Homework 3, Math 509 Spring 2018

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1: Let $M_n(\mathbb{C})$ be the C^* algebra of $n \times n$ matrices with the usual involution $*$ and the usual norm. A *density matrix* is a positive semi-definite matrix ρ such that $\mathrm{Tr}[\rho] = 1$. Show that $S(M_n(\mathbb{C}))$ consists of the $n \times n$ density matrices ρ, identifying ρ with the linear functional $A \mapsto \mathrm{Tr}[rhoA]$. Show that the set of extreme points of $S(M_n(\mathbb{C}))$ is the set of rank-one density matrices. .

4: Let $\mathscr{H}_n$ be the direct sum of n copies of $\mathbb{C}^n$. That is, a vector $\eta \in \mathscr{H}_n$ is an n-tuple $(\eta_!, \dots, \eta_n)$ of vectors in $\mathbb{C}^n$, and we define

$$\langle (\eta_!, \dots, \eta_n), (\zeta_!, \dots, \zeta_n) \rangle = \sum_{j=1}^{n} \langle \eta_j, \zeta_j \rangle .,$$

where the inner product on the right is the usual one in $\mathbb{C}^n$. Define a representation π of $M_n(\mathbb{C})$ on $\mathscr{H}_n$ by

$$\pi(A)(\eta_!, \dots, \eta_n) = (A\eta_!, \dots, A\eta_n) \ .$$

Show that $(A\eta_!, \dots, A\eta_n)$ is separating for this representation if and only if $\{\eta_1, \dots, \eta_n\}$ is linearly independent in $\mathbb{C}^n$. Show that $(A\eta_!, \dots, A\eta_n)$ is cyclic for this representation if and only if $\{\eta_1, \dots, \eta_n\}$ is linearly independent in $\mathbb{C}^n$.

3: We can identify $\mathscr{B}(\mathscr{H}_n)$ as the set of $n \times n$ matrices with entries in $M_n(\mathbb{C})$, as in the proof of the von Neumann Double Commutant Theorem: Given $T \in \mathscr{B}(\mathscr{H}_n)$ and $\xi \in \mathbb{C}^n$, for each $1 \le i, j \le n$, define $(\xi)_j$ to be the element of $\mathscr{H}_n$ whose jth entry is ξ, and all other entries are zero, and then define $T_{i,j}\xi$ to be the ith entry of $T(\xi)_j$. Then $T_{i,j}$ is the i,jth entry in the block-matrix representation of T.

Let $E_{i,j}$ denote the $n \times n$ matrix that has 1 in the i,j entry, and has 0 in all other entries. Then $\{E_{i,j}\}_{1 \le i,j \le n}$ is called the *matrix unit* basis of $M_n(\mathbb{C})$. Let

$$S = \sum_{i,j=1}^{n} S_{i,j} \otimes E_{i,j}$$

denote the block matrix whose i,j entry is $S_{i,j} \in M_n(\mathbb{C})$. What are $(\pi(A))'$ and $(\pi(A))''$?

Show that for every state φ on $M_n(\mathbb{C})$, there is a unit vector in $\xi_\varphi \in \mathscr{H}_n$ such that for all $A \in M_n(\mathbb{C})$,

$$\varphi(A) = \langle \xi_\varphi, \pi(A)\xi_\varphi \rangle \ .$$

That is, in this representation, every state is induced by a (non-unique) pure state on $\mathscr{B}(\mathscr{H}_n)$. Show also that if ρ is the density matrix for φ, we also have that for all $A \in M_n(\mathbb{C})$,

$$\varphi(A) = \mathrm{Tr}[\pi(SA)(\rho \otimes E_{1,1}]$$

where the traces is taken on $\mathscr{H}_n$, and that that state $T \mapsto \mathrm{Tr}[T(\rho \otimes E_{1,1}]$ on $\mathscr{B}(\mathscr{H}_n)$ is not pure if ρ is not pure. If φ is not pure, a vector state on $\mathscr{H}_n$ $\varphi(A) = \langle \xi_\varphi, \pi(A)\xi_\varphi \rangle$ is called a *purification* of φ. Purifications are not unique, but the following is significant: Let ρ be a density matrix on $\mathbb{C}^n$, and let $|\xi\rangle\langle\xi|$ be a purification of it on $\mathscr{H}_n$. Define a state on $(\pi(M_n(\mathbb{C}))'$ by $T \mapsto \langle \xi, T\xi \rangle$ and show (using your identification of $(\pi(M_n(\mathbb{C}))')$ that this is a state on $M_n(\mathbb{C})$ with a density matrix σ, and that ρ and σ have the same eigenvalues with the same multiplicities. (Hint: we can identify a vector $\xi \in \mathscr{H}_n$ with an operator on $\mathbb{C}_n$. How are the density matrices ρ and σ related to this operator?)

4: Let ρ be the density matrix of a faithful state on $M_n(\mathbb{C})$. Show that the GNS representation of $M_n(\mathbb{C})$ induced by φ is unitarily equivalent to the representation introduced above, and thus that all of the GNS representation induced by faithful state are unitarily equivalent. At the opposite extreme, show that if φ is a pure state on $M_n(\mathbb{C})$, then the representation produced by the GNS construction is unitarily equivalent to the identity representation.