Notes on Topological Vector Spaces for Math 502

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Contents

1 Topological Vector Spaces

1.1 Neighborhood bases for topological vector spaces

1.1 DEFINITION (Topological Vector Space)**.** A *topological vector space* is a vector space X over $\mathbb{C}$ that is equipped with a topology of open sets $\mathscr{O}$ such that the maps $(\alpha, x) \mapsto \alpha x$ and $(x, y) \mapsto x + y$ are continuous on $\mathbb{C} \times X$ and $X \times X$ respectively. A *real topological vector space* is defined in the same way except that the field $\mathbb{C}$ replaced by $\mathbb{R}$.

Let X be a vector space. For all $x \in X$ and $A, B \subset X$, define

$$x + B := \{x + y \ : \ y \in B\} \quad \text{and} \quad A + B := \{x + y \ : \ x \in A, y \in B\} \ . \tag{1.1}$$

Then $A + B$ is said to be the *Minkowski sum of* A *and* B. For $x \in X$, let T_x denote the map from X to X given by $T_x(y) = x + y$. Then $T_x^{-1} = T_{-x}$, and T_x vector space isomorphism from on X. The maps T_x, $x \in X$, are called *translations*.

Likewise, for $\alpha \in \mathbb{C}\backslash\{0\}$ let S_α denote the map form X to X given by $x \mapsto \alpha x$. Since $S_\alpha^{-1} = S_{\alpha^{-1}}$, each S_α is a vector space isomorphism on X. The maps S_α, $\alpha \in \mathbb{C}\backslash\{0\}$ are called *scale transformations*. For all $\alpha \in \mathbb{C}$ and all $A \subset X$, define

$$\alpha A := \{\alpha x \ : \ x \in A\} \ . \tag{1.2}$$

Whenever $(X, \mathscr{O})$ is a topological vector space, both T_x and T_x^{-1} are continuous. Hence each $T_x, x \in X$, is a homeomorphism on X. Likewise, each S_α, $\alpha \in \mathbb{C}\backslash\{0\}$, is a homeomorphism on X. It follows that $U \subset X$ is open if and only if $x + U = T_x(U)$ is open for each $x \in X$. If U is any non-empty set, and $-x \in U$, then $0 \in x + U = T_x(U)$. Therefore, *the sets in $\mathscr{O}$ are precisely the translates of the sets in $\mathscr{O}$ that contain* 0. Likewise, $U \subset X$ is open if and only $S_\alpha(U) \in \mathscr{O}$ for all $\alpha \in \mathbb{C}\backslash\{0\}$.

1.2 DEFINITION (Neigborhood base at 0)**.** Let $(X, \mathscr{O})$ be a topological vector space. A *neigborhood base at* 0 for the topology $\mathscr{O}$ is a set $\mathscr{V} \subset \mathscr{O}$ such that $0 \in V$ for all $V \in \mathscr{V}$, and if $0 \in U \in \mathscr{O}$, there exists some $V \in \mathscr{V}$ such that $V \subset U$.

A given topology $\mathscr{O}$ on X such that $(X, \mathscr{O})$ is a topological vector space will have infinitely many neighborhood bases at 0, but given a neighborhood base $\mathscr{V}$ at 0 for $(X, \mathscr{O})$, there does not exist any different topology $\widetilde{\mathscr{O}}$ such that $(X, \widetilde{\mathscr{O}})$ is a topological vector space and $\mathscr{V}$ is a neighborhood base at 0 for $\widetilde{\mathscr{O}}$.

1.3 LEMMA. *$(X, \mathscr{O})$ be a topological vector space, and let $\mathscr{V}$ be a neighborhood base at 0 for the topology $\mathscr{O}$. Then a non-empty $U \subset X$ is open if and only if for each $x \in U$, there is some $V \in \mathscr{V}$ such that $x + V \subset U$.*

Proof. Suppose that for each $x \in U$, there is some $V_x \in \mathscr{V}$ such that $x + V_x \subset U$. Then each $x + V_x$ is open and writing $U = \bigcup\{x + V_x \ : x \in U\}$ displays U as a union of open sets. Hence $U \in \mathscr{O}$.

Conversely, suppose that $U \in \mathscr{O}$. Then for each $x \in U$, $-x + U$ is an open set containing 0. Hence for some $V_x \in \mathscr{V}$, $V_x \subset -x + U$. But then $x + V_x \subset U$. □

There is a useful construction of topologies $\mathscr{O}$ on a vector space X such that $(X, \mathscr{O})$ is a topological vector space. The construction is closely related to the previous lemma.

1.4 LEMMA. *Let X be a vector space, and let $\mathscr{W}$ be a family of subsets of X that is closed under finite intersections.*

Define $\mathscr{O}$ to be the set of subsets U of X given by

$$\mathscr{O} = \emptyset \cup \{U \subset X \ : \ \text{for all } x \in U \text{ there exists } W_x \in \mathscr{W} \text{ such that } x + W_x \subset U\} \ . \tag{1.3}$$

Then $\mathscr{O}$ is a topology on X, and under this topology, each of the maps T_x, $x \in X$, is continuous.

Proof. Evidently, $\emptyset \in \mathscr{O}$, and evidently an arbitrary union of sets U such that for each $x \in U$, there is some $W_x \in \mathscr{W}$ such that $x + W_x \subset U$ also has this same property. Hence $\mathscr{O}$ is closed under arbitrary unions.

