B \otimes 1 \leq 1$ in $\mathcal{B} \otimes M_n(\mathbb{C})$. Then for all $X \geq 0$ in $\mathcal{B} \otimes M_n(\mathbb{C})$, $\psi((B \otimes 1)X) = \psi(X^{1/2}(B \otimes 1)X^{1/2}) \leq \psi(X)$. Since $\psi$ is pure, $(B \otimes 1)\psi$ is a multiple of $\psi$, and evaluating at the identity, the multiple is $\psi(B \otimes 1))$. That is $(B \otimes 1)\psi = \psi(B \otimes 1)\psi$. Then for all $A \in M_n(\mathbb{C})$,

\[
\psi(B \otimes A) = (B \otimes 1)\psi(1 \otimes A).
\]

If we define $\rho(B) = \psi(B \otimes 1)$ and $\sigma(A) = \psi(1 \otimes A)$, then $\rho$ and $\sigma$ are states on $\mathcal{B}$ and $M_n(\mathbb{C})$, respectively, and we have that $\psi = \rho \otimes \sigma$. In summary, every pure state $\psi$ on $\mathcal{B} \otimes M_n(\mathbb{C})$ is a product state; $\psi = \rho \otimes \sigma$. \qed

5.16 THEOREM (Stinespring’s Commutative Domain Theorem). Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$ algebras with $\mathcal{A}$ commutative, and let $\Phi : \mathcal{A} \to \mathcal{B}$ be unital positive. Then $\Phi$ is completely positive.

Proof. Since every commutative $C^*$ algebra is isometrically $*$-isomorphic to $C_0(X)$ for some locally compact Haussdorf space $X$. we may assume that $\mathcal{A}$ has this form. Then the general element of $M_n(\mathcal{A})$ is a continuous function $f$ from $X$ into $M_n(\mathbb{C})$ such that for all $\epsilon > 0$, there exists a compact set $K \subset X$ such that $\|f(x)\| < \epsilon$ for $x \notin K$. Cover $K$ by a finite collection $\{U_1, \ldots, U_m\}$ of open sets such that for some $x_1 \in U_i$ and all $y \in U_i$, $\|f(y) - f(x_i)\| < \epsilon$. Then we may choose a partition
of unity on $K$ subordinate to $\{U_1, \ldots, U_m\}$: There exist a sets of functions $\{h_1, \ldots, h_m\} \subset \mathcal{C}_0(X)$ such that $0 \leq h_i \leq 1$ for each $i$, the support of $h_i$ is contained in $U_i$ and with $h := \sum_{j=1}^m h_i(x)$, $h = 1$ on $K$, and $0 \leq h \leq 1$ on $X$.

For each $x \in X$, taking norms in $\mathcal{M}_n(\mathbb{C})$ for each fixed $x$,

$$\|f(x) - \sum_{i=1}^m h_i(x)f(x_i)\| \leq \|(1 - h)f(x)\| + \|\sum_{i=1}^m h_i(x)(f(x) - f(x_i))\|.$$ 

Since $h = 1$ on $K$, $\|(1 - h)f(x)\| < \epsilon$ for all $x$, and since $h_i(x)(f(x) - f(x_i))$ is supported in $U_i$, $\|h_i(x)(f(x) - f(x_i))\| \leq h_i(x)\epsilon$. Altogether, $\|f(x) - \sum_{i=1}^m h_i(x)f(x_i)\| \leq 2\epsilon$.

Then $\|\Phi_n(f) - \sum_{i=1}^m \Phi(h_i) \otimes f(x_i)\| \leq 2\epsilon\|\Phi\|$. Evidently, $\sum_{i=1}^m \Phi(h_i(x)) \otimes f(x_i)$ is positive in $\mathcal{B} \otimes \mathcal{M}_n(\mathbb{C})$, and then since $\epsilon > 0$ is arbitrary and the positive cone in $\mathcal{B} \otimes \mathcal{M}_n(\mathbb{C})$ is closed, $\Phi_n(f) \geq 0$.

**5.17 Lemma.** Let $\mathcal{A}$ be a $C^*$-algebra and let $\mathcal{H}$ be a Hilbert space. Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ be completely positive. Consider an arbitrary element of $\mathcal{A}$ (The Stinespring inner product). Let $\mathcal{A}$ be a norm one projection. Then $\Phi$ is non-negative if and only if $\Phi$ is completely positive.

**Proof.** Suppose that $\Phi$ is completely positive. By Theorem 5.6, $[A_j^*A_j]$ is positive in $\mathcal{M}_n(\mathcal{A})$, and hence $[\Phi(A_j^*A_j)]$ is positive in $\mathcal{M}_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}_n)$. Hence $\sum_{i,j=1}^n \langle \xi_i, \Phi(A_j^*A_j)\xi_j \rangle_{\mathcal{H}} \geq 0$ for all $(\xi_1, \ldots, \xi_n) \in \mathcal{H}_n$.

