Notes on Topology for Real Analysis

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1 Introduction

The basic strategy in real analysis is *approximation.* In particular, one often tries to approximate general elements of some infinite dimensional vector space of functions by elements of a subspace consisting of well-behaved functions, or one tries to construct solutions to equations as limits of solutions to "approximate" equations. The basic framework for making mathematical sense of "approximation" is provided by the theory of *metric spaces*, and more generally the theory of *topological spaces.* Methods of approximation are especially effective in a metric space that is *complete* and rich in *compact sets*, as we explain in this introductory chapter.

In the next chapter, we study the Lebesgue theory of integration. A fundamental advantage of it over previous integration theories is that it permits the construction of many complete metric spaces in which compact sets can be concretely described. This provides an extremely useful framework for solving a wide variety of equations. First, we introduce the fundamental topological theory, and illustrate it with examples that do not require the Lebesgue theory of integration.

2 Metric Spaces

2.1 DEFINITION (Metric Space)**.** *A* metric space (X, d) *consists of a set* X *and a function* $d : X \times X \to [0, \infty)$ *satisfying:*

(1) $d(x, y) \geq 0$ *for all* $x, y \in X$ *and* $d(x, y) = 0 \iff x = y$.

(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

(3) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

The inequality in (3) is called the triangle inequality, and a function d *satisfying (1), (2) and (3) is called a* metric *on* X.

2.1 EXAMPLE. *For real numbers* $a < b$*, let* $\mathcal{C}([a, b], \mathbb{R})$ *denote the set of of continuous real-valued functions on* $[a, b]$*. Given any two* $f, g \in \mathcal{C}([a, b], \mathbb{R})$*. Define*

$$d_\infty(f, g) = \sup_{t \in [a,b]} |f(t) - g(t)| \ .$$

It is easy to check that d_∞ *is a metric on* $\mathcal{C}([a, b], \mathbb{R})$*, called the* uniform metric*. Since* $\mathcal{C}([a, b], \mathbb{R})$ *has an obvious vector space structure, it is our first example of a metric space that is also a vector space of functions.*

2.1 Continuity in metric spaces

A function f from X to Y is continuous if a sufficiently small change in the input results in a small change in the output. In other words, $f(x)$ will be a close approximation of $f(x_0)$, if x is a sufficiently close approximation of x_0. Here is the precise version of this in the metric space setting:

2.2 DEFINITION (Continuous functions from one metric space to another)**.** *Let* (X, d_X) *and* (Y, d_Y) *be two metric spaces. Let* f *be a function from* X *to* Y*. Then* f *is continuous at* $x_0 \in X$ *in case for every* $\epsilon > 0$*, there is a* $\delta_\epsilon > 0$ *such that*

$$d_X(x, x_0) < \delta_\epsilon \quad \Rightarrow \quad d_Y(f(x), f(x_0)) < \epsilon \ . \tag{2.1}$$

The function f *is continuous in case it is continuous at each* $x_0 \in X$.

2.1 THEOREM (Continuity and sequences)**.** *Let* (X, d_X) *and* (Y, d_Y) *be two metric spaces. Let* f *be a function from* X *to* Y*. Then* f *is continuous at* $x_0 \in X$ *if and only if for every sequence* $\{x_k\}$ *in* X

$$\lim_{k \to \infty} x_k = x_0 \quad \Rightarrow \quad \lim_{k \to \infty} f(x_k) = f(x_0) \ . \tag{2.2}$$

Proof. Suppose that f is continuous, and that $\lim_{k\to\infty} x_k = x_0$. Pick any $\epsilon > 0$, and let δ_ϵ be as in (2.1). Choose N so large that for all $k > N$, $d(x_k, x_0) < \delta_\epsilon$. Then, for all $k > N$, $d(f(x), f(x_0)) < \epsilon$. Since ϵ is arbitrary, this shows that $\lim_{k\to\infty} f(x_k) = f(x_0)$.

Next suppose that f is not continuous at x_0. Then there exists some $\epsilon > 0$ so that for every $\delta > 0$, there is at least one point x satisfying $d_X(x, x_0) < \delta$ such that $d_Y(f(x), f(x_0)) > \epsilon$. Define a sequence $\{x_k\}$ as follows: For each k, choose x_k so that $d_X(x_k, x_0) < 1/k$ such that $d_Y(f(x_k), f(x_0)) > \epsilon$. Then $\lim_{k\to\infty} d_X(x_k, x_0) = 0$, but it is not the case that $\lim_{k\to\infty} f(x_k) = f(x_0)$. Thus, when f is not continuous at x_0, (2.2) does not hold true. Hence, whenever (2.2) does hold true, it must be the case that f is continuous at x_0. □

There is another characterization of continuity involving the notion of *open sets*, which we now define:

2.3 DEFINITION (Open sets in metric spaces)**.** *Let X be a metric space with metric d. Given a number $r > 0$, and a point $x \in X$, let $B_r(x)$ be defined by*

$$B_r(x) = \{\ y \in X \ :\ d(y, x) < r\ \}\ .$$

This set is called the open ball of radius r about x.

A subset U of X is open in case either:

(1) It is the empty set $\emptyset$, or else

(2) For each $x \in U$, there is an $r > 0$, depending on x, such that

$$B_r(x) \subset U\ .$$

It is possible, and as we shall see, useful to characterize the continuity of functions between two metric spaces simply in terms of open sets, without explicit reference to the the specific metrics themselves.

2.2 THEOREM (Continuity and open sets)**.** *Let X and Y be metric spaces with metrics d_X and d_Y respectively. Let f be a function from X to Y. Then f is continuous if and only if for every open set U in Y, $f^{-1}(U)$ is open in X.*

Proof. Suppose that f is continuous, and let U be an open set in Y. If $f^{-1}(U) = \emptyset$, then $f^{-1}(U)$ is open by *(1)*. Otherwise, if $f^{-1}(U) \neq \emptyset$, consider any $x \in f^{-1}(U)$. Then $f(x) \in U$, and since U is open, there exists a $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset U$. Then, since f is continuous at x_0, there is a $\delta > 0$ so that

$$d_X(\tilde{x}, x) < \delta \Rightarrow d_Y(f(\tilde{x}), f(x)) < \epsilon\ .$$

Hence

$$f(B_\delta(x)) \subset B_\epsilon(f(x)) \subset U\ .$$

But this means that

$$B_\delta(x) \subset f^{-1}(U)\ .$$

Since x was any point in $f^{-1}(U)$, we have shown that $f^{-1}(U)$ contains an open ball about each of its members, and hence is open.

Conversely, suppose that f has the property that whenever U is open in Y, $f^{-1}(U)$ is open in X. Fix any $x \in X$ and any $\epsilon > 0$. $f^{-1}(B_\epsilon(f(x)))$ is open and contains x. Therefore, there is some $\delta_\epsilon > 0$ such that

$$B_{\delta_\epsilon}(x) \subset f^{-1}(B_\epsilon(f(x))) \ .$$

But then

$$f(B_{\delta_\epsilon}(x)) \subset B_\epsilon(f(x)) \ ,$$

which is just another way to write (2.1). Since ϵ is arbitrary, f is continuous at x. Since x is arbitrary, f is continuous. □

2.2 Complete metric spaces

The metric space $(\mathcal{C}([a,b],\mathbb{R}), d_\infty)$ has another feature that is desirable for analysis: It is *complete.*

2.4 DEFINITION (Complete metric space)**.** *A metric space (X,d) is complete in case whenever $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in X, there exists an $x \in X$ such that*

$$\lim_{k\to\infty} d(x_k, x) = 0 \ .$$

2.3 THEOREM. *$(\mathcal{C}([a,b],\mathbb{R}), d_\infty)$ is complete.*

Proof. Suppose that $\{f_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{C}([a,b],\mathbb{R}), d_\infty)$. Then for any $\epsilon > 0$, there is an N_ϵ so that

$$k,\ell \geq N_\epsilon \Rightarrow d_\infty(f_k, f_\ell) \leq \epsilon \ .$$

Fix any $t \in [a,b]$, Since $|f_k(t) - f_\ell(t)| \leq d_\infty(f_k, f_\ell)$,

$$k,\ell \geq N_\epsilon \to |f_k(t) - f_\ell(t)| \leq \epsilon \ , \tag{2.3}$$

and hence $\{f_k(t)\}$ is a Cauchy sequence in $\mathbb{R}$. By the completeness of the real numbers, it has a limit which we denote by $f(t)$. This defines a real valued function f on $[a,b]$, and it remains to show that $f \in \mathcal{C}([a,b],\mathbb{R})$, and that $\lim_{k\to\infty} d_\infty(f_k, f) = 0$.

By the definition of $f(t)$, $|f_k(t) - f(t)| = \lim_{\ell\to\infty} |f_k(t) - f_\ell(t)|$. For any $s,t \in [0,1]$,

$$|f(t) - f(s)| \leq |f(t) - f_k(t)| + |f_k(t) - f_k(s)| + |f_k(s) - f(s)| \ ,$$

and so

$$k \geq N_\epsilon \to |f_k(t) - f(t)| \leq \epsilon \ . \tag{2.4}$$

For $k = N_\epsilon$, this becomes

$$|f(t) - f(s)| \leq |f_{N_\epsilon}(t) - f_{N_\epsilon}(s)| + 2\epsilon \ .$$

Since f_{N_ϵ} is continuous, there is a $\delta > 0$ so that $|s - t| \leq \delta \Rightarrow |f_{N_\epsilon}(s) - f_{N_\epsilon}(t)| \leq \epsilon$. Therefore,

$$|s - t| \leq \delta \Rightarrow |f(s) - f(t)| \leq 3\epsilon \ .$$

Since $\epsilon > 0$ is arbitrary, this proves that $f \in \mathcal{C}([a,b],\mathbb{R})$. Finally, since (2.4) is valid uniformly in t,

$$\lim_{k\to\infty} d_k(f_k, f) = 0 \ .$$

□

The next theorem provides an example of the importance of completeness.

2.4 THEOREM (Banach Contraction Mapping Theorem)**.** *Let (X, d) be a complete metric space. Let $\Phi : X \to X$ have the property that for some $\lambda < 1$, $d(\Phi(x), \Phi(y)) \leq \lambda d(x, y)$ for all $x, y \in X$. Then there is a unique $x_0 \in X$ such that*

$$x_0 = \Phi(x_0) \ . \tag{2.5}$$

Moreover, for all $x \in X$, let the sequence $\{x_k\}_{k\in\mathbb{N}}$ be defined by $x_1 = x$ and for $x_{k+1} = \Phi(x_k)$. Then

$$\lim_{k\to\infty} d(x_0, x_k) = 0 \ . \tag{2.6}$$

Proof. Pick any $x \in X$, and construct the sequence $\{x_k\}_{k\in\mathbb{N}}$ as described in the theorem. Define $R := d(x_2, x_1) = d(\Phi(x), x))$. By the hypothesis

$$d(x_3, x_2) = d(\Phi(x_2), \Phi(x_1)) \leq \lambda d(x_2, x_1) = \lambda R \ .$$

Then by a simple induction,

$$d(x_{k+2}, x_{k+1}) \leq \lambda^k R$$

for all $k \in \mathbb{N}$. By the triangle inequality, for all $\ell > k$,

$$d(x_\ell, x_k) \leq \sum_{j=0}^{\ell-k-1} d(x_{k+j+1}, x_{k+j} \leq \sum_{j=0}^{\infty} \lambda^{k+j-1} R = \frac{\lambda^{k-1}}{1-\lambda} R \ .$$

Since $\lim_{k\to\infty} \lambda^k = 0$, this proves that $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is complete, this sequence has a limit x_0. Clearly

$$\Phi(x_0) = \lim_{k\to\infty} \Phi(x_k) = \lim_{k\to\infty} x_{k+1} = x_0$$

so that x_0 is a fixed point of Φ. Finally if y_0 is any fixed point of Φ, then

$$d(x_0, y_0) = d(\Phi(x_0), \Phi(y_0)) \leq \lambda d(x_0, y_0) \ .$$

The only $t \in [0, \infty)$ satisfying $t \leq \lambda t$ is $t = 0$. By property *(1)* in the definition of a metric, this implies that $y_0 = x_0$ which proves the uniqueness. □

2.2 EXAMPLE. *The previous theorem is the basis of the basic existence and uniqueness theorem for ordinary differential equations; we now sketch the main points in the simplest case. Let $v(x, t)$ be a continuous function of $\mathbb{R} \times \mathbb{R}$ such that for some $L < \infty$,*

$$|v(x, t) - v(y, t)| \leq L|x - y|$$

for all x, y, t. For $x(\cdot) \in \mathcal{C}([0, (2L)^{-1}], \mathbb{R})$ and $x_0 \in \mathbb{R}$ define

$$\Phi(x(\cdot))(t) = x_0 + \int_0^t v(x(s), s)\mathrm{d}s \ .$$

Note that $x(\cdot)$ is a fixed point of Φ is and only if for all t,

$$x(t) = x_0 + \int_0^t v(x(s), s)\mathrm{d}s \ .$$

Suppose that such a fixed point exists. Then, since the right hand side is continuously differentiable, $x(t)$ is continuously differentiable, and differentiating both sides,

$$x'(t) = v(x(t), t) \tag{2.7}$$

for all t, and moreover, $x(0) = x_0$. Conversely, any solution of (2.7) with $x(0) = x_0$ is a fixed point of Φ. Hence, proving existence and uniqueness of fixed points of Φ is tantamount to proving the existence and uniqueness of solutions of the ordinary differential equation (2.7), at least on this time interval. To apply the Banach Contraction Mapping Theorem, consider the complete metric space $\mathcal{C}([0,(2L)^{-1}],\mathbb{R})$ equipped with the d_∞ metric.