To see that $\mathscr{O}$ is closed under finite intersections, let $\{U_1, \dots, U_n\} \subset \mathscr{O}$, and let $U = \cap_{j=1}^n U_j$. By definition, for each $x \in U$, $x \in U_j$, $j = 1, \dots, n$, and hence there is some $W_{x,j}$ such that $x + W_{x,j} \subset U_j$. But then

$$x + \bigcap_{j=1}^n W_{x,j} \subset U \ ,$$

and by the closure under finite intersections, $\cap_{j=1}^n W_{x,j} \in \mathscr{W}$, and hence $U \in \mathscr{O}$.

For the final statement, since $T_x^{-1} = T_{-x}$, it suffices to show that each T_x is open. By definition, the general open set U has the form $U = \cup\{y + W_y \ : y \in U\}$ for some set $\{W_y \ : \ y \in U\} \subset \mathscr{W}$. Bu then $T_x(U) = \cup\{x + y + W_y \ : y \in U\}$ which is open. □

The condition that $(x, y) \mapsto x + y$ is continuous on $X \times X$ in the product topology is *stronger* than the condition that each of the maps T_x are continuous. The latter condition amounts to *separate continuity* of the map $(x, y) \mapsto x + y$, and for $(X, \mathscr{O})$ to be a topological vector space, we require *joint continuity.* To achieve this, and to ensure that the topology $\mathscr{O}$ provided by Lemma 1.4 is Hausdorff and has the other properties that would make $(X, \mathscr{O})$ is a topological vector space, we must impose further conditions on the sets in $\mathscr{W}$.

1.5 DEFINITION. Let X be a vector space over $\mathbb{C}$. a set $A \subset X$ is *absorbing* if for all $x \in X$, there is a $\delta_x > 0$ so that for $|t| < \delta_x$, $tx \in A$, or, what is the same thing, that $x \in tA$ for all $t > 1/\delta_x$. A set $A \subset X$ is *balanced* in case for all $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, $\alpha A \subset A$. (Hence every balanced set contains 0.)

1.6 LEMMA. *Let X be a vector space, and let $\mathscr{W}$ be a family of subsets of X such that that is closed under finite intersections, and let $\mathscr{O}$ be the topology on X defined in Lemma 1.4. Suppose that each $W \in \mathscr{W}$ is absorbing, balanced and convex, and that for all $W \in \mathscr{W}$, $\frac{1}{2}W$ belongs to $\mathscr{W}$. Then the map $(\alpha, x) \mapsto \alpha x$ is jointly continuous on $(\mathbb{C}\backslash\{0\}) \times (X, \mathscr{O})$, and the map $(x, y) \mapsto x + y$ is jointly continuous on $X \times X$.*

Proof. Let $\alpha_0 \in \mathbb{C}\backslash\{0\}$ and $x \in X$. It suffices to show that if $U \in \mathscr{O}$ and $\alpha_0 x \in U$, then there is a $\delta > 0$ and a $\widetilde{W} \in \mathscr{W}$ such that if $|\alpha - \alpha_0| < \delta$, then $\alpha(x + \widetilde{W}) \subset U$.

Since $\alpha_0 x \in U \in \mathscr{O}$, there is some $W \in \mathscr{W}$ such that $\alpha_0 x + W \subset U$. Then

$$\alpha x + \frac{1}{2}W = (\alpha - \alpha_0)x + (\alpha_0 x + \frac{1}{2}W) \ .$$

Since W is absorbing and balanced, for δ sufficiently small, if $|\alpha - \alpha_0| < \delta$, then $(\alpha - \alpha_0)x \in \frac{1}{2}W$. Then since W is convex, $\alpha x + \frac{1}{2}W \subset \frac{1}{2}W + (\alpha_0 x + \frac{1}{2}W) \subset \alpha_0 x + W \subset U$. Taking $\widetilde{W} = \frac{1}{2}W$, we have what we sought for the scalar multiplication. The vector additions is simpler: Let $x, y \in X$, and let $U \in \mathscr{O}$ be such that $x + y \in U$. Then there exists $W \in \mathscr{W}$ such that $x + y + W \subset U$. But then $(x + \frac{1}{2}W) + (y + \frac{1}{2}W) \subset U$. □

1.7 LEMMA. *Let X be a vector space, and let $\mathscr{W}$ be a family of subsets of X such that that is closed under finite intersections, and let $\mathscr{O}$ be the topology on X defined in Lemma 1.4. Suppose that each $W \in \mathscr{W}$ is absorbing, convex and balanced, and that for all $W \in \mathscr{W}$, $\frac{1}{2}W$ belongs to $\mathscr{W}$. Suppose also that for all $x \neq 0$, there exists $W \in \mathscr{W}$ such that $x \notin W$. Then $(X, \mathscr{O})$ is a Hausdorff topological space.*

Proof. Let $x, y \in X$, $x - y \neq 0$. Then there exists $W \in \mathscr{W}$ such that $x - y \notin W$. Then $x + \frac{1}{2}W$ is an open set containing x, and $y + \frac{1}{2}W$ is an open set containing y. If $z \in (x + \frac{1}{2}W) \cap (y + \frac{1}{2}W)$ there exist $w_x, w_y \in W$ such that $z = x + \frac{1}{2}w_x = y + \frac{1}{2}w_x$, and hence $x - y = \frac{1}{2}(w_x - w_y)$ Since W is balanced, $-w_y \in W$, and then since W is convex, $\frac{1}{2}(w_x - w_y) \in W$. Hence $x - y \in W$, which is a contradiction showing that no such z exists, and hence $(x + \frac{1}{2}W) \cap (y + \frac{1}{2}W) = \emptyset$. Thus, $(X, \mathscr{O})$ is a Hausdorff topological space. □