Next suppose that $\Phi$ is non-negative on $\mathcal{A} \otimes \mathcal{H}$. By Theorem 5.6, the general element of $\mathcal{M}_n(\mathcal{A})^+$ is a sum of at most $n$ terms of the form $[A_j^*A_j]$. Hence $\Phi_n$ is positive on $\mathcal{M}_n(\mathcal{A})$ if for each $[A_j^*A_j]$, $\Phi_n([A_j^*A_j])$ is positive in $\mathcal{M}_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}_n)$. For each $(\xi_1, \ldots, \xi_n) \in \mathcal{H}_n$,

$$\sum_{i,j=1}^n \langle \xi_i, \Phi(A_j^*A_j)\xi_j \rangle = q_\Phi \left( \sum_{j=1}^n A_j \otimes \eta_j \right) \geq 0,$$

showing that $\Phi_n([A_j^*A_j])$ is positive in $\mathcal{M}_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}_n)$. \(\square\)

**5.18 Definition** (The Stinespring inner product). Let $\mathcal{A}$ be a $C^*$-algebra and let $\mathcal{H}$ be a Hilbert space. Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ be completely positive. The *Stinespring inner product on $\mathcal{A} \otimes \mathcal{H}$* is the inner product $\langle \cdot, \cdot \rangle_{\Phi}$ on $\mathcal{A} \otimes \mathcal{H}$ given by the polarization of $q_{\Phi}$ as defined in (5.18).

That is,

$$\left\langle \sum_{j=1}^n A_j \otimes \eta_j, \sum_{j=1}^n B_j \otimes \xi_j \right\rangle_{\Phi} = \sum_{i,j=1}^n \langle \eta_i, \Phi(A_j^*B_j)\xi_j \rangle.$$ 

**5.19 Theorem** (Conditional expectations are completely positive). Let $\mathcal{A}$ be a $C^*$-algebra with unit 1, and let $\mathcal{B}$ be a $C^*$ subalgebra of $\mathcal{A}$ containing 1. Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a norm one projection. Then $\Phi$ is completely positive.
Proof. By Tomiyama’s Theorem, $\Phi$ has the property that for all $X,Y \in \mathcal{B}$ and all $A \in \mathcal{A}$, $\Phi(XAY) = X\Phi(A)Y$, and $\Phi$ is positive. We must show that for all $n$, and all $[A_i^*A_j] \in M_n(C)$, $\Phi_n([A_i^*A_j]) \geq 0$ in $M_n(\mathcal{B})$. We may regard $\mathcal{A}$ as a $C^*$ subalgebra of $\mathcal{B}(\mathcal{H})$, and then of course $\mathcal{B}$ is a $C^*$ subalgebra of $\mathcal{B}(\mathcal{H})$. Identifying $M_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}_n)$, it suffices to show that $\Phi_n([A_i^*A_j])$ is positive in $\mathcal{B}(\mathcal{H}_n)$, which means that for all $(\eta_1, \ldots, \eta_n)$ a dense subset of $\mathcal{H}_n$,

$$\sum_{i,j=1}^n \langle \eta \Phi(A_i^*A_j) \rangle_j \geq 0 \, .$$

Suppose that $\xi_0$ is a cyclic vector for the action of $\mathcal{B}$ on $\mathcal{H}$. Then the set of vectors of the form $(B_1\xi_0, \ldots, B_n\xi_0)$ is dense in $\mathcal{H}_n$ and then with $C = \sum_{j=1}^n A_j B_j$,

$$\sum_{i,j=1}^n \langle B_i \xi_0 \Phi(A_i^*A_j) B_j \xi_0 \rangle = \sum_{i,j=1}^n \langle \xi_0 \Phi(B_i^*A_i^*A_jB_j) \xi_0 \rangle = \langle \xi_0 \Phi(C^*C) \xi_0 \rangle \geq 0 \, .$$

In general, $\mathcal{H}$ is a direct sum, possibly uncountable, of subspaces generated by cyclic vectors. Every $\xi \in \mathcal{H}$ may be approximated arbitrarily well in norm by vectors in a finite sum of such spaces. Thus there is a dense set of vectors $(\eta_1, \ldots, \eta_n) \in \mathcal{H}_n$ such that each $\eta_j$ is contained in a finite direct sum of orthogonal cyclic subspaces of $\mathcal{H}$, invariant under $\mathcal{B}$. Since each of these subspaces is invariant under $\mathcal{B}$, we need only check the positivity for the components of $\eta_1, \ldots, \eta_n$ in each of these subspaces, and this is the case that we have dealt with above. \qed

5.4 Kadison’s inequality

5.20 LEMMA. Let $\mathcal{A}$ be a $C^*$ algebra with identity 1. Then for all $A, B \in \mathcal{A}$, $A^*A \leq B$ if and only if $\begin{bmatrix} 1 & A \\ A^* & B \end{bmatrix}$ is positive in $M_2(\mathcal{A})$.