Let $x(\cdot), y(\cdot) \in \mathcal{C}([0,(2L)^{-1}],\mathbb{R})$. Then for all $0 \le t \le (2L)^{-1}$,

$$\begin{aligned}
|\Phi(x(\cdot))(t) - \Phi(y(\cdot))(t)| &= \left|\int_0^t [v(x(t),t) - v(y(t),t)]\mathrm{d}t\right| \\
&\le \int_0^t |v(x(t),t) - v(y(t),t)]|\mathrm{d}t \\
&\le L\int_0^t |x(t) - y(t)|\mathrm{d}t \\
&\le Lt \sup_{0\le s\le t} |x(t) - y(t)| \le \frac{1}{2} d_\infty(x(\cdot), y(\cdot))
\end{aligned}$$

Thus,

$$d_\infty(\Phi(x(\cdot)), \Phi(y(\cdot))) \le \frac{1}{2} d_\infty(x(\cdot), y(\cdot)) \ ,$$

and the Banach Contraction Mapping Theorem yields the existence of a unique fixed point of Φ. This yields the existence and uniqueness of a solution to (2.7) on the interval $[0,(2L)^{-1}]$. The basic idea explained here can be used to prove much more.

We have just explained how *completeness* can be applied to solve equations. We shall do the same for compactness after developing more of the theory.

2.3 Compactness in metric spaces

Alongside completeness, the other fundamental concept pertaining to approximation in analysis is that of compactness:

2.5 DEFINITION (Sequentially compact subset of metric space)**.** *Let (X,d) be a metric space, and $A \subset X$. Then A is* sequentially compact *in case every sequence $\{x_k\}_{k\in\mathbb{N}}$ in A contains a subsequence converging to an element of A. A metric space (X,d) is* sequentially compact *in case X itself is sequentially compact.*

Note that if (X,d) is a metric space, and $A \subset X$, and d_A denotes the restriction of d to $A \times A$, the (A, d_A) is itself a metric space, and A is sequentially compact subset of X if an only if (A, d_A) is a compact metric space. Thus, characterizing compact subsets of a metric space reduces to a question about whether or not a metric spaces is compact.

Later in this chapter we shall prove the *Arzela-Ascoli Theorem* which characterizes compact sets in $(\mathcal{C}([a,b]), d_\infty)$, and we shall give an example of the application of this to solving equations. There is an equivalent formulation of sequential compactness in a metric space that will be useful in proving the Arzela-Ascoli Theorem and other theorems characterizing compact sets.

2.6 DEFINITION (Totally bounded)**.** *A metric space (X, d) is totally bounded if and only if for every $\epsilon > 0$, there is a finite set $\mathcal{U}_\epsilon$ consisting of finitely many open balls of radius ϵ; i.e.,*

$$\mathcal{U}_\epsilon = \{B_\epsilon(x_1), \ldots, B_\epsilon(x_{n_\epsilon})\}\ ,$$

that covers X; i.e, $X = \bigcup_{j=1}^{n_\epsilon} B_\epsilon(x_j)$.

2.5 THEOREM (Sequential compactness and total boundedness)**.** *A metric space (X, d) is sequentially compact if and only if it is complete and totally bounded.*

Proof. Suppose that (X, d) is not totally bounded. We shall show that then it is not sequentially compact. To do this, we construct a sequence that has no convergent subsequence. By hypothesis, for some $\epsilon > 0$, there does not exist any finite cover of X by open balls of radius ϵ. Thus given any set $\{x_1, \ldots, x_n\}$ of X, $\bigcup_{j=1}^{n_\epsilon} B_\epsilon(x_j) \neq X$, and so we can select x_{n+1} so that

$$d(x_{n+1}, x_j) \geq \epsilon\ , \quad \text{for all } j = 1, \ldots, n\ .$$

Thus, starting from an arbitrary choice of x_1, using a simple induction we can construct an infinite sequence $\{x_k\}_{k\in\mathbb{N}}$ such that

$$d(x_k, x_\ell) \geq \epsilon$$

for all $k \neq \ell$. Clearly, such a sequence has no convergent subsequence.

Next, suppose that (X, d) is totally bounded and complete. Let $\{x_k\}_{k\in\mathbb{N}}$ be any infinite sequence in X. For each $m \in \mathbb{N}$, there exists a set $\{B_{1/m}(x_1), \ldots B_{1/m}(x_{n_m})\}$ of open balls of radius $1/m$ such that

$$X = \bigcup_{j=1}^{n_{1/m}} B_\epsilon(x_j)\ .$$

By the pigeon-hole principle, at least one of these balls contains infinitely many elements of any infinite sequence in X.

By what we have explained above, there exists an infinite subsequence $\{x_k^{(1)}\}_{k\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$ such that all elements of this subsequence lie in some open ball of radius 1. Next, for the same reason, exists an infinite subsequence $\{x_k^{(2)}\}_{k\in\mathbb{N}}$ of $\{x_k^{(1)}\}_{k\in\mathbb{N}}$ such that all elements of this subsequence lie in some open ball of radius $1/2$. Continuing the obvious induction, we obtain a sequence $\{x_k^{(j)}\}_{k\in\mathbb{N}}$ of sequences such that each $\{x_k^{(j+1)}\}_{k\in\mathbb{N}}$ is a subsequence of $\{x_k^{(j)}\}_{k\in\mathbb{N}}$, and all elements in $\{x_k^{(j)}\}_{k\in\mathbb{N}}$ lie in some open ball of radius $1/j$.

Now we use Cantor's "diagonal sequence" construction: define $y_j = x_j^{(j)}$. Then $\{y_j\}_{j\in\mathbb{N}}$ is a subsequence of $\{x_k\}_{k\in\mathbb{N}}$, and for all m,

$$j, k \geq m \Rightarrow d(y_j, y_k) \leq 1/m\ .$$

Thus, $\{y_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is complete, it is also convergent. We have thus shown the existence of a convergent subsequence of an arbitrary sequence of X. $\square$

Many important theorems can be proved by using the fact that real valued continuous functions on a sequentially compact metric space have maxima and minima. That is, if f is a real valued continuous function on a sequentially compact metric space, the there exist x_0 and x_1 such that

$$f(x_0) \leq f(x) \leq f(x_1)$$

for all $x \in X$.

While most often in the chapters that follow, we shall be working with the notions of continuity and compactness in a metric space setting, this is not always possible, and it is not always convenient even when it is possible. It is advantageous to develop these notions in a more general setting, that of *topological spaces*. We now introduce this more general setting.

3 Topological spaces

Since by Theorem 2.2 we can characterize continuous functions from one metric space to another in terms of open sets, without explicitly mentioning either metric at all, it is sometimes useful to "strip away" the metric structure, and only refer to the open sets.

3.1 DEFINITION (Topological Spaces, Hausdorff Topological Spaces)**.** *Let X be any set, and let $\mathcal{O}$ be any collection of sets in X satisfying:*

(1) The empty set $\emptyset$ belongs to $\mathcal{O}$, as does X itself.

(2) The union of any arbitrary set of sets in $\mathcal{O}$ belongs to $\mathcal{O}$.

(3) The intersection of any finite set of sets in $\mathcal{O}$ belongs to $\mathcal{O}$.

In this case, $\mathcal{O}$ is said to be a topology on X, and the sets belonging to $\mathcal{O}$ are called open *sets in X (for the topology in question). A subset A of X is closed in case its complement, A^c is open.*

The pair $(X, \mathcal{O})$ is said to be a topological space. It is a Hausdorff *if whenever x, y are any two distinct elements in X, there exists disjoint open sets sets U and V with $x \in U$ and $y \in V$.*

Note that by De Morgan's laws, the intersection of any *arbitrary* set of closed sets in X is itself closed.

3.1 EXAMPLE. *It is left as an easy exercise to show that if X is any metric space, and $\mathcal{O}$ is the collection of all open sets in X, as defined above in terms of open balls, $\mathcal{O}$ does indeed constitute a topology on X.*

If (X, d) is any metric space and x, y are distinct in X, then $r := d(x, y) > 0$. If $w \in B_{r/3}(x)$ and $z \in B_{r/3}(y)$, then

$$r = d(x, y) \leq d(x, w) + d(w, z) + d(z, y) < d(w, z) + 2r/3 \ ,$$

so that $d(w, z) > r/3$, In particular, $B_{r/3}(x) \cap B_{r/3}(y) = \emptyset$. Since $B_{r/3}(x)$ and $B_{r/3}(y)$ are open sets containing x and y respectively, this proves that every metric space is Hausdorff.

Not every topology is Hausdorff. If X is an set, the trivial topology *on X is given by $\mathcal{O} = \{\emptyset, X\}$. This is evidently a topology, and when X contains more than a single element, it cannot be Hausdorff.*

3.2 EXAMPLE (Relative topology on a subset)**.** *Let $(X, \mathcal{O})$ be a topological space. Let A be any subset of X. The* relative topology *on A induced by $\mathcal{O}$ is denoted by $\mathcal{O}_A$ and is given by*

$$\mathcal{O}_A = \{ \ U \cap A \ : \ U \in \mathcal{O} \ \} \ .$$

It is readily checked that this is a topology.

3.2 DEFINITION (Metrizable Topology). *Let $(X, \mathcal{O})$ be a topological space. The topology $\mathcal{O}$ is* metrizable *if there exists some metric d on $X \times X$ such that the set of open sets for this metric is exactly $\mathcal{O}$.*

The remarks in Example 3.1 show that non-Hausdorff topologies are never metrizable. We shall be (almost) exclusively concerned with Hausdorff topologies, but as we shall see, there are useful Hausdorff topologies that are not metrizable.

The next definitions introduces some more useful terminology

3.3 DEFINITION (Interior, closure and neighborhoods). *Let $(X, \mathcal{O})$ be a topological space, and A a subset of X.The* interior *of A, A^o, is the union of all of the open sets contained in A. The* closure of *A, $\overline{A}$, is the intersection of all of the closed sets containing A. Finally for any $x \in X$, the set $\mathcal{N}_x$ of* neighborhoods of *x consists of all sets B such that $x \in B^o$.*

3.1 Continuity in topological spaces

3.4 DEFINITION (Continuous functions between topological spaces). *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces. A function f from X to Y is* continuous at *$x \in X$ in case for every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*

A function f from X to Y is continuous *whenever U is open in Y, $f^{-1}(U)$ is open in X.*

It is easy to see that $f : X \to Y$ is continuous if and only if it is continuous at each $x \in X$.

3.1 Remark. By Example 3.1 and by Theorem 2.2, the definition of continuity that we make next is consistent with our existing notion of continuity in the metric space setting.

We now turn to the notion of *approximation* in topological spaces. The following definition is the starting point.

3.5 DEFINITION (Limit points in a topological space). *Let $(X, \mathcal{O}_X)$ be a topological space. If A is any set in X, a point $x \in X$ is a limit point of X in case every for every open set U that contains x,*

$$A \cap (U \backslash \{x\}) \neq \emptyset \ .$$

That is, every open set U that contains x also contains some point in A *other than* x. Note that x itself may or may not be in A.

