Course description for Math 509,
Positive Maps on Operator Algebras and Quantum Markov Semigroups

Lectures by Eric Carlen

This course will provide an introduction to the theory of operator algebras, with a focus on topics concerning positive and completely positive maps and semigroups of completely positive maps. Such semigroups are known as *quantum Markov semigroups*, and they are the subject of a very active field of current research, with many new question arising in quantum information theory and quantum statistical mechanics.

Part One

The first part of the course will consist of a thorough introduction to the fundamentals of C^* algebra theory, which provides an abstract setting in which operator algebras can be investigated. A Banach algebra $\mathcal{A}$ is a Banach space equipped with a multiplication such that for all A, B in $\mathcal{A}$, $\|AB\| \le \|A\|\|B\|$. A Banach $*$-algebra is Banach algebra equipped with a conjugate-linear involution $* : \mathcal{A} \to \mathcal{A}$, $* : A \mapsto A^*$, such that $(AB)^* = B^*A^*$. An example is the algebra of bounded linear transformations on a Hilbert space, equipped with the operator norm, and where $*$ is the Hermitian adjoint. A C^* $\mathcal{A}$ algebra is a Banach $*$-algebra in which the norm, the product and the involution are related by *the C^* algebra identity*:

$$\|A^*A\| = \|A^*\|\|A\| \ . \tag{1}$$

The algebra is *unital* if it has a multiplicative identity. It is easy to verify that this is satisfied in the example of bounded operators on a Hilbert space. Therefore, any self-adjoint and norm closed subalgebra of the algebra of bounded operators on a Hilbert space is a C^* algebra.

The converse is also true: In a ground-breaking paper in 1943, Gelfand and Neumark proved that every "abstract" C^* algebra could be isometrically and $*$-isomorphically embedded in the algebra of bounded operators on some Hilbert space provided some additional assumptions held: *(i)* the algebra is unital *(ii)* The involution is an isometry; i.e.,$\|A^*\| = \|A\|$ for all A and *(iii)* for all A, $1 + A^*A$ is invertible. They conjectured, at least in the unital case, which is all that they discussed, that the hypotheses *(ii)* and *(iii)* were redundant. That is, they conjectured that *(ii)* and *(iii)* were true in any unital Banach $*$-algebra in which the C^* algebra identitiy is valid. It took 10 years for *(iii)* to be shown to be redundant in the unital case, 17 years for *(ii)* to be shown to be redundant in the unital case, and 24 years for it to be shown that existence of a multiplicative identity was not needed for any of this. The amazing and really beautiful part of the story is that the insights that made this achievement possible are very simple, clean and clear.

In the meantime, many people took the stronger hypothesis that

$$\|A^*A\| = \|A\|^2 \ , \tag{2}$$

discussed also by Gelfand and Neumark, as their starting point. This easily yields the isometry property $\|A^*\| = \|A\|$. Then, by way of the isometry property, (2) implies (1).

And now that it is known that (1) implies the isometry property, it is known that (1) implies (2). Thus, (1) and (2) are equivalent, and there is no loss of generality in defining a C^*-algebra to be a Banach $*$-algebra in which (2) is true for all A in the algebra. If one does so, one can beat a slightly shorter path from the initial definition to the Theorem of Gelfand and Neumark that provides an isometric $*$-isomorphism of any C^* algebra into the algebra of bounded operators on a Hilbert space than one can if one starts from (1). Here we start from (1), and this will prove useful later on, since the simple clean and clear insights that 24 years later finally provided the proof of the Gelfand and Neumark conjecture in full generality, are, as often happens with simple, incisive ideas, useful elsewhere.

In the first four chapters of the notes, we present the fundamentals of the theory of C^* algebras, including the equivalence of (1) and (2), In the third chapter, we turn to operator algebras *per se*, introducing three important non-norm topologies on the space of bounded linear transformations on a Hilbert space, the weakest of which is the *weak operator topology*. A *von Neumann algebra* is a C^* subalgebra of the bounded operators on a Hilbert space that is closed not only in the norm topology, but also the weak operator topology. This gives von Neumann algebras a particularly rich structure. In the fourth chapter, we turn to representations of C^* algebras and give the proof of the Gelfand-Neumark Theorem. We then go on to prove Sherman's Theorem, which which uses the Gelfand-Neumark Theorem to embed any C^* algebra into its *enveloping von Neumann algebra*. This provides a powerful technique for studying maps between C^* algebras, and we close the fourth Chapter with a proof of Tomiyama's Theorem which says that any norm one projection of a unital C^* algebra $\mathcal{A}$ onto a unital C^* subalgebra $\mathcal{B}$ behaves like a "conditional expectation"' in classical probability theory, and in particular, preserves positivity. This sets the stage for Part Two.