Proof. By the Gelfand-Neumark Theorem, we may suppose that $\mathcal{A}$ is a $C^*$ subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Suppose $B \geq A^*A$, and define $C = B - A^*A$. Then

$$\begin{bmatrix} 1 & A \\ A^* & B \end{bmatrix} = \begin{bmatrix} 1 & A \\ A^* & A^*A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} 1 & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & A \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \, ,$$

which displays the left hand side as a sum of positive operators.

For the converse, suppose that $\begin{bmatrix} 1 & A \\ A^* & B \end{bmatrix}$ is positive. Then for all $\xi, \eta \in \mathcal{H}$,

$$\left\langle \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \begin{bmatrix} 1 & A \\ A^* & B \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \right\rangle_{\mathcal{H} \otimes \mathcal{H}} = \|\xi\|^2_{\mathcal{H}} + 2\Re\langle \xi, A\eta \rangle_{\mathcal{H}} + \langle \eta, B\eta \rangle_{\mathcal{H}} \geq 0 \, .$$

For $\xi = -A\eta$, this becomes $\langle \eta, [B - A^*A]\eta \rangle_{\mathcal{H}} \geq 0$, and this shows that $B \geq A^*A$. \qed

5.21 THEOREM (Kadison’s Inequality). Let $\mathcal{A}$ and $\mathcal{B}$ be unital $C^*$ algebras Let $\Phi$ be a unital 2-positive map from $\mathcal{A}$ to $\mathcal{B}$. Then for all $A \in \mathcal{A}$,

$$\Phi(A)^*\Phi(A) \leq \Phi(A^*A) \, .$$

(5.20)
Proof. Since \( \begin{bmatrix} 1 & A \\ A^* & A^*A \end{bmatrix} \geq 0 \) in \( M_2(\mathcal{A}) \), \( \Phi_2 \left( \begin{bmatrix} 1 & A \\ A^* & A^*A \end{bmatrix} \right) = \begin{bmatrix} 1 & \Phi(A) \\ \Phi(A)^* & \Phi(A^*A) \end{bmatrix} \geq 0 \), and by Lemma 5.20, this implies (5.20).

5.22 COROLLARY. Let \( \mathcal{A} \) and \( \mathcal{B} \) be unital \( C^* \) algebras Let \( \Phi \) be a unital positive map from \( \mathcal{A} \) to \( \mathcal{B} \). Then for all normal \( A \in \mathcal{A} \),

\[
\Phi(A^*) \Phi(A) \leq \Phi(A^*A) .
\] (5.21)

In particular, for all unitary \( U \in \mathcal{A} \), \( \Phi(U)^* \Phi(U) \leq 1 \), with the consequence that \( \|\Phi\| \leq 1 \).

Proof. for \( A \) normal, \( A \) belongs to the commutative \( C^* \) algebra \( C(\{1, A\}) \) and by Stinespring’s Commutative Domain Theorem, \( \Phi \) is completely positive on \( C(\{1, A\}) \). This proves (5.21).

Next, if \( U \in \mathcal{A} \) is unitary \( U \) is normal and \( U^*U = 1 \). Therefore \( \Phi(U)^* \Phi(U) \leq 1 \), and hence \( \|\Phi(U)\| \leq 1 \). Then by the Russo-Dye Theorem, \( \|\Phi(A)\| \leq 1 \) for all \( A \in B_\mathcal{A} \).

5.23 LEMMA. Let \( \mathcal{A} \) be a \( C^* \) algebra. Then \( \begin{bmatrix} 0 & A \\ A^* & B \end{bmatrix} \geq 0 \) in \( M_2(\mathcal{A}) \) if and only if \( A = 0 \) and \( B \geq 0 \). Likewise, \( \begin{bmatrix} B & A \\ A^* & 0 \end{bmatrix} \geq 0 \) in \( M_2(\mathcal{A}) \) if and only if \( A = 0 \) and \( B \geq 0 \).

Proof. We compute \( \left( \begin{bmatrix} \eta & 0 \\ 0 & A \end{bmatrix} \right) \left( \begin{bmatrix} 0 & A \\ A^* & B \end{bmatrix} \right) \left( \begin{bmatrix} \eta \\ \xi \end{bmatrix} \right) = 2\Re \langle \eta, A\xi \rangle + \langle \xi, B\xi \rangle \).