Combining Lemmas 1.4, 1.6 and 1.7, we have proved:

1.8 THEOREM. *Let X be a vector space, and let $\mathscr{W}$ be a family of subsets of X such that that is closed under finite intersections. Suppose further that: each $W \in \mathscr{W}$ is absorbing, convex and balanced, and that for all $W \in \mathscr{W}$, $\frac{1}{2}W$ belongs to $\mathscr{W}$. Let $\mathscr{O}$ be defined by*

$$\mathscr{O} = \emptyset \cup \{U \subset X \ : \ \text{for all } x \in U \text{ there exists } W_x \in \mathscr{W} \text{ such that } x + W_x \subset U\} \ .$$

Then $(X, \mathscr{O})$ is a topological vector space, and $\mathscr{W}$ is a neighborhood base at 0 for this topology. Moreover, if for all $x \neq 0$, there exists $W \in \mathscr{W}$ such that $x \notin W$, then $(X, \mathscr{O})$ is a Hausdorff topological vector space.

Let X be a vector space and let $\mathscr{V}$ be a set of of absorbing, convex and balanced subsets of X. Since any finite intersection of absorbing, convex and balanced sets is again absorbing, convex and balanced, if we define $\mathscr{W}$ to be the set of all finite intersections of sets in $\mathscr{V}$, then by Theorem 1.8, $\mathscr{W}$ is a neighborhood base at 0 of a uniquely determined topology $\mathscr{O}$ on X such that $(X, \mathscr{O})$ is a topological vector space.

The vector spaces topologies that we consider below are always generated this way, and we will often need to determine when two such topologies are comparable. The following lemma facilitates this.

1.9 LEMMA. *Let X be a vector space. Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be two sets of absorbing, convex and balanced subsets of X, both also closed under multiplication by 2^{-k}, $k \in \mathbb{N}$. Let $\mathscr{W}_1$ and $\mathscr{W}_2$ be the sets of all finite intersections of sets in $\mathscr{V}_1$ and $\mathscr{V}_2$ respectively. Let $\mathscr{O}_1$ and $\mathscr{O}_2$ be the topological vector space topologies on X that have $\mathscr{W}_1$ and $\mathscr{W}_2$ respectively as neighborhood bases at 0. Then $\mathscr{O}_1 \subset \mathscr{O}_2$ if each $V_1 \in \mathscr{V}_1$ contains some $V_2 \in \mathscr{V}_2$.*

Proof. Let $U \in \mathscr{O}_1$. By definition, for each $x \in U$, there exists $W_x \in \mathscr{W}_1$ such that $x + W_x \subset U$. Suppose that each $W \in \mathscr{W}_1$ contains some $\widetilde{W} \in \mathscr{W}_2$. In particular, writing $U = \cup_{x \in U}\{x + W_x\}$, and letting $\widetilde{W}_x$ denote some element of $\mathscr{W}_2$ contained in W_x, we have $U = \cup_{x \in U}\{x + \widetilde{W}_x\}$. Thus $U \in \mathscr{O}_2$.

Therefore, it suffice to show that whenever each $V_1 \in \mathscr{V}_1$ contains some $V_2 \in \mathscr{V}_2$, then each $W_1 \in \mathscr{W}_1$ contains some $W_2 \in \mathscr{W}_2$.

Let $W_1 \in \mathscr{W}_1$. Then W_1 is a finite intersection of elements of $\mathscr{V}_1$. Each of these contains a set in $\mathscr{V}_2$. The intersection over the later sets belongs to $\mathscr{W}_2$, and is contained in $\mathscr{W}_1$. □

1.2 Seminorms, norms and normed vector spaces

1.10 DEFINITION. Let X be a vector space over $\mathbb{C}$ or $\mathbb{R}$. A function $f : X \to \mathbb{R}$ is *convex* in case for all $\lambda \in (0, 1)$ and all $x, y \in X$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \ , \tag{1.4}$$

and is *subadditive* in case for all $x, y \in X$,

$$f(x + y) \leq f(x) + f(y) \ , \tag{1.5}$$

and is *homogenous of degree one* in case for all $x \in X$ and $\alpha \in \mathbb{C}$

$$f(\alpha x) = |\alpha| f(x) \ . \tag{1.6}$$

1.11 LEMMA. *Let X be a vector space over $\mathbb{C}$ or $\mathbb{R}$ and let $f : X \to \mathbb{R}$ be homogeneous of degree one. Then f is convex if and only if f is subadditive.*

Proof. Suppose that f is homogeneous of degree one and convex. Then

$$f(x+y) = 2f\left(\frac{x+y}{2}\right) \leq 2\left(\frac{1}{2}f(x) + \frac{1}{2}f(y)\right) = f(x) + f(y) \ .$$

Conversely, Suppose that f is homogeneous of degree one and subadditive. Then for $\lambda \in (0,1)$

$$f((1-\lambda)x + \lambda y) \leq f((1-\lambda)x) + f(\lambda y) = (1-\lambda)f(x) + \lambda f(x) \ .$$

□

1.12 DEFINITION (Seminorm and norm)**.** Let X be a vector space over $\mathbb{C}$ or $\mathbb{R}$. A *seminorm* on X is a function $p : X \to [0,\infty)$ such that p is homogeneous of degree one and convex, or, what is the same thing, homogeneous of degree one and subadditive. A seminorm p is a *norm* in case $p(x) = 0$ implies that $x = 0$. A different notation, namely $x \mapsto \|x\|$ is usually used for norm functions, also sometimes for seminorms.