Note that if $(X, \mathcal{O}_X)$ is Hausdorff, and x is a limit point of $A \subset X$, with $x \notin A$, then not only is $A \cap U$ non-empty for every neighborhood U of x: *$A \cap U$ must contain infinitely many points.* To see this, suppose $y \in A \cap U$. Let V_x and V_y be disjoint open sets containing x and y respectively. Then $V_x \cap U$ is another neighborhood of x, contained in U, so $A \cap (V_x \cap U) \neq \emptyset$. Since $A \cap (V_x \cap U) \subset A \cap U$, and is missing at least y. Repeating this procedure, it is clear that we can repeatedly remove elements from $A \cap U$ without ever emptying it, and so it must contain infinitely many points.

3.6 DEFINITION (convergent sequence). *Let $(X, \mathcal{O})$ be a topological space. Let $\{x_k\}$ be a sequence of elements of X. Then $\{x_k\}_{k \in \mathbb{N}}$ is convergent to x in case every open set U containing x also contains all but finitely many terms in the sequence $\{x_k\}_{k \in \mathbb{N}}$.*

Note the differences between the notions of limit point the notion of the limit of a sequence. One difference is that a sequence $\{x_k\}_{k\in\mathbb{N}}$ is a *function* from $\mathbb{N}$ to X, though it is common practice to identify the sequence with its range, which is a subset of X. Apart from this, there is the essential difference between "infinitely many" and "all but finitely many".

In particular, in a Hausdorff space, a sequence can have at most one limit, but (identified with its range, and considered as a set), but it may have more than one limit point, since x is a limit point of $\{x_n\}_{n\in\mathbb{N}}$ if and only if every open set U containing x also infinitely many terms in the sequence $\{x_k\}$, while $\lim_{n\to\infty} x_n = x$ if and only if every neighborhood U of x contains all but finitely many terms in the sequence.

In a metric space, there is of course a characterization of limit points in terms of sequences; x is a limit point of A if and only if there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements in A such that $\lim_{n\to\infty} x_{k_n} = x$.

3.3 EXAMPLE (The right order topology on $\mathbb{R}$)**.** *Let $\mathcal{O}_r$ be the set of all subsets of $\mathbb{R}$ of the form (a,∞), $a\in\mathbb{R}$, together with $\emptyset$ and $\mathbb{R}$. It is readily checked that this is a topology, called the* right order topology*. Conisder the function from $\mathbb{R}$ to $\mathbb{R}$ defined by*

$$f(x) := \begin{cases} 1 & x > 0 \\ 0 & x \le 0\ . \end{cases}$$

Then

$$f^{-1}((a,\infty)) = \begin{cases} \emptyset & a \ge 1 \\ (0,\infty) & 0 \le a < 1 \\ \mathbb{R} & a < 0\ . \end{cases}$$

Therefore, if we equip the range with the right order topology and the domain with the usual topology, f is conitnuous, though it is not continuous if both the domain and range are equipped with the usual topology.

Let $(X.\mathcal{O})$ be a topological space. A functions that is continuous from $(X.\mathcal{O})$ to $(\mathbb{R},\mathcal{O}_r)$ is called lower semicontinuous*. The left order topology and upper semicontinuous functions are defined in the analogous way using the sets $(-\infty,b)$.*

We are now ready for the theorem that justifies the terminology "closed":

3.2 THEOREM (Closed sets and limit points)**.** *Let $(X,\mathcal{O})$ be any topological space. A subset A of X is closed if and only if A contains all of its limit points.*

Proof. Suppose that A is closed, and $x\in A^c$. Since A^c is open, there is an open set U containing x that has an empty intersection with A. Thus, x is not a limit point of A. Since x was an arbitrary point outside A, A must contain all of its limit points.

On the other hand, suppose that A contains all of its limit points. We must show that A is closed, or, what is the same thing, that A^c is open. Consider any point $x\in A^c$. Since it is not a limit point of x, there is an open set U_x containing x that has empty intersection with A. For each $x\in A^c$, chose such a U_x. But then, since U_x contains x,

$$A^c \subset \bigcup_{x\in A^c} U_x$$

On the other hand, since each $U_x \subset A^c$,

$$\bigcup_{x\in A^c} U_x \subset A^c$$

Thus, $A = \displaystyle\bigcup_{x\in A^c} U_x$, and by *(2)* in the definition of topological spaces, $\displaystyle\bigcup_{x\in A^c} U_x$ is open. □

We close this subsection with one more definition:

3.7 DEFINITION (Density)**.** *Let $(X, \mathcal{O})$ be a topological space. Let $A \subset B \subset X$. Then A* is dense in *B in case the closure of A contains B.*

By Theorem 3.2, A is dense in B if and only if every point in B is a limit point in A; i.e, if every point in B can be approximated arbitrarily well by points in A.

3.2 Compactness in topological spaces

3.8 DEFINITION (Compact Sets)**.** *Let $(X, \mathcal{O}_X)$ be a topological space. A subset K is called compact in X in case for* open cover *$\mathcal{U}$ of K; that is, for every collection $\mathcal{U}$ of open sets such that*

$$K \subset \bigcup_{u\in\mathcal{U}} U \ ,$$

there is a finite subset $\mathcal{G}$ of $\mathcal{U}$ that also covers K:

$$K \subset \bigcup_{u\in\mathcal{G}} U \ .$$

$\mathcal{G}$ is called a finite subcover of A. The topological space $(X, \mathcal{O})$ is called compact in case X itself is compact.

3.3 THEOREM. *Let $(X, \mathcal{O})$ be any compact topological space. Then every closed subset K of X is compact. Conversely, if X is Hausdorff, every compact subset of X is closed.*

Proof. Let $\mathcal{U}$ be any open cover of K. Then $\mathcal{U} \cup \{K^c\}$ is an open cover of X. Thus, there exists a finite set of the sets in sets in $\mathcal{U} \cup \{K^c\}$ that covers all of X, and K^c is not needed to cover K. Hence K is compact.

Conversely, suppose K is compact and $(X, \mathcal{O})$ is Hausdorff. Suppose that $y \notin K$, but that y is a limit point of K. For each $x \in K$, there exist open sets U_x and V_x such that $U_x \cap V_x = \emptyset$, $x \in U_x$ and $y \in V_x$. Then $\{U_x \ : \ x \in K\}$ is an open cover of K, so there exists a finite subcover $\{U_{x_1}, \dots, U_{x_n}\}$ that covers K. However,

$$y \in V := \bigcap_{j=1}^{n} V_{x_j}$$

and V is open. Since y is a limit point of K, $K \cap V \neq \emptyset$. But this is impossible since $V \cap U_{x_j} = \emptyset$ for each j, and $K \subset \cup_{j=1}^n U_{x_j}$. □

Our first example of a convergence theorem involving compactness is the classical result known as *Dini's Theorem.* In proving this, we shall make use of the following fact: If $\mathcal{U}$ is any set of open subsets of X, then by De Morgan's laws,

$$\left(\bigcup_{U\in\mathcal{U}} U\right)^c = \bigcap_{U\in\mathcal{U}} U^c \ ,$$

Thus $\mathcal{U}$ is an open cover if and only if $\{\ U^c \ :\ U\in\mathcal{U}\ \}$ is a set of closed subsets of X with empty intersection.

Therefore, X is compact if and only if whenever $\mathcal{K}$ is a set of closed subsets of X such that

$$\bigcap_{K\in\mathcal{K}} K = \emptyset \ ,$$

there is a finite subset $\{K_1,\dots,K_n\}\subset\mathcal{K}$ such that

$$\bigcap_{j=1}^{n} K_j = \emptyset \ .$$

This analysis is often summarized by saying that X *is compact if and only if* X *has the "finite intersection property".*

3.4 THEOREM (Dini's Theorem)**.** *Let* $(X,\mathcal{O})$ *be a compact topological space, and let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of real valued continuous functions on* X*, and suppose that there is a continuous real valued function* f *on* X *such that for each* $x\in X$*, the sequence* $\{f_n(x)\}_{n\in\mathbb{N}}$ *is monotone non-decreasing, and*

$$\lim_{n\to\infty} f_n(x) = f(x) \ .$$

Then

$$\lim_{n\to\infty} f_n = f$$

uniformly.

In other words, pointwise convergence, together with compactness and monotonicity, imply uniform convergence. Also note that replacing each f_n by $-f_n$, one converts a monotone non-decreasing sequence into a monotone non-increasing sequence, and so the theorem remains true if one replaces "monotone non-decreasing" by "monotone non-increasing".

Proof. Fix $\epsilon>0$. Define the sets K_ℓ, $\ell\in\mathbb{N}$, by

$$K_\ell := \{\ x\in X \ :\ f(x)-f_\ell(x)\geq\epsilon\ \} \ .$$

Since f and f_ℓ are continuous, K_ℓ is closed. Since $\{f_n(x)\}_{n\in\mathbb{N}}$ is monotone non-decreasing,

$$\ell\geq k \Rightarrow K_\ell\subset K_k \ . \tag{3.1}$$

Also, since for each x, $\lim_{\ell\to\infty} f_\ell(x) = f(x)$, $\displaystyle\bigcap_{k=1}^{\infty} K_\ell = \emptyset$ Then, by the compactness of X, there is some $n\in\mathbb{N}$ such that

$$\bigcap_{\ell=1}^{n} K_\ell = \emptyset. \tag{3.2}$$

Combining (3.1) and (3.2), we see that $K_\ell = \emptyset$ for all $\ell \geq n$. Hence, for all $\ell > n$, and all x, $|f_\ell(x) - f(x)| < \epsilon$. which proves the uniform convergence. $\square$

Next, we turn to one of the main theorems on compactness.

3.5 THEOREM (Compactness, Continuity, and Minima)**.** *Let $(X, \mathcal{O})$ be any topological space, and let K be a compact subset of X. Let f be a function from X to $\mathbb{R}$ that is continuous when $\mathbb{R}$ is equipped with its usual metric topology. Then there exists and $x \in K$ so that*

$$f(x) \leq f(y) \qquad \text{for all} \qquad y \in K\ . \tag{3.3}$$

Proof. Consider the open sets $(-n, \infty)$ in $\mathbb{R}$, since f is continuous,

$$\mathcal{U} = \{\ f^{-1}((-n,\infty))\ \ : n \in \mathbb{N}\ \}$$

is an open cover of X, and hence K. Since K is compact, there exists an open subcover. But for $n > m$, $f^{-1}((-m,\infty)) \subset f^{-1}((-n,\infty))$, so there is an n with $K \subset f^{-1}((-n,\infty))$. In particular, f is bounded from below on K.

Now let a be the greatest lower bound of the numbers $f(y)$ for $y \in K$. We claim that there exists an $x \in K$ with $f(x) = a$. If so, then plainly (3.3) is true.

To prove this, let us suppose that there is no such x. Then

$$\mathcal{U} = \{\ f^{-1}((a + 1/n,\infty))\ \ : n \in \mathbb{N}\ \}$$

is an open cover of K. This means that there is a finite subcover, and again, since the sets in the open cover are nested, a single one of them, say $f^{-1}((a+1/n,\infty))$, covers K. But this would mean that $f(y) \geq a + 1/n$ for each y in k, which is not possible since a is the greatest lower bound. $\square$

Any point x for which (3.3) is true is called a *minimizer of f on X*. Likewise, any point x for which

$$f(x) \geq f(y) \qquad \text{for all} \qquad y \in K\ . \tag{3.4}$$

is called a *maximizer of f on X*.

There are several important conclusions to be drawn from this proof. First, if f is continuous, so is $-f$, and a minimizer of $-f$ is a maximizer of f. Hence the theorem implies the existence of both minimizers and maximizers for continuous functions on compact sets.

Now suppose we have a real valued function f defined on a sets K, and we want to know if f has a minimizer in K. If we can find a topology on K that makes f continuous, and makes K compact, then we can apply the previous theorem.

However, the demands of continuity and compactness pull in opposite directions when we look for our topology: The topology has to have sufficiently many open sets in it for f to be continuous, since we need $f^{-1}(U)$ to be open for every open set U in $\mathbb{R}$. On the other hand, the more open sets we include in our topology, the more open covers we have to worry about when showing that every open cover has a finite subcover.

Very often, one is stuck between a rock and a hard place, and there is *no* topology that both makes f continuous, and K compact. Indeed, there are many very nice functions f – such as the exponential function of $\mathbb{R}$ – that simply do not have minimizers or maximizers. While $\mathbb{R}$ is compact under the trivial topology $\mathcal{O} = \{\emptyset, \mathbb{R}\}$, and while the exponential function is continuous under the usual metric topology on $\mathbb{R}$, the fact that the exponential function does not have either a maximizer or a minimizer shows that there is *no* topology on $\mathbb{R}$ under which $\mathbb{R}$ is compact and the exponential function is continuous.