If \( A \neq 0 \), choose \( \xi \) so that \( A\xi \neq 0 \), and then, for \( t > 0 \), \( \eta = -tA\xi \). Then

\[
2\Re \langle \eta, a\xi \rangle + \langle \xi, b\xi \rangle = -2t\|A\xi\|^2 + \langle \xi, B\xi \rangle.
\]

For sufficiently large \( t \), the right hand side is negative. Hence positivity of \( \begin{bmatrix} 0 & A \\ A^* & B \end{bmatrix} \) implies that \( A = 0 \), and then it is clear that \( B \geq 0 \). The converse is evident, and the statement for the matrix with \( 0 \) in the lower right position follows in the same way.

Let \( \mathcal{A} \) and \( \mathcal{B} \) be unital \( C^* \) algebras, and let \( \Phi : \mathcal{A} \to \mathcal{B} \) be completely positive and unital. Since \( \Phi_2 \) is 2-positive, Kadison’s inequality applied to \( \Phi_2 \) at \( \begin{bmatrix} A & B^* \\ 0 & 0 \end{bmatrix} \) yields

\[
\left( \Phi_2 \left( \begin{bmatrix} A & B^* \\ 0 & 0 \end{bmatrix} \right) \right)^* \Phi_2 \left( \begin{bmatrix} A & B^* \\ 0 & 0 \end{bmatrix} \right) \leq \Phi_2 \left( \begin{bmatrix} A^*A & A^*B^* \\ BA & BB^* \end{bmatrix} \right) ,
\]

and this is

\[
\begin{bmatrix} \Phi(A^*A) - \Phi(A)^* \Phi(A) & \Phi(A^*B^*) - \Phi(A)^* \Phi(B)^* \\ \Phi(BA) - \Phi(B)^* \Phi(A) & \Phi(BB^*) - \Phi(B)^* \Phi(B)^* \end{bmatrix}.
\]

By Lemma 5.23, if either \( \Phi(A^*A) = \Phi(A)^* \Phi(A) \) or \( \Phi(BB^*) = \Phi(B)^* \Phi(B)^* \), then \( \Phi(BA) = \Phi(B)^* \Phi(A) \). Conversely, if for some \( A \in \mathcal{A} \), \( \Phi(BA) = \Phi(B)^* \Phi(A) \) for all \( B \), then taking \( B = A^* \), we have \( \Phi(A^*A) = \Phi(A)^* \Phi(A) \), and if for some \( B \in \mathcal{A} \), \( \Phi(BA) = \Phi(B)^* \Phi(A) \) for all \( A \), then taking \( A = B^* \), \( \Phi(BB^*) = \Phi(B)^* \Phi(B)^* \). We obtain:
5.24 THEOREM. Let $\mathcal{A}$ and $\mathcal{B}$ be unital $C^*$ algebras, and let $\Phi: \mathcal{A} \to \mathcal{B}$ be completely positive and unital. Then

(i) \( \{ A \in \mathcal{A} : \Phi(A^*A) = \Phi(A)^*\Phi(A) \} = \{ A \in \mathcal{A} : \Phi(BA) = \Phi(B)\Phi(A) \text{ for all } B \in \mathcal{A} \} \), and this set is a subalgebra of $\mathcal{A}$, and $\Phi$ is a homomorphism when restricted to this set.

(ii) \( \{ A \in \mathcal{A} : \Phi(AA^*) = \Phi(A)\Phi(A)^* \} = \{ A \in \mathcal{A} : \Phi(AB) = \Phi(A)\Phi(B) \text{ for all } B \in \mathcal{A} \} \), and this set is a subalgebra of $\mathcal{A}$, and $\Phi$ is a homomorphism when restricted to this set.

(iii) the set

\[ \{ A \in \mathcal{A} : \Phi(A^*A) = \Phi(A)^*\Phi(A) \} \cap \{ A \in \mathcal{A} : \Phi(AA^*) = \Phi(A)\Phi(A)^* \} \tag{5.22} \]

is a $C^*$ subalgebra of $\mathcal{A}$, and $\Phi$ is a $*$-homomorphism when restricted to this set.

5.25 DEFINITION (Multiplicative domain). The set in (i) of Theorem 5.24 is the left multiplicative domain of $\Phi$, and the set (ii) of Theorem 5.24 is the right multiplicative domain of $\Phi$. Their intersection is the multiplicative domain of $\Phi$.