1.13 LEMMA. *Let p be a seminorm on the vector space X. Define the set B_p by*

$$B_p := \{x \in X \ \ p(x) \leq 1\} \ . \tag{1.7}$$

Then B_p is absorbing, balanced and convex. Moreover,

$$B_p = \cap_{r>1} rB_p \ . \tag{1.8}$$

Proof. For $x \in X$, $\lambda > 0$, $p(\lambda x) = \lambda p(x) < 1$, so that $\lambda x \in B_p$, for all $\lambda < 1/p(x)$. Therefore, B_p is absorbing. For $x \in B_p$, $\alpha \in \mathbb{C}$, $|\alpha| \leq 1$, $p(\alpha x) = |\alpha| p(x) \leq 1$, so that $\alpha x \in B_p$. Therefore, B_p is balanced. For all $x, y \in B_p$, and all $\lambda \in (0,1)$, $p((1-\lambda)x + \lambda y) \leq (1-\lambda)p(x) + \lambda p(y) \leq 1$. Therefore, B_p is convex.

To prove (1.8), note that $x \in rB_p$ if and only if $r^{-1}x \in B_p$ if and only if $p(r^{-1}x) \leq 1$. By the homogeneity of p, $p(r^{-1}x) \leq 1$ is the same as $p(x) \leq r$. Hence $x \in \cap_{r>1} rB_p$ if and only if $p(x) \leq r$ for all $r > 1$, and this means that $p(x) \leq 1$. □

It turns out that there is a one-to-one correspondence between absorbing, balanced, convex sets V in X with the property that $V = \cap_{r>1} rV$, and seminorms p on X, as the lemmas show.

1.14 LEMMA. *Let V be any absorbing, balanced, convex set in X. Define a function $p_V : X \to [0,\infty)$ by*

$$p_V(x) = \inf\{t > 0 \ : \ x \in tV\} \ . \tag{1.9}$$

Then p_V is a seminorm on X.

Proof. Since V is absorbing, $p_V(x) < \infty$ for all $x \in X$. Next, let $x, y \in X$. Let t, s be such that $x \in tV$ and $y \in sV$. That is, $t^{-1}x, s^{-1}y \in V$. Since V is convex, $(1-\lambda)t^{-1}x + \lambda s^{-1}y \in V$ for all $\lambda \in (0,1)$. Define $\lambda = s/(t+s)$. Then we have $x+y \in (t+s)V$. Therefore, $p_V(x+y) \leq p_V(x) + p_V(y)$.

Let $\alpha \in \mathbb{C}$, $\alpha = 1$, Since V is balanced, $S_\alpha V \subset V$, and then $V \subset S_\alpha(S_{\overline{\alpha}}V) \subset S_\alpha V$. Thus $V = \alpha V$. Therefore, for all $x \in X$ and all $t > 0$,

$$\alpha x \in tV \iff x \in t\overline{\alpha}V \iff x \in tV \ .$$

Therefore, $p_V(\alpha x) = p_V(x)$ when $|\alpha| = 1$. Also, for all $r, t > 0$,

$$t(rx) \in V \iff (tr)x \in V$$

and therefore $p_V(rX) = rp_V(x)$. Then for $r > 0$ and $\alpha \in \mathbb{C}$, $\alpha = 1$, for all $x \in X$,

$$p_V(r\alpha x) = rp_V(\alpha x) = rp_V(x) = |r\alpha|p_v(x) \ .$$

This shows that p_V is homogeneous of degree one. Altogether, we have shown that p_V is a seminorm.

□

1.15 LEMMA. *Let V be any absorbing, balanced, convex set in X. Suppose also that $V = \cap_{r>1} rV$. Let p_V be the seminorm defined by (1.9), and then let B_{p_V} be the absorbing, balanced, convex set defined by (1.7) with p_V in place of V. Then $V = B_{p_V}$.*

Proof. If $x \in V$, then $p_V(x) \leq 1$, and hence $x \in B_{p_V}$. If $x \in B_{p_V}$, $p_V(x) \leq 1$, and then $x \in rV$ for all $r > 1$. Since $V = \cap_{r>1} rV$, $x \in V$. Thus, $V = B_{p_V}$. □

1.16 LEMMA. *Let $V \subset X$ be absorbing, balanced and convex, and let p_V be the seminorm defined in (1.9). Then p_V is a norm if and only if for all non-zero $x \in X$, there is some $t > 0$ such that $x \notin tV$*

Proof. If for all non-zero $x \in X$, there is some $t_0 > 0$ such that $x \notin tV$, $p_V(x) = \inf\{t > 0 \ : \ x \in tV\} \geq t_0$, and hence $p_V(x) > 0$ for all $x \neq 0$. Conversely, if $p_V(x) > 0$ for all $x \neq 0$, then for all x and all $0 < t < p_V(x)$, $x \notin tV$. □

Summarizing, we have the following reuslt:

1.17 THEOREM. *Let X be a vector space over $\mathbb{C}$. For each absorbing, balanced, convex set V in X, the function p_V defined by (1.9) is a seminorm. Conversely, for each seminorm p, the set is absorbing, balanced and convex. Moreover the map $p \mapsto B_p$, with B_p defined in (1.7), is a bijection between the set of seminorms on X and the set of absorbing, balanced, convex sets $V \subset X$ with the property that $V = \cap_{r>1} rV$. Finally, for any absorbing, balanced and convex $V \subset X$, the seminorm p_V defined in (1.9) is a norm if and only if for all non-zero $x \in X$, there is some $t > 0$ such that $x \notin tV$.*

As a consequence of Theorem 1.8 and Theorem 1.17, there is a topology $\mathscr{O}$ on a vector space X associated to any set $\mathscr{P}$ of seminorms p on X, and $(X, \mathscr{O})$ is a topological vector space such that for each $p \in \mathscr{P}$ and each $\epsilon > 0$, $\epsilon B_p \in \mathscr{O}$, and consequently, such that each $p \in \mathscr{P}$ is continuous.