A situation that is frequently encountered in applications is that a function f on X does have, say, a minimizer, but not a maximizer. Also in this situation, it is impossible to find a topology for which f is continuous and X is compact, since them both minima and maxima would exist.

However, if we are just looking for minima, it is worth noticing that in our proof of Theorem 3.5, we did not use the full strength of the continuity hypothesis. The same proof yields the same conclusion if we assume only the property that $f^{-1}((t,\infty))$ is open for each t in $\mathbb{R}$.

3.9 DEFINITION (Upper and lower semicontinuous function)**.** *Let $(X, \mathcal{O}_X)$ be a topological space. A function f from X to $\mathbb{R}$ is called lower semicontinuous in case for all t in $\mathbb{R}$, $f^{-1}((t,\infty))$ is open. It is called upper semicontinuous in case for all t in $\mathbb{R}$, $f^{-1}((-\infty,t))$ is open. As has been explained in Example 3.3, lower semicontinuity of f is the same as conitnuity of f from $(X, \mathcal{O}_X)$ to $(\mathbb{R}, \mathcal{O}_r)$, where $\mathcal{O}_r$ is the right order topology on $\mathbb{R}$.*

Summarizing the discussion above, we have the following variant of Theorem 3.5:

3.6 THEOREM. *Let $(X, \mathcal{O})$ be a topological space, and let $K \subset X$ be either compact or sequentially compact. Let f be a lower semicontinous real valued function on $(X, \mathcal{O})$. Then there exists $x_0 \in K$ such that $f(x_0) \leq f(x)$ for all $x \in K$.*

Thus, we can prove existence of minimizers for f on X by finding a topology that makes f lower semicontinuous, and K compact. This turns out to be a very useful strategy, as we shall see.

Still, to use either Theorem 3.1 or Theorem 3.5, we need criteria for compactness. How can we tell if a set X is compact? In metric spaces, we can reduce this to a question about sequences.

3.10 DEFINITION (Sequential compactness)**.** *A topological space $(X, \mathcal{O})$ is* sequentially compact *in case every sequence $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$.*

3.7 THEOREM (Compactness and subsequences in a Metric space)**.** *Let (X, d) be any metric space, and let K be any subset of X. Then K is compact if and only if every infinite sequence $\{x_k\}$ of elements of K has an infinite subsequence $\{x_{k_n}\}$ that converges to some x in K.*

In other words, a metric space is compact if and only if it is sequentially compact. In the broader setting of topological spaces, there is no relation between compactness and sequential compactness. There are topological spaces that are compact, but not sequentially compact, and there are sequentially compact spaces that are not compact.

The notion of compactness as we have defined it in terms of open covers is a 20th century notion. In the 19th century, mathematicians thought about compactness issues in terms of sequential compactness.

It is important to note that this theorem is *not true* in the general setting of topological spaces; it is important that the topology be a metric topology. Likewise, in the general topological setting, it is not true that a function f is continuous if and only if it takes convergent sequences to convergent sequences.

The close connection between sequences and continuity and compactness that one has in metric spaces does not carry over to the more general topological setting at all. Fortunately, almost all of the topologies that we shall encounter are metric topologies.

Proof of Theorem 3.7. The fact that compactness implies sequential compactness in a metric space is relatively easy. Suppose there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ that has no convergent subseqeunce. Then there is no $y \in K$ such that for all $r > 0$, $B_r(y)$ contains x_k for infinitely many k, since otherwise there would be a subsequence converging to y. (Consider the balls $B_{1/n}(y)$, and apply Cantor's diagonal sequences argument to the sequence of subsequences coinained in these balls.) Hence, for each $y \in K$ there exists an open set U_y that contains y, but which cointains x_k for at most finitely many k. Then $\{U_y \ : \ y \in K\}$ is an open cover of K. Since K is compact, there exists a finite subcover $\{U_{y_1}, \dots, U_{y_n}\}$. Thus, every k, x_k belongs to U_{y_j} for some j, but each U_{y_j} contains x_k for only finitely many k, which is impossible. Hence a convergent subsequence exists.

Fot the other implication, we assume sequention compactness, and shall prove compactness in four steps.

Step 1: K is bounded: We first show that K is bonded, which means that

$$\sup_{x,y\in K} d(x,y) < \infty \ .$$

This supremum is called the *diameter* of K.

To see that the diameter is finite, suppose that it is not. Under this hypothesis, we construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ as follows. First, fix any $x \in X$. Now for each $n \in N$, choose some $x_n \in K\backslash B_n(x)$. The set $K\backslash B_n(x)$ is not empty when the diameter of K is infinite.

Then, by hypothesis, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ and some $y \in K$ such that

$$\lim_{k\to\infty} x_{n_k} = y \ . \tag{3.5}$$

Then by the triangle inequality, we would have

$$d(x, x_{n_k}) \le d(x,y) + d(y, x_{n_k}) \ .$$

But this cannot be: By construction, $d(x, x_{n_k}) > n_k$, while $d(x,y)$ is some fixed, finite number, and for all sufficiently large k, $d(y, x_{n_k}) \le 1$, by (3.5). This contradiction shows that K must be bounded.

Step 2: K contains a dense sequence: We next show that there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ that is *dense* in K; i.e., that for every $\epsilon > 0$, and every $x \in K$, there is some n such that $d(x_n, x) < \epsilon$.

In other words, the sequence $\{x_n\}_{n\in\mathbb{N}}$ passes arbitrarily close to every point in K. Here is how to construct it:

Pick the first term x_1 arbitrarily. We then define the rest of the sequence recursively as follows: Suppose that $\{x_1, \dots, x_k\}$ have been chosen. For each $y \in K$, define

$$d_k(y) := \min_{1\le j\le k}\{d(y, x_j)\} \ .$$

This is, by definition, the distance from y to the set $\{x_1, \dots, x_k\} \subset K$, and of course, this is no greater than the diameter of K, which is finite by the first step.

Therefore, d_k, defined by

$$d_k := \sup_{y\in K} d_k(y)$$

is no greater than the diameter of K.

Armed with this knowledge, we are ready to choose x_{k+1}: We choose x_{k+1} to be any element of K with

$$d_k(x_{k+1}) \geq \frac{1}{2}d_k \ .$$

We now claim that $\lim_{k\to\infty} d_k = 0$. It should be clear that $\{x_n\}_{n\in\mathbb{N}}$ is dense if and only if this is the case. So, to complete Step 2, we need to prove that $\lim_{k\to\infty} d_k = 0$.

Towards this end, the first thing to observe is that $\{d_k\}_{k\in\mathbb{N}}$ is a monotone decreasing sequence, bounded below by zero: Indeed, for any sets $A \subset B \subset K$, the distance from y to B is no greater than the distance from y to A. Therefore, we only have to show that *some subsequence* of $\{d_k\}_{k\in\mathbb{N}}$ converges to zero.

To do this, let $\{x_{k_n}\}_{n\in\mathbb{N}}$ be a convergent subsequence of $\{x_k\}_{k\in\mathbb{N}}$, and let y be the limit; i.e.,

$$\lim_{n\to\infty} x_{k_n} = y \ .$$

Then of course since by the triangle inequality

$$d(x_{k_n}, x_{k_{n+1}}) \leq d(x_{k_n}, y) + d(y, x_{k_{n+1}}) \ ,$$

and since $\lim\limits_{n\to\infty} d(x_{k_n}, y) = \lim\limits_{n\to\infty} d(y, x_{k_{n+1}}) = 0$,

$$\lim_{n\to\infty} d(x_{k_n}, x_{k_{n+1}}) = 0 \ .$$

But since

$$x_{k_n} \in \{x_1, \ldots, x_{k_{n+1}-1}\} \ ,$$

$$d(x_{k_n}, x_{k_{n+1}}) \geq d_{k_{n+1}-1}(x_{k_{n+1}}) \geq \frac{1}{2}d_{k_{n+1}-1} \ .$$

Therefore,

$$\lim_{n\to\infty} d_{k_{n+1}-1} = 0 \ ,$$

and then, since the entire sequence is monotone decreasing, $\lim_{k\to\infty} d_k = 0$. Hence, the sequence we have constructed is dense.

Step 3: Given any open cover of K, there exists a countable subcover. To prove this, consider any open cover $\mathcal{G}$ of K. Consider the set of open balls $B_r(x_k)$ where $r > 0$ is rational, and $\{x_k\}_{k\in\mathbb{N}}$ is the dense sequence that we have constructed in Step 2. This set of balls is countable since a countable union of countable sets is countable.

The countable subcover is constructed as follows: For each rational $r > 0$ and each $k \in \mathbb{N}$, choose $U_{r,k}$ to be *some* open set in $\mathcal{G}$ that contains $B_r(x_k)$ if there is such a set, and otherwise, do not define $U_{r,k}$. Let $\mathcal{U}$ be the set of open sets defined in this way; clearly $\mathcal{U}$ is countable by construction.

We now claim that $\mathcal{U}$ is an open cover of K. Clearly the sets in $\mathcal{U}$ are open. To see that they cover, pick any $x \in K$. Since $\mathcal{G}$ is an open cover of K, $x \in V$ for some $V \in \mathcal{G}$. Then, since V is open, for some rational $r > 0$, $B_{2r}(x) \subset V$.

Then, since $\{x_k\}_{k\in\mathbb{N}}$ is dense, there is some k with $x_k \in B_r(x)$. But then $x \in B_r(x_k)$ and

$$B_r(x_k) \subset B_{2r}(x) \subset V \ ,$$

(where the first containment holds by the triangle inequality). This shows that for the pair (r,k), there is *some* $V \in \mathcal{G}$ containing $B_r(x_k)$. Therefore, by construction, $U_{r,k} \in \mathcal{U}$ contains $B_r(x_k)$, and hence $x \in U_{r,k}$. Since x is an arbitrary element of K, $\mathcal{U}$ covers K.

Step 4: Some finite subcover of the countable cover is a cover. Now order the sets in our countable cover $\mathcal{U}$ into a sequence of open sets $\{U_k\}_{k\in\mathbb{N}}$ that covers K.

Suppose that for each n, it is *not* the case that

$$K \subset \bigcup_{k=1}^{n} U_k \ . \tag{3.6}$$

Then we can construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ be choosing $x_n \in K\backslash \left(\bigcup_{k=1}^{n} U_k\right)$.

Let $\{x_{n_j}\}_{j\in\mathbb{N}}$ be a subsequence with $\lim_{j\to\infty} x_{n_j} = y \in K$. Then, since $\mathcal{U}$ is an open cover of K, there is some U_k with $y \in U_k$. But then all but finitely many terms of the sequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ lie in U_k, and so the whole sequence lies in some finite union of the sets in $\mathcal{U}$. This is a contradiction, and so (3.6) is true for some $n \in \mathbb{N}$. □

Part of the proof made use of a dense sequence in our sequentially compact metric space. The existence of a dense sequence is often useful, and so we make the following definition.

3.11 DEFINITION (Separable topological space)**.** *A topological space $(X,\mathcal{O})$ is* separable *in case it contains a countable dense subset.*

We have seen the a compact metric space is always separable, but also many non-compact spaces are separable. We shall se examples shortly.

We shall soon prove two powerful theorems on approximation and compactness in an infinite dimensional vector space, $\mathcal{C}(X,\mathbb{R})$, the space of real valued continuous functions on a compact Hausdorff space X, equipped with the uniform metric

$$d_\infty(f,g) = \sup_{x\in X}\{|f(x) - g(x)|\} \ .$$

Note that by Theorem 3.5 and the fact that X is compact, there exists an $x_0 \in X$ such that

$$|f(x_0) - g(x_0)| = \sup_{x\in X}\{|f(x) - g(x)|\} \ .$$

It is then very easy to see that d_∞ is indeed a metric on $\mathcal{C}(X,\mathbb{R})$, called the *uniform metric.* It is left as an exercise to generalize Theorem 2.3 and show that $(\mathcal{C}(X,\mathbb{R}), d_\infty)$ is complete.

3.3 Generated topologies

When working a class of functions $\mathcal{F}$ on a set X with values in a topological space $(Y,\mathcal{U})$, it is often useful to introduce a topology that makes every function $f \in \mathcal{F}$ continuous, but which contains the minimal number of open sets for this purpose. given two topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ on a sets X, we say that $\mathcal{O}_1$ is weaker than $\mathcal{O}_2$ in case $\mathcal{O}_1 \subset \mathcal{O}_2$. In the case at hand, there is a unique *weakest possible* topology $\mathcal{O}_{\mathcal{F}}$ on X with respect to which each $f \in \mathcal{F}$ is continuous, and such that $\mathcal{O}_{\mathcal{F}}$ is weaker than any other topology with this property.