5.26 DEFINITION (Invariant states). Let $\mathcal{A}$ be a unital $C^*$ algebras and let $\Phi: \mathcal{A} \to \mathcal{B}$ be completely positive and unital. A state $\phi$ on $\mathcal{A}$ is invariant under $\Phi$ in case for all $A \in \mathcal{A}$,

\[ \phi(\Phi(A)) = \phi(A) . \tag{5.23} \]

5.27 THEOREM. Let $\mathcal{A}$ be a unital $C^*$ algebras and let $\Phi: \mathcal{A} \to \mathcal{B}$ be completely positive and unital, and suppose that $\phi$ is a faithful state on $\mathcal{A}$ that is invariant under $\Phi$. Define

\[ \mathcal{C} = \{ C \in \mathcal{A} : \Phi(C) = C \} . \]

Then $\mathcal{C}$ is a $C^*$ algebra of $\mathcal{A}$, and for all $A, B \in \mathcal{A}$ and all $C \in \mathcal{C}$,

\[ \Phi(ACB) = \Phi(A)C\Phi(B) . \tag{5.24} \]

Proof. By Kadison’s inequality, for $C \in \mathcal{C}$,

\[ \Phi(C^*C) - C^*C \geq 0 . \]

Applying the faithful invariant state $\phi$ yields

\[ \phi(\Phi(C^*C) - C^*C) = \phi(C^*C) - \phi(C^*C) = 0 . \]

Since $\phi$ is faithful, $\Phi(C^*C) = C^*C$, and so $C^*C \in \mathcal{C}$. We now have that $\Phi(C^*C) = \Phi(C)^*\Phi(C)$. Since $\mathcal{C}$ is closed under the involution, the same applies with $C$ replaced by $C^*$. Thus $\mathcal{C}$ is in the multiplicative domain of $\Phi$, and is a $C^*$ subalgebra of $\mathcal{A}$ by Theorem 5.23, which then also yields (5.24).
5.5 Stinespring’s Theorem

5.28 DEFINITION. Let \( \pi \) be a representation of a unital \( C^* \)-algebra \( \mathcal{A} \) on the Hilbert space \( \mathcal{H} \). Then a subspace \( \mathcal{K} \) is cyclic for \( \pi \) in case \( \{ \pi(A)\xi : A \in \mathcal{A}, \xi \in \mathcal{K} \} \) is dense in \( \mathcal{K} \).

5.29 THEOREM. Let \( \mathcal{A} \) be a \( C^* \)-algebra with identity 1, and let \( \Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) be a completely positive map. Then there exists a Hilbert space \( \mathcal{K} \), a representation \( \pi \) of \( \mathcal{A} \) on \( \mathcal{K} \), and a bounded operator \( V : \mathcal{K} \to \mathcal{H} \) such that \( V\mathcal{K} \) is cyclic for \( \pi \), and \( \|V\|^2 = \|\Phi(1)\| \) and for all \( A \in \mathcal{A} \),

\[
\Phi(A) = V^*\pi(A)V \quad \text{and} \quad \|\pi(A)\| \leq \|A\|. \tag{5.25}
\]

When \( \Phi \) is unital, \( V \) is an isometry, and

\[
\|\pi(A)\|^2 = \Phi(A^*A). \tag{5.26}
\]

Proof. Equip the vector space \( \mathcal{A} \otimes \mathcal{H} \) with the Stinespring inner product \( \langle \cdot, \cdot \rangle_{\Phi} \). Define

\[
\mathcal{N} = \{ \xi \in \mathcal{A} \otimes \mathcal{H} : \langle \xi, \xi \rangle_{\Phi} = 0 \}.
\]

Then \( \mathcal{N} \) is a subspace of \( \mathcal{A} \otimes \mathcal{H} \), and for all \( \xi, \xi' \in \mathcal{A} \otimes \mathcal{H} \) we write \( \xi \sim \xi' \) in case \( \xi - \xi' \in \mathcal{N} \).

Define an inner product, again denoted \( \langle \cdot, \cdot \rangle_{\Phi} \) on the quotient space \( \mathcal{A} \otimes \mathcal{H}/\mathcal{N} \) by

\[
\langle \{\xi\}, \{\zeta\} \rangle_{\Phi} = \langle \xi, \zeta \rangle_{\Phi}.
\]

By the Cauchy-Schwarz inequality, the inner product is independent of the choice of representatives. Let \( \mathcal{H} \) be the Hilbert space completion of \( \mathcal{A} \otimes \mathcal{H}/\mathcal{N} \) in the metric associated to this inner product.

Every \( \xi \in \mathcal{A} \otimes \mathcal{H} \) has the form \( \xi = \sum_{j=1}^{n} B_j \otimes \zeta_j \) where \( \{\zeta_1, \ldots, \zeta_n\} \subset \mathcal{H} \) is orthonormal.