Part Two

Let A be a self-adjoint element in a unital C^* algebra. Then A is *positive* ($A \geq 0$) in case $t1 + A$ is invertible for all $t > 0$. It is true, but not obvious, that A is positive if and only if $A = B^*B$ for some B in the algebra. A linear transformation (map) Φ from one C^* algebra $\mathcal{A}$ to another $\mathcal{B}$ is positive in case $\Phi(A) \geq 0$ whenever $A \geq 0$.

In 1955, Stinespring introduced the class of *completely positive maps*, which has turned out to be fundamentally important in both mathematics and physics. The definition is as follows. Let M_n denote the C^* algebra of $n \times n$ complex matrices. (This is a C^* algebra with the usual operator norm.) Let $E_{i,j}$ be the $n \times n$ matrix with 1 in the i, j place, and 0 everywhere else. Then $\{E_{i,j}\}_{1 \leq i,j \leq n}$ is a basis for M_n regarded as a vector space, and for any C^* algebra $\mathcal{A}$, we can form the tensor product $\mathcal{A} \otimes M_n$, first as a tensor product of vector spaces. The general element of $\mathcal{A} \otimes M_n$ can be written as

$$\sum_{i,j=1}^{n} A_{i,j} \otimes E_{i,j} \ .$$

Then giving $\mathcal{A} \otimes M_n$ the obvious multiplication and involution, there is one natural norm on it making it a C^* algebra. A map $\Phi : \mathcal{A} \to \mathcal{B}$ is *completely positive* in case $\sum_{i,j=1}^{n} \Phi(A_{i,j}) \otimes E_{i,j} \geq 0$ in $\mathcal{A} \otimes M_n$ whenever $\sum_{i,j=1}^{n} A_{i,j} \otimes E_{i,j} \geq 0$ in $\mathcal{A} \otimes M_n$. Stinespring's Theorem give a necessary and sufficient condition for Φ to be completely positive, and the nature of this condition explains its physical significance in quantum mechanics.

We prove Stinespring's Theorem, and then turn to a number of other results on both positive and completely positive maps the build on it. A number of these provide inequalities, such as the *Operator Jensen Inequality*, that are useful in quantum mechanics and elsewhere. We shall be especially interested in completely positive maps on unital C^* algebras $\mathcal{A}$ that preserve the identity. The are the non-commutative analogs of Markov operators from classical probability theory. In this context we discuss *conditional expectations in operator algebras*, a topic that has been developed as pure mathematics, but which also has applications in quantum mechanics. Already in this part of the course, the lectures will present some open problems, some of which surely have simple solutions.

Part Three

In the third part of the course, we turn to quantum Markov semigroups, which are semigroups $\{\Phi_t\}_{t\geq 0}$ of completely positive maps that also preserve the identitiy. A fundamental result for this part of the course comes from a 1975 paper of Lindblad who gave necessary and sufficient conditions on an operator L on certain types of C^* algebras (including the case of all bounded operators on a Hilbert space) for L to be the generator of a semigroup $\{\Phi_t\}_{t\geq 0}$ such that $t \mapsto \Phi_t$ is continuous in operator norm (for operators on the algebra). One would like to have such a result in greater generality; Lindblad remarks on this in his paper, and many examples arising naturally in physics lie outside the scope of his theorem. There is much more remains to be done here, despite the many more recent accomplisments.

This is an active field of current research, with fresh questions coming from quantum information theory. I will present some results from my work with Jan Maas and also by other people currently working in the field. A number of open research problems will be pointed out. Exactly what get covered in this part of the course will depend in part of the particular interests of the students, but in any event, students who follow the course will come away with solid grounding in methods and problems of an actively developing field of research.