The weaker a topology is, the more compact sets there will be, and so such topologies are useful when we wish to apply theorems requiring both continuity and compactness.

3.8 THEOREM (Generated topologies)**.** *Given a class of functions $\mathcal{F}$ on a set X with values in a topological space $(Y,\mathcal{U})$, define*

$$\mathcal{E} = \{f^{-1}(U) \ : \ f \in \mathcal{F} \ , \ U \in \mathcal{U} \ \},$$

and define $\widehat{\mathcal{E}}$ to be the set of all finite intersections of sets in $\mathcal{E}$. Finally define $\mathcal{O}_{\mathcal{F}}$ to be the set of arbitrary unions of sets in $\widehat{\mathcal{E}}$. Then $\mathcal{O}_{\mathcal{F}}$ is a topology, and it is the weakest topology with respect to which each $f \in \mathcal{F}$ is continuous.

Proof. Since each $x \in X$ lies in $f^{-1}(Y) \in \mathcal{E}$ for any $f \in \mathcal{F}$, it is clear that $X \in \mathcal{O}_{\mathcal{F}}$, and taking the empty union, $\emptyset \in \mathcal{O}_{\mathcal{F}}$. Evidently $\mathcal{O}_{\mathcal{F}}$ is closed under arbitrary unions. Thus, if $\mathcal{O}_{\mathcal{F}}$ is closed under finite intersections, is is a topology. Let $E, F \in \mathcal{O}_{\mathcal{F}}$. If $x \in E \cap F$, then, by definition, there exist E_x, F_x in $\widehat{\mathcal{E}}$ such that $E_x \subset E$ and $F_x \subset F$. Since $\widehat{\mathcal{E}}$ is closed under finite intersections, $x \in E_x \cap F_x \in \widehat{\mathcal{E}}$, and $E_x \cap F_x \subset E \cap F$. Making such a construction for each $x \in E \cap F$, we have

$$E \cap F = \bigcup_{x \in E \cap F} E_x \cap F_x \ ,$$

showing that $E \cap F \in \mathcal{O}_{\mathcal{F}}$. Thus, $\mathcal{O}_{\mathcal{F}}$ is a topology.

Now let $\mathcal{O}$ be any topology with respect to which each $f \in \mathcal{F}$ is continuous. Evidently, $\mathcal{O}$ must contain all of the sets

$$\mathcal{E} = \{f^{-1}(U) \ : \ f \in \mathcal{F} \ , \ U \in \mathcal{U} \ \}.$$

Certainly also any such topology must contain, $\widehat{\mathcal{E}}$, the set of all finite intersections of sets in $\mathcal{E}$, and then it must contain all arbitrary unions of sets in $\widehat{\mathcal{E}}$. Hence $\mathcal{O}$ must contain $\mathcal{O}_{\mathcal{F}}$. □

3.12 DEFINITION (Weak topology)**.** *Given a set X and a family $\mathcal{F}$ of functions from X to a topological space $(Y, \mathcal{O})$, the* weak topology on X generated by $\mathcal{F}$.

4 Some frequently used theorems

4.1 Baire's Theorem

4.1 DEFINITION (Nowhere dense)**.** *Let $(X, \mathcal{O})$ be a topological space. A subset$X$ is* nowhere dense *in case the closure of A has empty interior; i.e., $(\overline{A})^\circ = \emptyset$.*

Note that if A is closed, it is nowhere dense if and only if $A^\circ = \emptyset$, which is the case iff and only if $A^c \cap U \neq \emptyset$ for all $U \in \mathcal{O}$. Therefore, a closed set A is nowhere densre if and only if its complement is an open dense set in X.

4.1 LEMMA (Baire's Lemma)**.** *Let (X, d) be a complete metric space. Let $\{U_n\}_{n \in \mathbb{N}}$ be a sequence of open dense sets in X. Then $\bigcap_{n \in \mathbb{N}} U_n$ is dense in X.*

Proof. Let W be any open subset of X. It suffices to show that

$$\bigcap_{n\in N} U_n \cap W \neq \emptyset \ . \tag{4.1}$$

Now construct a Cauchy seequence as follows: Since U_1 is open and dense, and W is open, $U_1 \cap W$ is open and non-empty. Hence for some $r_1 > 0$ and some $x_1 \in X$, $\overline{B_{r_1}(x_1)} \subset U_1 \cap W$. (Note that if $B_{s_1}(x_1) \subset U_1 \cap W$, then for any $r_1 < s_1$, $\overline{B_{r_1}(x_1)} \subset U_1 \cap W$ because $\overline{B_{r_1}(x_1) \subset B_{s_1}(x_1)}$.) Without loss of generality, we may suppose that $r_1 < 1$. Since U_2 is open and dense and $B_{r_1}(x_0)$, is open, there exist $r_2 > 0$ and some $x_2 \in X$ so that $B_{r_2}(x_2) \subset B_{r_1}(x_1) \cap U_2$. Without loss of generality, we may suppose that $r_2 < r_1/2$.

Now supposing that $\{(r_1, x_1), \dots (r_n, x_n)\}$ are chosen, pick (r_{n+1}, x_{n+1}) so that

$$\overline{B_{r_{n+1}}(x_{n+1})} \subset B_{r_n}(x_n) \cap U_{n+1} \qquad \text{and} \qquad r_{n+1} < r_n/2 \ .$$

By contruction

$$\overline{B_{r_n}(x_n)} \subset B_{r_m}(x_m) \subset W \quad \text{for all} \quad n > m \tag{4.2}$$

and

$$\overline{B_{r_n}(x_n)} \subset \bigcap_{m=1}^{n} U_m \cap W \ . \tag{4.3}$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy since (4.2) says that for all $j, k \geq m$, $d(x_j, x_k) < r_m$ and $\lim_{m\to\infty} r_m = 0$. Since (X, d) is complete, there exists $x_0 \in X$ such that $\lim_{n\to\infty} x_n = x_0$. By (4.2) again, for all $n \geq m$, $x_n \in \overline{B_{b_m}(x_m)}$, and hence $x_0 \in \overline{B_{b_m}(x_m)}$ for all m. By (4.3), $x_0 \in \bigcap_{n=1}^{\infty} U_n \cap W$. □

4.2 THEOREM (Baire's Theorem)**.** *A complete metric space (X, d) is never the countable union of nowhere dense sets.*

Proof. Let $\{E_m\}_{n\in\mathbb{N}}$ be a sequence of nowhere dense sets in X. It suffice to show $\bigcup_{n=1}^{\infty} \overline{E_n} \neq X$. By Baire's Lemma,

$$\left(\bigcup_{n=1}^{\infty} \overline{E_n}\right)^c = \bigcap_{n=1}^{\infty} (\overline{E_n})^c \neq \emptyset$$

since each $(\overline{E_n})^c$ is open and dense. □

Notice that to prove Baire's Theorem, we only needed to know that the inetrsection of a sequence of open dense sets is non-empty, while Baire's lemma tells us that not only is $\cap_{n=1}^{\infty} U_n$ non-empty, it is even dense. This stronger information is sometimes useful in applications.

In fact, in a metric space (X, d) that has no isoolated points, $\cap_{n=1}^{\infty} U_n$ is uncountable. To see this, suppose on the contrary that $\cap_{n=1}^{\infty} U_n$ is countable. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $\{x_n \ : \ n \in \mathbb{N}\} = \cap_{n=1}^{\infty} U_n$. (There will be repeats in the sequence if the latter set is finite.) Define $E_n = U_n^c \cup \{x_n\}$. Then each E_n is closed and has empty interior (since X has no isolated points). Then

$$\bigcup_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} U_n^c\right) \cup \left(\bigcap_{n=1}^{\infty} U_n\right) = \left(\bigcap_{n=1}^{\infty} U_n\right)^c \cup \left(\bigcap_{n=1}^{\infty} U_n\right) = X \ .$$

By Baire's Theorem, this is impossible.

4.2 The Arzelà-Ascoli Theorem

Let X be a compact topological space, and consider the metric space, and hence topological space, consisting of $\mathcal{C}(X,\mathbb{R})$ equipped with the uniform metric.

4.2 DEFINITION (Equicontinuous, pointwise bounded)**.** *Let $\mathcal{F} \subset \mathcal{C}(X,\mathbb{R})$. Then $\mathcal{F}$ is* equicontinuous *in case for each $\epsilon > 0$ and each $x \in X$, there is a neighborhood U_ϵ of x such that for all $f \in \mathcal{F}$,*

$$y \in U_\epsilon \Rightarrow |f(y) - f(x)| < \epsilon \ .$$

Also, $\mathcal{F}$ is pointwise bounded *in case for each $x \in X$, $\{f(x) \ : f \in \mathcal{F}\ \}$ is a bounded subset of $\mathbb{R}$.*

The first thing to observe is that if $\mathcal{F}$ is a compact subset of $\mathcal{C}(X,\mathbb{R})$, then $\mathcal{F}$ is both pointwise bounded and equicontinuous.

Indeed, suppose that $\mathcal{F}$ is compact. Then evidently $\mathcal{F} \subset \bigcup_{f\in\mathcal{F}} B_f(1)$, and hence there exists a finite set $\{f_1,\dots,f_n\}$ in $\mathcal{F}$ such that

$$\mathcal{F} \subset \bigcup_{j=1}^n B_{f_j}(1) \ .$$

Let M_j denote the maximum of $|f_j|$, which is continuous on X, and hence bounded. Let $M = \max_{j=1,\dots,n} M_j$. Then for any $f \in \mathcal{F}$, there is some j such that $d_\infty(f,f_j) < 1$, and hence for all x,

$$|f(x)| \leq |f_j(x)| + |f(x) - f_j(x)| \leq M_j + d_\infty(f,f_j) \leq M + 1 \ .$$

This even shows that $\mathcal{F}$ uniformly bounded.

To show that $\mathcal{F}$ is equicontinuous, fix $x \in X$ and $\epsilon > 0$. For each $f \in \mathcal{F}$, define V_f by

$$V_f = \{\ g \in \mathcal{F} \ : \ d(g,f) < \epsilon/3\ \} \ .$$

Clearly, each V_f is open, and $\bigcup_{f\in\mathcal{F}} V_f = \mathcal{F}$. Hence there exists a finite set $\{f_1,\dots,f_n\} \subset \mathcal{F}$ such that

$$\bigcup_{j=1}^n V_{f_j} = \mathcal{F} \ .$$

Next, define $U_j \subset X$ by $U_j = \{\ y \in X \ : \ |f_j(y) - f_j(x)| < \epsilon/3\}$. Evidently $U = \cap_{j=1}^n U_j$ is a neighborhood of x.

For any $f \in \mathcal{F}$, $d_\infty(f,f_j) < \epsilon/3$ for some j, and so for $y \in U$,

$$|f(y) - f(x)| \leq |f(y) - f_j(y)| + |f_j(y) - f_j(x)| + |f_j(x) - f(x)| \leq |f_j(y) - f_j(x)| + 2d_\infty(f,f_j) < \epsilon \ .$$

Thus, U is a neighborhood of x such that $|f(y) - f(x)| < \epsilon$ for all $y \in U$ and all $f \in \mathcal{F}$.

Thus, a necessary condition for $\mathcal{F}$ to be compact in $\mathcal{C}(X,\mathbb{R})$ is that $\mathcal{F}$ be equicontinuous and pointwise bounded. The Arzelà-Ascoli Theorem says that these conditions are essentially sufficient as well:

4.3 THEOREM (Arzelà-Ascoli)**.** *Let X be a compact topological space space, and let $\mathcal{F}$ be an equicontinuous and pointwise bounded subset of $\mathcal{C}(X,\mathbb{R})$. Then the closure of $\mathcal{F}$ is compact.*

Proof. The first thing to observe is that if $\mathcal{F}$ is equicontinuous and pointwise bounded, then so is the closure of $\mathcal{F}$. Hence, let us assume that $\mathcal{F}$ is closed as well as equicontinuous and pointwise bounded. We shall then show that $\mathcal{F}$ is compact.