Let \( A \in \mathcal{A} \), and define \( \pi(A)\xi = \sum_{j=1}^{n} A^*B_j \otimes \zeta_j \). Then \( \|\pi(A)\xi\|^2_{\mathcal{H}} = \langle \zeta, \Phi(B_jA^*AB_j)\zeta \rangle_{\mathcal{H}} \). Since \( \{\zeta_1, \ldots, \zeta_n\} \) is orthonormal, \( [B_jA^*AB_j] \in M_n(\mathcal{A}) \) is given by

\[
[B_j^*A^*AB_j] = \begin{bmatrix}
1_{\mathcal{A}} & 0 & \cdots & 0 \\
0 & 1_{\mathcal{A}} & \cdots & 0 \\
0 & 0 & \cdots & 1_{\mathcal{A}}
\end{bmatrix}^* \begin{bmatrix}
A^*A & 0 & \cdots & 0 \\
0 & A^*A & \cdots & 0 \\
0 & 0 & \cdots & A^*A
\end{bmatrix} \begin{bmatrix}
1_{\mathcal{A}} & 0 & \cdots & 0 \\
0 & 1_{\mathcal{A}} & \cdots & 0 \\
0 & 0 & \cdots & 1_{\mathcal{A}}
\end{bmatrix} \geq 0
\tag{5.27}
\]

Since

\[
\|A\|^2 = \begin{bmatrix}
1_{\mathcal{A}} & 0 & \cdots & 0 \\
0 & 1_{\mathcal{A}} & \cdots & 0 \\
0 & 0 & \cdots & 1_{\mathcal{A}}
\end{bmatrix},
\]

the factorization (5.27) implies that \( \|A\|^2[B_j^*B_j] \geq [B_j^*A^*AB_j] \), and then since \( \Phi \) is completely positive, \( \|A\|^2\Phi_n([B_j^*B_j]) \geq \Phi_n([B_j^*A^*AB_j]) \). Therefore

\[
\|A\|^2_{\Phi} \langle \xi, \xi \rangle_{\Phi} = \sum_{i,j=1}^{n} \|A\|^2 \langle \zeta_i, \Phi(B_i^*B_j)\zeta_j \rangle_{\mathcal{H}} \geq \sum_{i,j=1}^{n} \langle \zeta_i, \Phi(B_i^*A^*AB_j)\zeta_j \rangle_{\mathcal{H}} = \langle \pi(A)\xi, \pi(A)\xi \rangle_{\Phi}. \tag{5.28}
\]
In particular, whenever $\xi \in \mathcal{N}$, then $\pi(A)\xi \in \mathcal{N}$, and consequently for all $\xi, \xi' \in \mathcal{A} \otimes \mathcal{H}$, $\xi \sim \xi' \Rightarrow \pi(A)\xi \sim \pi(A)\xi'$. Therefore, $\pi(A)$ induces a linear transformation on $\mathcal{A} \otimes \mathcal{H} / \mathcal{N}$ through the definition

$$\pi(A)\{\xi\} = \{\pi(A)\xi\},$$

and as a further consequence of (5.28), this linear transformation on $\mathcal{A} \otimes \mathcal{H} / \mathcal{N}$ is bounded with norm no greater than $\|A\|$. It therefore extends to an element of $\mathcal{B}(\mathcal{H})$, still denoted by $\pi(A)$, that has norm $\|A\|$. It is easy to show, mimicking the corresponding argument from the GNS construction, that $A \mapsto \pi(A)$ is a $*$-homomorphism from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$, and we have proved the inequality on the right in (5.25).

Define $V : \mathcal{H} \to \mathcal{H}$ by $V\xi = \{1_{\mathcal{A}} \otimes \zeta\} \in \mathcal{A} \otimes \mathcal{H} / \mathcal{N}$. Since $\|V\xi\|^2_{\mathcal{H}} = \langle \xi, \Phi(1_{\mathcal{A}})\xi \rangle_{\mathcal{H}},$

$$\|V\| = \sup_{\|\xi\| = 1} \{\|V\xi\|^2_{\mathcal{H}}\} = \sup_{\|\xi\| = 1} \{\langle \xi, \Phi(1_{\mathcal{A}})\xi \rangle_{\mathcal{H}}\} = \|\Phi(1_{\mathcal{A}})\|.$$

For all $\zeta_1, \zeta_2 \in \mathcal{H}$, $\langle \zeta_1, V^*\pi(a)V\zeta_2 \rangle_{\mathcal{H}} = \{1_{\mathcal{A}} \otimes \zeta_1, \pi(A)\{1_{\mathcal{A}} \otimes \zeta_2\}\} = \{\zeta_1, \Phi(A)\zeta_2\},$ and this proves the rest of (5.25). Since $\{\sum_{j=1}^n B_j \otimes \zeta_j\} = \{\sum_{j=1}^n \pi(B_j)V\zeta_j\}$, $V\mathcal{H}$ is cyclic in $\mathcal{H}$ for $\pi$.