Let p be any seminorm on the vector space X. Consider the nested family of absorbing, balanced, convex sets $\mathscr{W} = \{2^k B_p \ : \ k \in \mathbb{Z}\}$. Since this set is nested, it is closed under finite intersections. Then by Theorem 1.8, the set $\mathscr{O}$ defined by (1.3) is a topology on X such that $(X, \mathscr{O})$ is a topological vector space. This topology is Hausdorff if and only if p is a norm.

To get a Hausdorff topology out of seminorms, one must use a sufficiently large family of them. Let $\mathscr{P}$ be a set of seminorms on X. Define

$$\mathscr{W}_{\mathscr{P}} := \{2^k B_p \ : \ k \in \mathbb{Z} \ , \ p \in \mathscr{P}\} \ . \tag{1.10}$$

Then $\mathscr{W}$ is closed under finite intersections and each set $W \in \mathscr{W}$ is balanced, convex and absorbing and by Theorem 1.8, the set $\mathscr{O}$ defined by (1.3) is a topology on X such that $(X, \mathscr{O})$ is a topological vector space. This topology is Hausdorff if and only if for each $x \in$, there is some $p \in \mathscr{P}$ such that $p(x) \neq 0$.

1.18 DEFINITION. Let $\mathscr{P}$ be a family of seminorms on a vector space X. The weakest topology on X that contains all translates of all of the sets in $\mathscr{W}_{\mathscr{P}}$, as defined in (1.10) is called the topology on X generated by $\mathscr{P}$.

If $\mathscr{P}$ is a *countable* set of seminorms on X, then topology on X generated by $\mathscr{P}$ is metrizable. This is the main content of the next theorem.

1.19 THEOREM. *Let $\mathscr{P}$ be a countable family of seminorms on a vector space X, and suppose that for each $x \in X$, there is some $p \in \mathscr{P}$ such that $p(x) \neq 0$. Then there is a translation is a translation invariant metric ρ on X such that the topology induced on X by this metric coincides with the topology on X generated by $\mathscr{P}$.*

Proof. Order the elements of $\mathscr{P}$ in a sequence $\{p_n\}_{n\in\mathbb{N}}$. Define the function $\phi : [0, \infty) \to [0, 1)$ by $\phi(t) = t/(1 + t)$. Then for $s, t \geq 0$,

$$\phi(s + t) = \frac{s}{1 + s + t} + \frac{t}{1 + s + t} \leq \phi(s) + \phi(t) \tag{1.11}$$

and ϕ is strictly monotone increasing. For $x, y \in X$ define

$$\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n}\phi(p_n(x - y)) \ , \tag{1.12}$$

and note that the sum converges absolutely and in fact $\rho(x, y) < 1$ for all $x, y \in X$. It is evident that $\rho(x, y) = \rho(y, x)$, and since if $x \neq y$, there is some n such that $p_n(x - y) \neq 0$, then $\rho(x, y) \neq 0$. Finally, by the subadditivity of each p_n, for all $x, y, z \in X$,

$$p_n(x - z) = p_n((x - y) + (y - z)) \leq p_n(x - y) + p_n(y - z) \ ,$$

and then by (1.11), $\phi(p_n(x - z)) \leq \phi(p_n(x - y)) + \phi(p_n(x - z))$. It follows that ρ satisfies the triangle inequality, and therefore is a metric on X. Notice that by definition, for all $x, y, z \in X$,

$$\rho(T_x y, T_x z) = \rho(y, z) \ ,$$

so that ρ is translation invariant. For $r > 0$, let $B_\rho(r, 0) := \{x \in X \ : \ \rho(x, 0) < r\}$.

It remains to show that for each $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, $2^k B_{p_n}$ contains $B_\rho(r, 0)$ for some $r > 0$, and that for each $r > 0$, $B_\rho(r, 0)$ contains a set that is open in the topology generated by $\mathscr{P}$.

Consider the set $2^k B_{p_n} = \{x \ : \ p_n(x) < 2^k\}$. Since for each n, and each x, $\rho(x, 0) \geq 2^{-n}\phi(p_n(x))$,

$$p_n(x) \leq \frac{2^n \rho(x, 0)}{1 - 2^n \rho(x, 0)}$$

for all x such that $\rho(x,0) < 2^{-n}$. Hence for each $n \in N$, and each $\epsilon > 0$, there is an $r_{n,\epsilon} > 0$ such that $p_n(x) < \epsilon$ whenever $x \in B(r_{n,\epsilon}, 0)$. Taking $\epsilon = 2^k$, this can be written as

$$B(r_{n,2^k}, 0) \subset 2^k B_{p_n} \ .$$

Next, fix $r > 0$ and $n \in \mathbb{N}$. Fix $N \in N$ such that $2^{-N} < r/2$. Then

$$\rho(x,0) = \sum_{n=1}^{N} 2^{-n}\phi(p_n(x)) + \sum_{n=N+1}^{\infty} 2^{-n} \le \sum_{n=1}^{N} 2^{-n}\phi(p_n(x)) + r/2$$

