Homework 1, Math 509 Fall 2018

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1: Let $\mathscr{A}$ be a Banach Algebra. As usual, define $A \in \mathscr{A}$ to be quasi regular in case there exists $B \in \mathscr{A}$ such that $A + B + AB = A + B + BA = 0$, and in this case, let A' denote the quasi inverse B, which is unique. Show that for all quasi regular $X, Y \in \mathscr{A}$ such that $(-X'Y')$ is quasi regular, $X \circ (-X'Y') \circ Y = X + Y$.

2: Let $\mathscr{A}$ be a C^*-Algebra, and let $-\lambda^{-1} = x + iy$ with $y \neq 0$. Show that for all self adjoint A that the quasi inverse of $-\lambda^{-1}A$, $(\lambda^{-1}A)'$, satisfies $\|((x+iy)A)'\| \leq (|y| + |x|)/|y|$, (Use the spectral calculus and fact that the quasi inverse $A' = f(A)$ where $f(t) - t/(1+t)$). Use this to show that if B is self adjoint and $\|B - A\|$ is sufficiently small, $\lambda 1 - B$ is invertible, and then give another proof of the fact that $\mathscr{A}_{\text{s.a.}}$ is closed.

3: Let $\mathscr{A}$ be a C^*-Algebra. Let $A \in \mathscr{A}$ satisfy $\|A\| \leq 1$. For $\alpha \in (0, 1/2)$, and for $n \in \mathbb{N}$, define $U_n := A(1/n + (A^*A))^{-1/2}(A^*A)^\alpha$. Show that $U := \lim_{n\to\infty} U_n$ exists, with convergence in norm, and that $\|U\| \leq 1$. Show that this gives us a decomposition $A = U|A|^{1-2\alpha}$ for $\alpha \in (0, 1/2)$, with $\|U\| < 1$. This "pseudo polar decomposition" will be useful in studying extreme points of the unit ball in $\mathscr{A}$.

Hint: Define $V_{m,n} = (1/n + (A^*A))^{-1/2} - (1/m + (A^*A))^{-1/2}$, and note that

$$U_n - U_m = AV_{m,n}(A^*A)^\alpha \ .$$

At this point we know that the involution is continuous, so we know that there is some constant c such that $\|U_n - U_m\|^2 \leq c\|(U_n - U_m)^*(U_n - U_m)\|$. Use this to bound $\|U_n - U_m\|^2$ by $\|V_{m,n}(A^*A)^\gamma\|^2$ for some positive γ, and now use spectral theory – everything is expressed in terms of the positive self adjoint operator A^*A. (Dini's Theorem may be useful for the uniform convergence you will need.)