Now suppose that $\Phi$ is unital. For all $\zeta \in \mathcal{H}$,

$$\|V\zeta\|^2_{\mathcal{H}} = \|\{1_{\mathcal{A}} \otimes \zeta\}\|^2_{\mathcal{H}} = \langle \zeta, \Phi(1_{\mathcal{A}})\zeta \rangle_{\mathcal{H}} = \|\zeta\|^2_{\mathcal{H}},$$

showing that $V$ is an isometry. Moreover, $\|\pi(A)\{1_{\mathcal{A}} \otimes \zeta\}\|^2_{\mathcal{H}} = \langle \zeta, \Phi(A^*A)\zeta \rangle_{\mathcal{H}}$. Hence for all non-zero $\zeta \in \mathcal{H},$

$$\frac{\|\pi(A)\{1_{\mathcal{A}} \otimes \zeta\}\|^2_{\mathcal{H}}}{\|\{1_{\mathcal{A}} \otimes \zeta\}\|^2_{\mathcal{H}}} = \frac{\langle \zeta, \Phi(A^*A)\zeta \rangle_{\mathcal{H}}}{\|\zeta\|^2_{\mathcal{H}}},$$

and this proves (5.26).

\[\Box\]

5.30 DEFINITION (Minimal Stinespring representation). Let $\mathcal{A}$ be a unital $C^*$ algebra, and let $\Phi$ be a completely positive map from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. A triple $(\pi, V, \mathcal{H})$ where $\pi$ is a unital $*$-representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$, and $V$ is a bounded map from $\mathcal{H} \to \mathcal{H}$ such that $V\mathcal{H}$ is cyclic in $\mathcal{H}$ for $\pi$, and such that $\Phi = V^*\pi V$ is called a **minimal Stinespring representation** of $\Phi$. If we drop the cyclicity requirement, it is simply a Stinespring representation of $\Phi$, but the construction in the proof of Stinespring’s Theorem always yields a minimal representation.

The **cannonical minimalis representetation** is the one provided by the proof of Stinespring’s Theorem. That is, $q$ is the quadratic form on $\mathcal{A} \otimes \mathcal{H}$ defined by the Stinespring inner product, $\mathcal{N} := \{\xi \in \mathcal{A} \otimes \mathcal{H} : q(\xi) = 1\}$, $\mathcal{H}$ is the completion of $\mathcal{A} \otimes \mathcal{H} / \mathcal{N}$ in the inner product induced by the Stinespring inner product, $\pi$ is the representation $\pi(A)\{\sum_{j=1}^n B_j \otimes \zeta_j\} = \{\sum_{j=1}^n AB_j \otimes \zeta_j\},$ and $V\zeta = \{1_{\mathcal{A}} \otimes \eta\}$, where $\{\xi\}$ denotes equivalence class of $\xi \in \mathcal{A} \otimes \mathcal{H}$ mod $\mathcal{N}$.

5.31 LEMMA. Let $\mathcal{A}$ be a unital $C^*$ algebra, and let $\Phi$ be a completely positive map from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. For every minimal Stinespring representation $(\pi, V, \mathcal{H})$ of $\Phi$,

$$\left\|\sum_{j=1}^n \pi(B_j)V\zeta_j\right\|^2 = \sum_{i,j=1}^n \langle \zeta_i, \Phi(B_i^*B_j)\zeta_j \rangle.$$

(5.29)
5.32 THEOREM. (Uniqueness of minimal Stinespring representations) Let \( \mathcal{A} \) be a unital C*-algebra, and let \( \Phi \) be a completely positive map from \( \mathcal{A} \) into \( \mathcal{B}(\mathcal{H}) \). Let \((\pi_i, V_i, \mathcal{H}_i), i = 1, 2\) be two minimal Stinespring representations of \( \Phi \). Then there exists a unitary map \( U \) from \( \mathcal{H}_1 \) onto \( \mathcal{H}_2 \) such that \( UV_1 = V_2 \) and \( U^* \pi_1 U = \pi_2 \).

Proof. If such a unitary \( U \) exists, then necessarily \( U \left( \sum_{j=1}^{n} \pi_1(B_j)\zeta_j \right) = \sum_{j=1}^{n} \pi_2(B_j)V_2\zeta_j \) for all \( n \in \mathbb{N}, \{\zeta_1, \ldots, \zeta_n\} \subset \mathcal{H} \) and \( \{B_1, \ldots, B_n\} \subset \mathcal{A} \). We need only check that this does, in fact, define an isometry, since by the minimality, \( U \) will have dense range. But this is clear from Lemma 5.31 since the right hand side of (5.29) is independent of the particular minimal Stinespring representation of \( \Phi \).

5.33 THEOREM (Arveson’s Radon-Nikodym Theorem). Let \( \mathcal{A} \) be a unital C*-algebra, and let \( \Phi, \Psi \) be completely positive maps from \( \mathcal{A} \) into \( \mathcal{B}(\mathcal{H}) \) such that \( \Phi - \Psi \) is completely positive. Let \((\pi, V, \mathcal{H})\) be a minimal Stinespring representation of \( \Phi \). There exists a unique operator \( R \in \pi(\mathcal{A})' \) with \( 0 \leq R \leq 1 \) such that \( \Psi(A) = V^* R \pi(A) V \) for all \( A \in \mathcal{A} \).