Since for all $t > 0$, $\phi(t) \le t$, if $x \in (r/2)B_{p_n}$, then $\phi(p_n(x)) \le (r/2)$. Now pick $\ell \in \mathbb{N}$ so that $2^{-\ell} < r/2$, and note that

$$W := \bigcap_{n=1}^{N} 2^{-\ell} B_{p_n}$$

is in the canonical neighborhood base at 0 of the topology generated by $\mathscr{P}$. By what we have noted above, for all $x \in \mathscr{P}$, $\rho(x,0) < r$. That is $W \subset B(r,0)$. □

A single seminorm that is not a norm cannot generate a Hausdorff topology: If p is a seminorm but not a norm, there is some non-zero $x \in X$ such that $p(x) = 0$, and therefore $tx \in B_p$ for all $t > 0$, or what is the same, $x \in 2^k B_p$ or all $k \in \mathbb{Z}$. Hence x belongs to each open set containing 0. However, when the seminorm p is norm, this problem is eliminated. A very important class of topological vector space topologies arises this way.

1.3 Normed vector spaces

1.20 DEFINITION (Normed vector space)**.** A *normed vector space* $(X, \|\cdot\|)$ is a vector space X equipped with a norm function $\|\cdot\|$. For each $r > 0$ and $x_0 \in X$, define

$$B(r, x_0) = \{x \in X \ : \ \|x - x_0\| < r\} \ . \tag{1.13}$$

For each $r > 0$, $B(r,0)$ is balanced, convex and absorbing. Define $\mathscr{W} = \{B(2^k, 0) \ : \ k \in \mathbb{Z}\}$, which is nested and therefore closed under finite intersections. Then by Theorem 1.8, $\mathscr{W}$ is a neigborhood base for a topology on X that makes it a topological vector space. This topology is called the *norm topology*. The set $B(r, x_0)$ is call the *open ball of radius r centered at x_0*.

Let $(X, \|\cdot\|)$ be a vector space, and define a function $\rho : X \times X \to [0,\infty)$ by $\rho(x,y) = \|x - y\|$. Then evidently $\rho(x,y) = \rho(y,x)$ for all x, y, and $\rho(x,y) = 0$ if and only if $x = y$. Moreover, by the subadditivity of the norm, for all $x, y, z \in X$,

$$\rho(x,z) = \|x - z\| = \|(x-y) + (y-z)\| \le \|x - y\| + \|y - z\| = \rho(x,y) + \rho(y,z) \ .$$

Therefore, the triangle inequality is satisfied, and hence ρ is a metric on X called the *norm metric*.

For all $x_0 \in X$, the metric open ball of radius r about is precisely the set $B(r, x_0)$ defined in (1.13), and $U \subset X$ is open in the metric topology if and only if for each x_0 in U, there is some $r > 0$ such that $B(r, x_0) \subset U$. Decreasing r if need be, we may assume that $r = 2^k$ for some $k \in Z$, and then since $B(2^k, x_0) = x_0 + B(2^k, 0)$, the metric topology is precisely the topology determined by the neighborhood base $\mathscr{W} = \{B(2^k, 0) \ : \ k \in \mathbb{Z}\}$.

A normed vector space is therefore not only a Hausdorff topological vector space, but the norm topology is a metric topology, and it has a countable neighborhood base.

1.21 DEFINITION (Equivalent norms)**.** Let $\|\cdot\|_0$ and $\|\cdot\|_1$ be two norms on a vector space X. These norms are *equivalent* in case for some constant $0 < C < \infty$,

$$C\|x\|_1 \le \|x\|_0 \le \frac{1}{C}\|x\|_1 \qquad \text{for all } x \in X \ . \tag{1.14}$$

$\|\cdot\|_0$ and $\|\cdot\|_1$ be two norms on a vector space X such that (1.14) is satisfied. For $j = 0, 1$, let $B_j(r, 0)$ be the centered open ball of radius r for the norm $\|\cdot\|_j$. Then is equivalent to

$$B_1(Cr, 0) \subset B_0(r, 0) \subset B_1(r/C, 0) \tag{1.15}$$

for all $r > 0$. Therefore, two norms generate the same topology if and only if they are equivalent.

1.22 THEOREM. *Let $(X, \|\cdot\|)$ be a normed vector space, and let Y be a finite dimensional subspace of X. Then Y is closed. Moreover, all norms on any finite dimensional vector space are equivalent to one another.*

Proof. Let V be a finite dimensional vector space over $\mathbb{C}$, and let $\{x_1, \dots, x_n\}$ be any basis for it. Define a linear map $T : \mathbb{C}^n \to V$ by $T(\alpha_1, \dots, \alpha_n) = \displaystyle\sum_{j=1}^n \alpha_j x_j$, which is a vector space isomorphism. Let $|\!|\!|\cdot|\!|\!|$ denote the Euclidean norm on $\mathbb{C}^n$; i.e.,

$$|\!|\!|(\alpha_1, \dots, \alpha_n)|\!|\!| = \left(\sum_{j=1}^n |\alpha_j|^2\right)^{1/2} .$$