By Theorem 3.7, it suffices to show that for any infinite sequence $\{f_\ell\}_{\ell\in\mathbb{N}}$ in $\mathcal{F}$, there is a convergent subsequence, and then by the completeness of $\mathcal{C}(X,\mathbb{R})$ it suffices to show that for any infinite sequence $\{f_\ell\}_{\ell\in\mathbb{N}}$ in $\mathcal{F}$, there is a Cauchy subsequence. Therefore, fix any infinite sequence $\{f_\ell\}_{\ell\in\mathbb{N}}$ in $\mathcal{F}$. We must prove that there is a subsequence $\{f_{\ell_j}\}_{j\in\mathbb{N}}$ such that for all $\epsilon > 0$, there is an $N_\epsilon \in \mathbb{N}$ such that

$$j,k \geq N_\epsilon \Rightarrow |f_{\ell_j}(x) - f_{\ell_k}(x)| < \epsilon \quad \text{for all } x \in X. \tag{4.4}$$

Fix $\epsilon > 0$. Use the compactness of X and the equicontinuity of $\mathcal{F}$ to select a finite set of points $\{x_1,\dots,x_m\}$ and neighborhoods $\{U_1,\dots,U_m\}$ that cover $\mathcal{F}$ and are such that

$$x \in U_j \Rightarrow |f(x) - f(x_j)| < \frac{\epsilon}{3} \qquad \text{for all } f \in \mathcal{F}\ . \tag{4.5}$$

Since for each i, the set $\{f(x_i)\ :\ f \in \mathcal{F}\ \}$ is a bounded subset of $\mathbb{R}$, we can choose a subsequence of $\{f_\ell\}_{\ell\in\mathbb{N}}$ along which $f_{\ell_k}(x_i)$ converges for each i. Since convergent sequences are Cauchy, it follows that there exists $N_\epsilon \in \mathbb{N}$ such that

$$j,k \geq N_\epsilon \Rightarrow |f_{\ell_j}(x_i) - f_{\ell_k}(x_i)| \leq \frac{\epsilon}{3}\ . \tag{4.6}$$

But then since each $x \in X$ belongs to U_i for some i, we have (for this i), and $j,k \geq N_\epsilon$,

$$\begin{aligned}
|f_{\ell_j}(x) - f_{\ell_k}(x)| &\leq |f_{\ell_j}(x) - f_{\ell_k}(x_i)| + |f_{\ell_j}(x_i) - f_{\ell_k}(x_i)| + |f_{\ell_j}(x_i) - f_{\ell_k}(x)| \\
&\leq \frac{\epsilon}{3} + |f_{\ell_j}(x_i) - f_{\ell_k}(x_i)| + \frac{\epsilon}{3} \\
&\leq \epsilon\ .
\end{aligned}$$

where the first inequality is the triangle inequality, the second is (4.5) and the third is (4.6). This proves (4.4). □

Another proof of the Arzela-Ascoli Theorem may be given using the fact that complete totally bounded subsets of a metric space are compact. This approach is useful in proving other compactness theorems, so we briefly explain it as well. The key is the following lemma of Hanche-Olsen and Holden:

4.4 LEMMA. *Let (X,d_X) be a metric space. Suppose that for all $\epsilon > 0$ there exists a $\delta > 0$ and a metric space (W,d_W) and a function $\Phi : X \to W$ such that:*

(1) $\Phi(X)$ is totally bounded in W.

(2) For all $x,y \in X$,

$$d_W(\Phi(x),\Phi(y)) < \delta \Rightarrow d_X(x,y) < \epsilon\ .$$

Then X is totally bounded.

4.5 Remark. If $\Phi : X \to W$ were invertible, then *(2)* could be rewritten as: For all $w,z \in W$,

$$d_W(w,z) < \delta \Rightarrow d_X(\Phi^{-1}(w),\Phi^{-1}(z)) < \epsilon\ ,$$

which would mean that Φ^{-1} is continuous. However, we do not assume that Φ is invertible. Nonetheless, even when Φ is not invertible, and A is any subset of W, we write $\Phi^{-1}(A)$ to denote the preimage of A under Φ, i.e., $\Phi^{-1}(A) = \{x \in X \ : \ \Phi(x) \in A\}$. Condition *(2)* then says that the diameter of the preimage of a set of diameter less than δ is less than ϵ.

Proof. Fix $\epsilon > 0$, Let δ, (W, d_W) and Φ be such that *(1)* and *(2)* are satisfied. Since $\Phi(X)$ is totally bounded, there is a finite cover $\{U_1, \dots, U_n\}$ of $\Phi(X)$ be balls of radius δ in W. It follows immediately that $\{\Phi^{-1}(U_1), \dots \Phi^{-1}(U_n)\}$ is a cover of X by sets of diameter 2ϵ. For each $i = 1, \dots, n$, pick $x_i \in \Phi^{-1}(U_i)$. Then $\Phi^{-1}(U_i) \subset B_{2\epsilon}(x_i)$, and so $\{B_{2\epsilon}(x_1), \dots, B_{2\epsilon}(x_n)\}$ is a finite cover of X by balls of radius 2ϵ . Since $\epsilon > 0$ is arbitrary, (X, d) is totally bounded. □

Second proof of the Arzela-Ascoli Theorem. We may suppose as before that $\mathcal{F}$ is closed and hence that $(\mathcal{F}, d_\infty)$ is a complete metric space. It therefore suffices to show that it is totally bounded. Starting as before, fix $\epsilon > 0$, and use the compactness of X and the equicontinuity of $\mathcal{F}$ to select a finite set of points $\{x_1, \dots, x_m\}$ and neighborhoods $\{U_1, \dots, U_m\}$ that cover $\mathcal{F}$ and are such that

$$x \in U_j \Rightarrow |f(x) - f(x_j)| < \frac{\epsilon}{3} \qquad \text{for all } f \in \mathcal{F} \ . \tag{4.7}$$

Define $\Phi : \mathcal{F} \to \mathbb{R}^m$ by

$$\Phi(f) = (f(x_1), \dots, f(x_m)) \ .$$

Since $\mathcal{F}$ is pointwise bounded, $\Phi(\mathcal{F})$ lies in some bounded rectangle in $\mathbb{R}^m$, and bounded sets in $\mathbb{R}^m$ are totally bounded. Thus, Φ satisfies condition *(1)* of Lemma 4.4. Next, consider any $f, g \in \mathcal{F}$. Denoting the Euclidean norm on $\mathbb{R}^m$ by $\|\cdot\|$,

$$\|\Phi(f) - \Phi(g)\| = \|(f(x_1), \dots, f(x_m)) - (g(x_1), \dots, g(x_m))\| \geq \max_{j=1}^{m} |f(x_j) - g(x_j)| \ . \tag{4.8}$$

fix any $x \in X$. Then for some j, $x \in U_j$, and for this j,

$$\begin{aligned} |f(x) - g(x)| &\leq |f(x) - f(x_j)| + |f(x_j) - g(x_j)| + |g(x_j) - g(x)| \\ &\leq \frac{\epsilon}{3} + |f(x_j) - g(x_j)| + \frac{\epsilon}{3} \\ &\leq \frac{\epsilon}{3} + \|\Phi(f) - \Phi(g)\| + \frac{\epsilon}{3} \end{aligned}$$

where the first inequality is the triangle inequality, the second is (4.7) and the third is (4.8). Thus, condition *(2)* of Lemma 4.4 is satisfied for $\delta = \epsilon/3$, and we have proved that $\mathcal{F}$ is totally bounded. □

4.3 The Stone-Wierstrass Theorem

The Stone-Wierstrass Theorem is an approximation theorem that generalizes the classical Wierstrass Approximation Theorem that we discussed at the beginning of these notes.

We begin with two definitions. Let X be a compact topological space, and let $\mathcal{C}(X, \mathbb{R})$ be the space of continuous real valued functions on X equipped with the uniform metric d_∞. A subset $\mathcal{A}$ of $\mathcal{C}(X, \mathbb{R})$ is an *algebra* in case $\mathcal{A}$ is a vector subspace over $\mathbb{R}$ of $\mathcal{C}(X, \mathbb{R})$ equipped with its usual rules of addition and scalar multiplication, and if, moreover, for every f and g in $\mathcal{A}$, the pointwise product fg also belongs to $\mathcal{A}$.

A subset $\mathcal{A}$ of $\mathcal{C}(X,\mathbb{R})$ is *separating* in case for pair of distinct points x, y in X, there is an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Notice that if X is not Hausdorff, not even $\mathcal{C}(X,\mathbb{R})$ the space of *all* continuous real valued functions on X is separating. Indeed, if X is not Hausdorff, there exist two distinct points x and y in X such that every neighborhood U of x contains y. But then for any continuous function f, $f(x) = f(y)$. Indeed, if $|f(x) - f(y)| := r > 0$, then $f^{-1}((f(x) - r.2, f(x) + r/2))$ would be an open neighborhood of x that excluded y. Thus, for all continuous f, $f(x) = f(y)$, so not even $\mathcal{C}(X,\mathbb{R})$ separates, let alone any proper subset of $\mathcal{C}(X,\mathbb{R})$. Hence throughout this subsection, we shall only be concerned with Hausdorff topological spaces.

The primary example of a separating algebra to keep in mind is $X = [0,1]$, with $\mathcal{A}$ being the algebra of all polynomials in the real variable $x \in [0,1]$. To see that this algebra is separating, consider the polynomial $p(x) = x$. Then for $x_0 \neq x_1$ in X, $p(x_0) \neq p(x_1)$.

4.6 THEOREM (Stone-Wierstrass)**.** *Let X be a compact topological space, and let $\mathcal{A}$ be a subset of $\mathcal{C}(X,\mathbb{R})$ that is a separating algebra. Let $\mathcal{B}$ be the uniform closure of $\mathcal{A}$. Then either $\mathcal{B} = \mathcal{C}(X,\mathbb{R})$, or else $\mathcal{B}$ consists of all continuous functions on X that vanish at some fixed point x_0. In particular, if $\mathcal{A}$ contains the constant functions, $\mathcal{B} = \mathcal{C}(X,\mathbb{R})$.*

We will prove Theorem 4.6 as a consequence of two lemmas, and shall make use of the *partial order* in $\mathcal{C}(X,\mathbb{R})$: If $f, g \in \mathcal{C}(X,\mathbb{R})$, we write $f \leq g$ in case $f(x) \leq g(x)$ for all $x \in X$. With this partial order, $\mathcal{C}(X,\mathbb{R})$ is a *lattice*: Given any $f, g \in \mathcal{C}(X,\mathbb{R})$ there is a unique function $g \wedge f \in \mathcal{C}(X,\mathbb{R})$ such that $g \wedge f \leq f, g$, and such that $h \leq g \wedge f$ whenever $h \leq f, g$. Of course, $g \wedge f$ is given by

$$g \wedge f(x) = \min\{\ f(x)\ ,\ g(x)\ \}\ ,$$

which is continuous.

Likewise, given any $f, g \in \mathcal{C}(X,\mathbb{R})$ there is a unique function $g \vee f \in \mathcal{C}(X,\mathbb{R})$ such that $f, g \leq g \vee f$, and such that $g \vee f \leq h$ whenever $f, g \leq h$. Of course, $g \vee f$ is given by

$$g \vee f(x) = \max\{\ f(x)\ ,\ g(x)\ \}\ ,$$

which is continuous.

A subset $\mathcal{F}$ of $\mathcal{C}(X,\mathbb{R})$ is itself a lattice if and only if whenever $f, g \in \mathcal{F}$, then both $f \wedge g$ and $f \vee g$ belong to $\mathcal{F}$. Then observing that

$$f \wedge g = \frac{1}{2}(f + g - |f - g|) \qquad \text{and} \qquad f \vee g = \frac{1}{2}(f + g + |f - g|)\ , \tag{4.9}$$

we see that a subset $\mathcal{F}$ of $\mathcal{C}(X,\mathbb{R})$ that is a *vector space* is a lattice if and only if whenever $f \in \mathcal{F}$, then $|f| \in \mathcal{F}$.

4.7 LEMMA (Limit point criterion for lattices in $\mathcal{C}(X,\mathbb{R})$)**.** *Let X be a compact Hausdorff space. Let $\mathcal{F} \subset \mathcal{C}(X,\mathbb{R})$ be a lattice.*

If f is any element of $\mathcal{C}(X,\mathbb{R})$ with the property that for every $x, y \in X$, there exists a function $f_{x,y} \in \mathcal{F}$ for which

$$f_{x,y}(x) = f(x) \qquad \text{and} \qquad f_{x,y}(y) = f(y)\ . \tag{4.10}$$

Then f is a limit point of $\mathcal{F}$; i.e., it belongs to the closure of $\mathcal{F}$.

Proof. Fix any $f \in \mathcal{C}(X, \mathbb{R})$ with the property every $x, y \in X$, there exists a function $f_{x,y} \in \mathcal{F}$ such that (4.10) is satisfied. Fix any $\epsilon > 0$. We must show that there exists some $g \in \mathcal{F}$ with $|g(x) - f(x)| < \epsilon$ for all $x \in X$.