Proof. Let \((\pi_2, V_2, \mathcal{H}_2)\) be the canonical minimal Stinespring representation of \( \Psi \). By Theorem 5.32, without loss of generality, we may assume that \((\pi, V, \mathcal{H})\) is also canonical. By Lemma 5.31, for all \( n \in \mathbb{N}, \{B_1, \ldots, B_n\} \subset \mathcal{A} \) and all \( \{\zeta_1, \ldots, \zeta_n\} \subset \mathcal{H} \), define quadratic form \( q \) and \( q_2 \) on \( \mathcal{A} \otimes \mathcal{H} \) by

\[
q \left( \sum_{j=1}^{n} B_j \otimes \zeta_j \right) = \left\| \sum_{j=1}^{n} \pi(B_j)V_2\zeta_j \right\|^2 \quad \text{and} \quad q_2 \left( \sum_{j=1}^{n} B_j \otimes \zeta_j \right) = \left\| \sum_{j=1}^{n} \pi_2(B_j)V_2\zeta_j \right\|^2 .
\]

We now claim that \( q_2 \leq q \). To see this, we compute

\[
\left\| \sum_{j=1}^{n} \pi(B_j)V_2\zeta_j \right\|^2 = \sum_{i,j=1}^{n} \langle \zeta_i, \Phi(B_i^*B_j)\zeta_j \rangle \\
\geq \sum_{i,j=1}^{n} \langle \zeta_i, \Psi(B_i^*B_j)\zeta_j \rangle = \left\| \sum_{j=1}^{n} \pi_2(B_j)V_2\zeta_j \right\|^2 .
\]

where in the inequality we have used the fact that \( \Phi - \Psi \) is completely positive.

As in Stinespring’s construction, define \( \mathcal{N} := \{\xi \in \mathcal{A} \otimes \mathcal{H} : q(\xi) = 0\} \) and \( \mathcal{N}_2 := \{\xi \in \mathcal{A} \otimes \mathcal{H} : q_2(\xi) = 0\} \). Then \( \mathcal{N} \subset \mathcal{N}_2 \) and therefore if \( \xi - \xi' \in \mathcal{N}, \xi - \xi' \in \mathcal{N}_2 \). Let \( \{\xi\} \) denote the equivalence class of \( \xi \) mod \( \mathcal{N} \), and let \( \{\xi\}_2 \) denote the equivalence class of \( \xi \) mod \( \mathcal{N}_2 \). Then \( \{\xi\}_2 \) is a well-defined linear transformation, and since \( \|\{\xi\}_2\|_{\mathcal{N}_2} = q_2(\xi) \leq q(\xi) = \|\{\xi\}\|_{\mathcal{N}}, \) \( T \) is a contraction from \( \mathcal{H} \) to \( \mathcal{H}_2 \). Moreover, for all \( \zeta \in \mathcal{H}, V_2\zeta = \{1_\mathcal{A} \otimes \zeta\}_2 = T\{1_\mathcal{A} \otimes \zeta\} = TV\zeta \).

Finally, for all \( A, B \in \mathcal{A} \), and \( \zeta \in \mathcal{H} \),

\[
\pi_2(A)T\{B \otimes \zeta\} = \pi_2(A)\{B \otimes \zeta\}_2 = \{AB \otimes \zeta\}_2 = T\{AB \otimes \zeta\} = T\pi(A)\{B \otimes \zeta\} .
\]

Since the span of the set of vectors of the from \( \{B \otimes \zeta\} \) is dense in \( \mathcal{H}, T\pi(A) = \pi_2(A)T \) for all \( A \). In summary, the map \( T \) has the properties that

\[
T\pi(A) = \pi_2(A)T \quad \text{for all} \ A \in \mathcal{A} \quad \text{and} \quad V_2 = TV .
\]
Taking adjoint of the identity on the left in (5.30), $T^*\pi_2(A) = \pi(A)T^*$ for all $A \in \mathcal{A}$, and then

$$T^*T\pi(A) = T^*\pi_2(A)T = \pi(A)T^*T,$$

and this shows that $R := T^*T \in \pi(\mathcal{A})'$. Also, by the left side of (5.30) once more, $R\pi(A) = T^*\pi_2(A)T$, and hence for all $\zeta \in \mathcal{H}$,

$$\langle \zeta, V^*R\pi(A)V\zeta \rangle_\mathcal{H} = \langle \zeta, VT^*\pi_2(A)TV\zeta \rangle_\mathcal{H} = \langle \zeta, V^*_2\pi_2(A)V_2\zeta \rangle_\mathcal{H} = \langle \zeta, \Psi(A)\zeta \rangle_\mathcal{H}.$$

By polarization, $\Psi(A) = V^*R\pi(A)V$ for all $A \in \mathcal{A}$.

\[\Box\]

References