Define the norm $\|\cdot\|_0$ on V by $\|x\|_0 = |\!|\!|T^{-1}x|\!|\!|$; it is left to the reader to check that this is indeed a norm. Now let $\|\cdot\|_1$ be any other norm on V. Since

$$(\alpha_1, \dots, \alpha_n) \mapsto \left\|\sum_{j=1}^n \alpha_j x_j\right\|_1 = \|T(\alpha_1, \dots, \alpha_n)\|_1$$

is continuous, this function has a maximum and a minimum on the unit sphere in $\mathbb{C}^n$; i.e., the set of vectors $(\alpha_1, \dots, \alpha_n)$ such that $\sum_{j=1}^n |\alpha_j|^2 = 1$, since this set is compact in the Euclidean norm topology. The minimum is non-zero since $\{x_1, \dots, x_n\}$ is linearly independent and $\|x\|_1 = 0$ only for $x = 0$. Hence there exists $0 < C < \infty$ such that for all $(\alpha_1, \dots, \alpha_n)$ satisfying $|\!|\!|(\alpha_1, \dots, \alpha_n)|\!|\!| = 1$,

$$C \le \|T((\alpha_1, \dots, \alpha_n)\|_1 \le \frac{1}{C} .$$

However, writing $x = T(\alpha_1, \dots, \alpha_n)$, and recalling that $\|x\|_0 = |\!|\!|T^{-1}x|\!|\!|$ this is the same as

$$C\|x\|_0 \le \|x\|_1 \le \frac{1}{C}\|x\|_0 ,$$

whenever $\|x\|_0 = 1$, and by the homogeneity of the norms, this last condition can be dropped. Hence the arbitrary norm $\|\cdot\|_1$ is equivalent to the norm $\|\cdot\|_0$ induced by the Euclidean norm and any choice of basis.

Now let Y be any finite dimensional subspace of a normed vector space $(X, \|\cdot\|)$, and let $\|\cdot\|_0$ be the norm on Y induced by the Euclidean norm and any choice $\{x_1, \dots, x_n\}$ of a basis for Y, as in the first part of the proof. Let $\{y_m\}_{m\in\mathbb{N}}$ be any sequence in Y that converges to some $x \in X$. We must show that $x \in Y$. Expand each y_m in the basis $\{x_1, \ldots, x_n\}$:

$$y_m = \sum_{j=1}^{n} \alpha_{j,m} x_j \ .$$

By the equivalence proved above, since $\{y_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in Y with respect to the norm $\|\cdot\|$, $\{(\alpha_{1,n}, \ldots, \alpha_{n,m})\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}^n$, and by the completeness of $\mathbb{C}^n$, this Cauchy sequence has the limit $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$. By the equivalence once more, this implies that

$$\lim_{m\to\infty} y_m = \lim_{m\to\infty} \sum_{j=1}^{n} \alpha_{j,m} x_j = \sum_{j=1}^{n} \alpha_j x_j := x \ ,$$

showing that $x \in Y$. □

1.23 THEOREM. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. Let $B_X(r,0)$ and $B_Y(r,0)$ denote the open balls of radius $r > 0$ in X and Y respectively. Let T be a linear transformation from X to Y. Then T is continuous if and only if*

$$\|T\| := \sup\{\ \|Tx\|_Y\ :\ x \in B_X(1,0)\ \} < \infty\ . \tag{1.16}$$

Proof. Suppose that T is continuous. Then $T^{-1}(B_Y(1,0))$ contains $B_X(r,0)$ for some $r > 0$, and hence for all $x \in B_X(1,0)$, $\|Tx\|_Y \le 1/r$. Hence $\|T\| \le 1/r$. Conversely, suppose that (1.16) is valid. Then for all $x_1, x_2 \in X$ and all $\lambda > \|x_1 - x_2\|_X$,

$$\|Tx_1 - Tx_2\|_Y = \lambda \|T((x_2 - x_2)/\lambda)\|_Y \le \lambda \|T\|\ .$$

It follows that

$$\|Tx_1 - Tx_2\|_Y \le \|T\| \|x_1 - x_2\|_X\ .$$

Thus, T is not only continuous, it is Lipschitz continuous. □

On account of this theorem, the term *bounded linear transformation* is often used as a synonym for the term continuous linear transformation when referring to transformations from one normed vector space to another. The bounded transformations themselves are often called *operators*.

1.24 DEFINITION. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces. Let $\mathscr{B}(X,Y)$ denote the vector space of continuous (bounded) linear transformations from X to Y. The function $T \mapsto \|T\|$, with $\|T\|$ defined by (1.16) is called the *operator norm* on $\mathscr{B}(X,Y)$.

The terminology in the previous definition is appropriate; it is easy to see, and left to the reader, that the operator norm, is indeed a norm on the vector space $\mathscr{B}(X,Y)$

We shall be forced to consider topologies $\mathscr{O}$ in vector spaces X that are not topologies coming from norms on account a simple corollary of the following result of Riesz.

1.25 LEMMA (Riesz's Lemma)**.** *Let $(X, \|\cdot\|)$ be a normed vector space. Let Y be a proper, closed subspace of X. Then for all $\alpha \in (0,1)$, there exists $u \in X$, $\|u\| = 1$, such that*

$$\alpha \le \inf\{\|u - y\|\ :\ y \in Y\}\ .$$

Proof. Since Y is proper, there exists some $x_0 \notin Y$, and then since Y is closed, there exists some $r > 0$ such that $B(r, x_0) \cap Y = \emptyset$. Therefore, $d := \inf\{\|u - y\| \ : \ y \in Y\} \geq r > 0$. By the definition of d, for all $\alpha \in (0, 1)$, there exists $y_0 \in Y$ such that $\|x_0 - y_0\| < d/\alpha$. Then, by the definition of d,

$$\alpha \|x_0 - y_0\| \leq d \leq \|x_0 - y_0 - y\| \quad \text{for all} \quad y \in Y \ .$$

That is,

$$\alpha \leq \left\| \frac{x_0 - y_0}{\|x_0 - y_0\|} - \frac{y}{\|x_0 - y_0\|} \right\| \quad \text{for all} \quad y \in Y \ .$$