First, for each $(x, y) \in X \times X$, make some choice of $f_{x,y}$, and define the open set $U_{x,y} \subset X$ by

$$U_{x,y} = \{ \ z \ : \ f_{x,y}(z) < f(z) + \epsilon \ \} \ .$$

Evidently, $x, y \in U_{x,y}$. Therefore

$$X = \bigcup_{x \in X} U_{x,y} \ ,$$

and then, since X is compact, there exists a finite set $\{x_1, \dots, x_n\} \subset X$ such that

$$X = \bigcup_{j=1}^{n} U_{x_j,y} \ .$$

Now define the function f_y by

$$f_y = f_{x_1,y} \wedge f_{x_2,y} \wedge \cdots \wedge f_{x_n,y} \ .$$

Since $\mathcal{F}$ is a lattice, $f_y \in \mathcal{F}$, and

$$f_y \leq f + \epsilon$$

in the lattice order; i.e., everywhere on X.

Furthermore, since $f_{x_j,y}(y) = f(y)$ for each j, $f_y(y) = f(y)$. Therefore, defining the open set V_y by

$$V_y := \{ \ z \in X \ : \ f(z) - \epsilon < f_y(z) \ \} \ ,$$

we have $y \in V_y$, and hence

$$X = \bigcup_{y \in X} V_y \ ,$$

hen, since X is compact, there exists a finite set $\{y_1, \dots, y_m\} \subset X$ such that

$$X = \bigcup_{k=1}^{m} V_{y_k} \ .$$

Now define g by

$$g = f_{y_1} \vee f_{y_2} \vee \dots, \vee f_{y_m} \ .$$

Then since $\mathcal{F}$ is a lattice, $g \in \mathcal{F}$, and by construction,

$$f - \epsilon \leq g \leq f + \epsilon \ ,$$

which means that $|f(x) - g(x)| < \epsilon$ for all $x \in X$. $\square$

4.8 LEMMA (A closed algebra in $\mathcal{C}(X, \mathbb{R})$ is a lattice)**.** *Let X be a compact Hausdorff space. Let $\mathcal{B}$ be a closed subset of $\mathcal{C}(X, \mathbb{R})$ that is also a subalgebra of $\mathcal{C}(X, \mathbb{R})$. Then $\mathcal{B}$ is a lattice.*

Proof. By the remarks we have made concerning (4.9), it suffices to show that for all $f \in \mathcal{B}$, $|f| \in \mathcal{B}$. Since X is compact and f is continuous, f is bounded above and below, and hence there is a finite positive number c such that $|cf| \leq 1$. Then since $|cf| = c|f|$, we may freely suppose that $f| \leq 1$.

Therefore, fix any $f \in \mathcal{B}$ with $|f| \leq 1$, We shall complete the proof by showing that there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ so that

$$|f| = \lim_{n\to\infty} p_n(f^2) \tag{4.11}$$

in the uniform topology. Since $\mathcal{B}$ is an algebra, $p_n(f^2) \in \mathcal{B}$ for each n, and then since $\mathcal{B}$ is closed, $|f| \in \mathcal{B}$.

For any number $a \in [0,1]$, we define a sequence $\{b_n\}_{n\in\mathbb{N}}$ recursively as follows: We set $b_1 = 0$ and then for all $n \in \mathbb{N}$,

$$b_{n+1} = b_n + \frac{a - b_n^2}{2} \ .$$

Notice that

$$b_1 = 0 \ , \quad b_2 = \frac{a}{2} \ , \quad b_3 = a - \frac{a^2}{8} \ ,$$

and so forth. It is easy to see by induction that for each n, there is a polynomial p_n, independent of the value of a, so that such that $b_n = p_n(a)$.

We claim that $\sqrt{a} = \lim_{n\to\infty} b_n$. This will give us a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ such that for each $a \in [0,1]$,

$$\sqrt{a} = \lim_{n\to\infty} p_n(a) \ ,$$

and therefore, such that

$$|f(x)| = \lim_{n\to\infty} p_n(f^2(x))$$

for all x in X. Then, since X is compact, Dini's Theorem implies that (4.11) is true with uniform convergence.

Hence, we need only verify the claim that $\sqrt{a} = \lim_{n\to\infty} b_n$. To do this, note that

$$\sqrt{a} - b_{n+1} = \sqrt{a} - b_n - \frac{(\sqrt{a} - b_n)(\sqrt{a} + b_n)}{2} = (\sqrt{a} - b_n)\left(1 - \frac{\sqrt{a} + b_n}{2}\right) \ .$$

Since $a \leq 1$, as long as $b_n \leq \sqrt{a}$, the right hand side is non-negative, and therefore $b_{n+1} \leq \sqrt{a}$. Since $b_1 \leq \sqrt{a}$, it follows that $\sqrt{a}$ is an upper bound for the sequence $\{b_n\}_{n\in\mathbb{N}}$.

Now, knowing that $b_n^2 \leq a$ for all n, it is clear from the definition that $\{b_n\}_{n\in\mathbb{N}}$ is a monotone non-decreasing sequence. Therefore the limit $b = \lim_{n\to\infty} b_n$ exists and satisfies

$$b = b + \frac{a - b^2}{2} \ .$$

This means that $b^2 = a$, and since $b \geq 0$, $b = \sqrt{a}$. □

Proof of Theorem 4.6: Fix $x \neq y$ in X, and consider the linear transformation from $\mathcal{A}$ to $\mathbb{R}^2$ given by

$$f \mapsto (f(x), f(y)) \ .$$

The range of this linear transformation is a subspace S of $\mathbb{R}^2$.

Since $\mathcal{A}$ separates, there can be at most one point $x_0 \in X$ for which $g(x_0) = 0$ for all $g \in \mathcal{A}$.

Let us first assume first that neither x nor y is such a point. Since $\mathcal{A}$ is an algebra, and a vector space in particular, if g is in $\mathcal{A}$ so is very multiple of g. By assumption, there is some $g \in \mathcal{A}$ such that $g(x) \neq 0$, and by choosing an appropriate multiple, we may arrange that $g(x) = 1$.

Thus, S contains a vector of the form $(1, a)$. (Since $\mathcal{A}$ separates, we can choose $g \in \mathcal{A}$ so that $g(y) = a \neq 1$.)

Now there are two cases to consider. If also $a \neq 0$, then the two vectors $(1, a)$ and $(1, a^2)$ are linearly independent, and $(1, a^2)$ also belongs to S since $\mathcal{A}$ is an algebra (so that $g^2 \in \mathcal{A}$). On the other hand if $a = 0$ then S contains the vector $(1, 0)$, and, since there is some other g with $g(y) = 1$, there is some $b \in \mathbb{R}$ such that $(b, 1) \in S$. Hence in this case, S contains the two vectors $(1, 0)$ and $(b, 1)$ which are linearly independent. Either way, $S = \mathbb{R}^2$, and so we have proved that as long as $g(x) \neq 0$ and $h(y) \neq 0$ for *some* $g, h \in \mathcal{A}$, then S is all of $\mathbb{R}^2$.

This has the consequence that for *any* $f \in \mathcal{C}(X, \mathbb{R})$, we can find a function $f_{x,y} \in \mathcal{A}$ for which

$$(f(x), f(y)) = (f_{x,y}(x), f_{x,y}(y)) \ . \tag{4.12}$$

Now we have two cases once more: Suppose first that there is no point $x_0 \in X$ with $f(x_0) = 0$ for all $f \in \mathcal{A}$. Then the above argument applies for *all* x and y in X and all $f \in \mathcal{C}(X, \mathbb{R})$, we can find $f_{x,y} \in \mathcal{A}$ such that (4.12) is true. Moreover, by Lemma 4.8, $\mathcal{B}$ is a lattice. Therefore, by Lemma 4.7, f is a limit point of $\mathcal{B}$, and since $\mathcal{B}$ is closed, $f \in \mathcal{B}$. Since f is an arbitrary element of $\mathcal{C}(X, \mathbb{R})$, we see that in this case, $\mathcal{B} = \mathcal{C}(X, \mathbb{R})$.

The remaining case to consider is that in which there is one point x_0 such that $g(x_0) = 0$ for all $g \in \mathcal{A}$, and hence $\mathcal{B}$, so that $\mathcal{B}$ is certainly contained in the closed subset of $\mathcal{C}(X, \mathbb{R})$ consisting of continuous functions f on X such that $f(x_0) = 0$.

Let f be any such function. The argument made above show that as long as neither x nor y equals x_0, then there is some $g \in \mathcal{A}$, and hence $\mathcal{B}$, for which (4.12) is true. Now suppose that $x = x_0$, and $y \neq x_0$. Then we trivially have

$$f(x_0) = g(x_0) = 0$$

for all $g \in \mathcal{B}$. And since $\mathcal{A}$ separates, and is a vector space, we can choose g so that $f(y) = g(y)$. Therefore, for any $f \in \mathcal{C}(X, \mathbb{R})$ with $f(x_0) = 0$, no matter how x and y are chosen, we can can find $g_{x,y} \in \mathcal{B}$ so that (4.12) is true.

Then the argument made above shows that every $f \in \mathcal{C}(X, \mathbb{R})$ with $f(x_0) = 0$ is a limit point of $\mathcal{B}$, and hence belongs to $\mathcal{B}$. Therefore, in this second case, $\mathcal{B}$ is the subset of $\mathcal{C}(X, \mathbb{R})$ consisting of functions f with $f(x_0) = 0$. $\square$

In our proof of Theorem 4.6, we made use of the fact that our functions f were real valued, and not complex valued: The real numbers are ordered, while the complex numbers are not, and the order on the real numbers played a crucial role in the proof through our use of Lemma 4.7.

This is not simply an artifact of the proof: If in the statement of the theorem we replace $\mathcal{C}(X, \mathbb{R})$ by, $\mathcal{C}(X, \mathbb{C})$, the space of continuous complex valued functions on X, the statement becomes false.

To see this, take X to be the closed unit disc in the complex plane $\mathbb{C}$. Take $\mathcal{A}$ to be the algebra of all complex polynomials in the complex variable z, which clearly separates. Polynomials in z

are analytic, and uniform limits of analytic functions are analytic, and so the closure of $\mathcal{A}$ consists of functions that are analytic in the interior of the the unit disc. Obviously, not every continuous function of the closed unit disc is analytic in the interior of the disc; $f(z) = z^*$, the complex conjugate of z, is an example. Hence, the uniform closure of $\mathcal{A}$ is not the full set of continuous complex valued functions on the closed unit disc.

However, under one simple additional condition on the algebra $\mathcal{A}$, one can reduce the complex valued case to the real case.

A (complex) subalgebra $\mathcal{A}$ of the algebra of complex valued function on a compact Hausdorff space is called a *$*$-algebra* in case it is closed under complex conjugation. That is, whenever $f \in \mathcal{A}$, then $f^* \in \mathcal{A}$, where f^* is the function defined by $f^*(x) = (f(x))^*$ for all $x \in X$.

In this case, for every $f \in \mathcal{A}$, the real and imaginary parts of f both belong to $\mathcal{A}$. It is also easy to see that when $\mathcal{A}$ separates, so does the real algebra consisting of the real and imaginary parts of functions in $\mathcal{A}$. Applying the Stone-Wierstrass Theorem to this algebra, one can separately approximate, in the uniform metric, the real and imaginary parts of any continuous complex valued function on X by functions in $\mathcal{A}$.

In summary, we have:

4.9 THEOREM (Complex Stone-Wierstrass)**.** *Let X be a compact topological space, and let $\mathcal{A}$ be a subset of $\mathcal{C}(X,\mathbb{C})$ that is a separating $*$-algebra. Let $\mathcal{B}$ be the uniform closure of $\mathcal{A}$. Then either $\mathcal{B} = \mathcal{C}(X,\mathbb{C})$, or else $\mathcal{B}$ consists of all continuous functions on X that vanish at some fixed point x_0. In particular, if $\mathcal{A}$ contains the constant functions, $\mathcal{B} = \mathcal{C}(X,\mathbb{R})$.*

Here is one important application of Theorem 4.9: Let X be the unit circle in $\mathbb{C}$, with its usual topology. Let $\mathcal{A} \subset \mathcal{C}(x,\mathbb{C})$ be the set consisting of functions f of the form

$$f(z) = \sum_{j=-n}^{n} a_j z^n$$

for some $n \in \mathbb{N}$, and some numbers $a_{-n}, \ldots, a_n$ in $\mathbb{C}$. (Each element of X is a complex number z, and z^n denotes the nth power of z.) The elements of $\mathcal{A}$ are called *complex trigonometric polynomials*

It is easy to see that $\mathcal{A}$ is a $*$-algebra, and that $\mathcal{A}$ separates. Hence, by Theorem 4.9, $\mathcal{A}$ is dense in $\mathcal{C}(X,\mathbb{C})$. This proves:

4.10 THEOREM (Density of Complex Trigonometric Polynomials)**.** *Let X be the unit circle in $\mathbb{C}$, with its usual topology. Then the set of complex trigonometric polynomials is dense in $\mathcal{C}(X,\mathbb{C})$, with respect to the uniform metric.*

4.4 Tychonoff's Theorem

Let X be a set. The *Cartesian product of X with itself*, $X \times X$, is the set of all ordered pairs (x_1, x_2) of elements of X. Of course (x_1, x_2) is the graph of a unique function $f : \{1,2\} \to X$, namely the one with $f(1) = x_1$ and $f(2) = x_2$. (One can accommodate Cartesian products of two different sets Y and Z in this framework by considering $X = Y \cup Z$ and restricting attention to functions f such that $f(1) \in Y$ and $f(2) \in Z$. No real generality is lost in taking the sets to be the same, and the notation is much simpler, so that is how we shall proceed.)

More generally, given any set non-empty S, the Cartesian product of X indexed by S, denotes X^S, is the set of all functions from S to X. For example, $X^{\mathbb{N}}$ is the set of all infinite sequences $\{x_n\}_{n\in\mathbb{N}}$ of elements of X.

On any Cartesian product, there is a natural family of functions with values in X, namely the *coordinate functions*: For each $s \in S$, define

$$\varphi_s : X^S \to X$$

by

$$\varphi(f) = f(s) \ .$$

That is, one simply evaluates the function $f \in X^S$ at s.

Note that when $S = \{1, 2\}$, $\varphi_j((x_1, x_2)) = x_j$, which is why the φ_s are called coordinate functions.

4.3 DEFINITION. *Let $(X, \mathcal{O})$ be a topological space, and S and arbitrary set. The product topology on the Cartesian product X^S is the topology generated by the coordinate functions. That is, it is the weakest topology for which each of the coordinate functions is continuous.*

Now suppose that $(X, \mathcal{O})$ is a compact topological space. When is $(X, \mathcal{O})$ compact in the product topology? The answer, given by Tychonov's theorem is: "Always."

4.11 THEOREM (Tychonoff's Theorem)**.** *Let $(X, \mathcal{O})$ be a compact topological space, and S any non-empty set. Then X^S, equipped with the product topology, is compact.*

The special case of this theorem in which X is a compact metric space and S is countable (or finite) is fairly easy to prove using the theorems presented so far in these notes. This is developed in the exercises that follow. The general case involves either the theory of "nets" or the theory of "filters", and this would be a digression, since we shall not invoke the general case in this course, nor shall we have any other occasion to use the theory of nest of filters. Furthermore, the proof of the general case involves the axiom of choice in a much more subtle way than does the spacial case. This is not a problem, but discussion of these subtleties would take us far afield. (The axiom of choice enters the subject, even in the special case, in an essential way: It is the axion of choice which assures us that X^S is non-empty: one can always choose, for each $x \in s$, some $x(s) \in X$. Moreover, it is known that Tychanov's Theorem is logically equivalent to the Axiom of Choice.)

It is well worth knowing the general case nonetheless. It shows that advantage of the 20th century notion of *compactness*, as defined above, in terms of open covers, and the 19th century notion of sequential compactness. As shown in the exercises, if we take $X = [0, 1]$ with its usual topology, and equip X^X, the set of all functions from $[0, 1]$ to $[0, 1]$, then X^X is not sequentially compact, but is compact *by Tychonov's Theorem.* Many theorems in which compactness is an hypothesis remain true if this hypothesis is replaced by sequential compactness (see the exercises). Tychonov's Theorem is an important example for which this is not the case.

5 Exercises

1. Let (X, d) be a separable metric space. Let Y be any subset of X, and define d_Y to be the restriction of d to $Y \times Y$. Show that (Y, d_Y) is separable.

2. Suppose that (X, d) is a complete metric space with a finite diameter; i.e., there exists $D < \infty$ such that $d(x, y) \leq D$ for all $x, y \in X$. Is it true that every continuous real valued function on X is bounded? Prove this assertion or give a counterexample.

3. Let (X, d) be a compact metric space.

(a) Show that if $f : X \to X$ is continuous but not onto, there is some $x_0 \in X$ and some $r > 0$ so that $d(f(x), x_0) \geq r$ for all $x \in X$.

(b) Let f be an isometry from X into itself; i.e., a function with the property that

$$d(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$. Show that f is necessarily one to one and onto, and hence invertible.

4. Let (X, d) be a compact metric space, and let $f : X \to \mathbb{C}$ be continuous. Show that for all $\epsilon > 0$, there exists $L < \infty$ so that

$$|f(x) - f(y)| \leq Ld(x, y) + \epsilon$$

for all $x, y \in X$.

5. (a) Let (X, d) be a complete metric space in which bounded sets are totally bounded. Let $A \subset X$ be closed and $B \subset X$ be compact. Show that there exist $x_1 \in A$ and $x_2 \in B$ such that

$$d(x_1, x_2) \leq d(x, y) \qquad \text{for all } x \in A\ , y \in B\ .$$

(b) Show by example that this is false if we weaken the assumption to only suppose that B is closed.

6. Define ℓ_1 to be the set of complex valued sequences $\{x_j\}_{j\in\mathbb{N}}$ such that $\displaystyle\sum_{j=1}^{\infty} |x_j| < \infty$. Define a function on d_{ℓ_1} on ℓ_2 by

$$d_{\ell_1}(\{x_j\}\ ,\ \{y_j\}) = \sum_{j=1}^{\infty} |x_j - y_j|\ .$$

(a) Show that (ℓ_1, d_{ℓ_1}) is a metric space.

(b) Show that the metric space (ℓ_1, d_{ℓ_1}) is complete.

7. Let (ℓ_2, d_{ℓ_2}) Show that a bounded subset X of ℓ_2 is totally bounded if and only if for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\sum_{k>N_\epsilon} |x_j|^2 < \epsilon^2$$

for all $\{x_j\} \in X$.

8. Let (ℓ_1, d_{ℓ_1}) be defined as in Exercise 6. Show that $X \subset \ell_1$ is totally bounded if and only if for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\sum_{k>N_\epsilon} |x_j| < \epsilon$$

for all $\{x_j\} \in X$. Then show that $B_1(\{0\})$, the ball of radius 1 about the zero sequence, is not totally bounded, and hence that the closed ball of radius 1 about the zero sequence is not compact.

9. Let $X = [0,1]$. Each $x \in X$ has a binary expansion

$$x = \sum_{n=1}^{\infty} b_n(x) 2^{-n}$$

with each $b_n(x) \in \{0,1\}$. We stipulate that if x is a dyadic rational, only finitely many of the $b_n(x)$ are non-zero, and under this condition, the $b_n(x)$ are uniquely determined, so that $b_x : X \to \{0,1\} \subset X$ is a well-defined function for each n.

(a) Show that no subsequence of $\{b_n\}_{n\in\mathbb{N}}$ converges pointwise.

(b) Equip X^X with its product topology and note that each b_n is a function from X to X, and hence is an element of X^X. Show that no subsequence of $\{b_n\}_{n\in\mathbb{N}}$ converges in the product topology, and thus that the analog of Tychonov's Theorem for sequential compactness is false.

10. Let (X,d) be a compact metric space. Then $X^{\mathbb{N}}$ consists of all sequences $\{x_k\}_{k\in\mathbb{N}}$ in X. Define a function d on $X^{\mathbb{N}} \times X^{\mathbb{N}}$ by

$$d(\{x_k\}_{k\in\mathbb{N}}, \{y_k\}_{k\in\mathbb{N}}) = \sum_{k=1}^{\infty} 2^{-k} d(x_k, y_k) \ .$$

(a) Show that d is a metric on $X^{\mathbb{N}} \times X^{\mathbb{N}}$.

(b) Show that the metric topology in $X^{\mathbb{N}}$ induced by d is at least as strong as the product topology.

(c) Show that with the metric topology induced by d, $X^{\mathbb{N}}$ is sequentially compact.

(d) Show directly, without invoking Tychonov's Theorem that $X^{\mathbb{N}}$ compact in the product topology.

11. Let (X,d_X) and (Y,d_Y) be two compact metric spaces. Let $\mathcal{C}(X\times Y,\mathbb{R})$ be the set of all valued functions on $X \times Y$ continuous real that are continuous with respect to the product topology. Let $\mathcal{A}$ be the set of functions f on $X \times Y$ of the form

$$f(x,y) = \sum_{j=1}^{n} g_j(x) h_j(y)$$

for some $n \in \mathbb{N}$ and some $\{g_1,\dots,g_n\} \subset \mathcal{C}(X,\mathbb{R})$ and some $\{h_1,\dots,h_n\} \subset \mathcal{C}(Y,\mathbb{R})$. Show that $\mathcal{A}$ is dense in $\mathcal{C}(X\times Y,\mathbb{R})$ in the uniform topology. (Note: The usual notation for $\mathcal{A}$ is $\mathcal{C}(X,\mathbb{R})\otimes\mathcal{C}(Y,\mathbb{R})$, and it is called the tensor product of $\mathcal{C}(X,\mathbb{R})$ and $\mathcal{C}(Y,\mathbb{R})$.)

12. A topological space is *locally compact* in case every point has a neighborhood whose closure is compact. Let (X,d_X) and (Y,d_Y) be locally compact metric spaces, and suppose that $f : X \to Y$ is continuous and bijective. Show that f^{-1} is continuous if and only if $f^{-1}(K)$ is compact for all compact $K \subset Y$.

13. Let $(X,\mathcal{O})$ be a compact topological space. Let A and B be non-empty closed and disjoint subsets of X. Suppose that for every $b \in B$, there exist a continuous function $f_b : X \to [0,1]$ such that $f_b(b) = 1$ and $f_b(a) = 0$ for all $a \in A$. Show that there exist open sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

14. Let $(X,\mathcal{O})$ be a compact topological space, and let $\mathcal{F}$ be a set of functions real valued on X that is equicontinuous and uniformly bounded. Define

$$g(x) = \sup_{f\in\mathcal{F}} f(x) \ .$$

Is $g(x)$ necessarily continuous? Prove that your answer is correct.

15. Let (X, d_X) and (Y, d_Y) be metric spaces with Y complete. For $L \in (0, \infty)$, a function $f : X \to Y$ is L-Lipschitz in case $d_Y(f(x), f(y)) \leq Ld_X(x, y)$ for all $x, y \in X$.

Let S be a dense subset of X. Let $g : S \to Y$ satisfy

$$d_Y(g(x), g(y)) \leq Ld_X(x, y) \quad \text{for all} \quad x, y \in S\ .$$

Show that there exists a unique L-Lipschitz function $f : X \to Y$ such that the restriction of f to S is g.

16. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous real valued functions on $[0, 1]$ that are continuously differentiable on $(0, 1)$. Suppose that $f_n(0) = 0$ for all n and that there is a continuous function $g : [0, 1] \to [0, \infty)$ such that $|f_n'(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and all $x \in (0, 1)$. Show that there exists a uniformly convergent subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$.

17. Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $f : X \to Y$ be continuous uand surjective, and supose that

$$d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2))$$

for all $x_1, x_2 \in X$.

(a) If (X, d_X) complete, must (Y, d_Y) be complete? Prove this or give a counterexample.

(b) If (Y, d_Y) complete, must (X, d_X) be complete? Prove this or give a counterexample.

18. Let (X, d) be a metric space. If every real-vauled conttinuous function f on X has a maximum, does this mean that X is compact? Prove your answer is correct.

19. Let (X, d) be a compact metric space. Ptove that if $\{U_1, \dots, U_k\}$ is an open cover of X, there there exists a closed cover $\{C_1, \dots, C_k\}$ with $C_j \subset U_j$ for $j = 1, \dots, k$.

20. Let A and B be compact subsets of a Hausdorff topological space. Prove that there exist open sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

21. Let (X, d_X) and (Y, d_Y) be metric spaces and let $F : X \to Y$ be a continuous functions such that f maps closed sets to closed sets and such that the inverse image of any point in Y is compact. Show that $f^{-‘}(K)$ is compact whenever K is compact.

22. Ler $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that f_n converges uniformly to f. Must f be uniformly continuous?