Let $u = \|x_0 - y_0\|^{-1}(x_0 - y_0)$. Notice that $\|x_0 - y_0\|^{-1} y$ ranges over all of Y as y ranges over Y. □

1.26 COROLLARY. *In every infinite dimensional normed vector space $(X, \|\cdot\|)$, there exists an infinite sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $\|u_n\| = 1$ for all n, but from which no convergence subsequence can be extracted. In other words, the closed unit ball in an infinite dimensional normed vector space is never sequentially complete.*

Proof. The sequence $\{u_n\}_{n\in\mathbb{N}}$ is constructed inductively as follows: Pick some $u_1 \in X$ with $\|u_1\| = 1$. For the induction, suppose that we have found vectors $\{u_1, \dots, u_n\}$ such that $\|u_j\| = 1$ for each j and if $j \neq k$, then $\|u_j - u_k\| \geq 1/2$. Let $V_n := \text{span}(\{u_1, \dots, u_n\})$ which is is closed by Theorem 1.22 proper since X is infinite dimensional. By Riesz's Lemma, we may choose u_{n+1} with $\|u_{n+1}\| = 1$ snd $\|u_{n+1} - y\| \geq 1/2$ for all $y \in Y$. In particular we have $\|u_j - u_j\| \geq 1/2$ for all $1 \leq j < k \leq n+1$. □

1.4 Topologies induced by sets of linear transformations

The usual way of constructing seminorms on X is to consider linear transformations T from X to a normed vector space $(Y, \|\cdot\|)$. (A particularly important example is that in which $(Y, \|\cdot\|) = (\mathbb{C}, |\cdot|)$.) Then evidently the function p_T on X defined by

$$p_T(x) = \|Tx\| \tag{1.17}$$

is a seminorm, and is a norm if and only if T is injective.

Let $(Y, \|\cdot\|)$ be a normed space and let $\mathscr{T}$ be any set of linear maps from $X \to Y$. For each $T \in \mathscr{T}$, let p_T be defined as in (1.17). Let $\widetilde{\mathscr{O}}$ be any translation invariant topology on X. Then T is continuous from $(X, \widetilde{\mathscr{O}})$ to Y equipped with the norm topology if and only if for each $x_0 \in X$, and each $\epsilon > 0$, there is a set $U \in \widetilde{\mathscr{O}}$ such that for all $x - x_o \in U$,

$$\|Tx - Tx_0\| = \|T(x - x_0)\| < \epsilon \ .$$

Since $p_T(x - x_0) = \|T(x - x_0)\|$, for all $k \in Z$,

$$x \in x_0 + 2^{-k} B_{p_T} \iff x - x_0 \in 2^{-k} B_{p_T} \iff \|Tx - Tx_0\| < 2^{-k} \ .$$

Hence T is continuous on $(X, \widetilde{\mathscr{O}})$ if and only if if $\widetilde{\mathscr{O}}$ contains each of the sets $x_0 + 2^k B_{p_T}$, $k \in \mathbb{Z}$, $x_0 \in X$. This proves:

1.27 THEOREM. *Let $\mathscr{T}$ be any set of linear transformations from a vector space X to a normed space $(Y, \|\cdot\|)$. Let $\mathscr{P}$ be the set of seminorms on X given by $\mathscr{P} = \{p_T : T \in \mathscr{T}\}$ where $p_T(x) = \|T(x)\|$. Let $\mathscr{O}$ be the topology generated by $\mathscr{P}$, as in Definition 1.18. Then $(X, \mathscr{O})$ is a topological vector space such that each $T \in \mathscr{T}$ is continuous from $(X, \mathscr{O})$ to $(Y, \|\cdot\|)$, and $\mathscr{O}$ is the weakest topology of any sort on X such that each $T \in \mathscr{T}$ is continuous from $(X, \mathscr{O})$ to $(Y, \|\cdot\|)$.*

In what follows, the case in which $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two normed vector spaces, and $\mathscr{T} = \mathscr{B}(X, Y)$, the set of all bounded linear transformations from X to Y, is especially important. In many cases we shall be able to show that norm closed, norm bounded, convex sets $K \subset X$ are compact in the weak topology on X generated by $\mathscr{T}$. (To say that K is *norm bounded*, or simply *bounded*, means that there exists $C < \infty$ such that $\|x\| \le C$ for all $x \in K$.) This will be more useful to us if we have not only compactness, but sequential compactness, which is the same thing if this weak topology, or at least the relative weak topology on K is metrizable.

The following theorem gives a useful sufficient condition for weak topology generated by $\mathscr{T}$ to be metrizable on bounded subsets of X.

1.28 THEOREM. *$(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two normed vector spaces, and let $\mathscr{T}$ be any subset of $\mathscr{B}(X, Y)$. Suppose that $\mathscr{T}_0$ is a countable subset of $\mathscr{T}$ that is dense in $\mathscr{T}$ in the operator norm. Then restricted to bounded subsets the weak topologies generated by $\mathscr{T}_0$ and $\mathscr{T}$ coincide.*

Proof. Let K be a bounded subset of X. Let $\mathscr{O}_0$ and $\mathscr{O}$ be the relative weak topologies generated by $\mathscr{T}_0$ and $\mathscr{T}$ respectively. Evidently, $\mathscr{O}_0 \subset \mathscr{O}$.

Let $C < \infty$ be such that $\|x\|_X \le C$ for all $x \in K$. For any $T \in \mathscr{T}$ and any $\epsilon > 0$, there is $T_0 \in \mathscr{T}_0$ such that $\|T - T_0\| < \epsilon$, and then for all $x \in K$,

$$|\|Tx\|_Y - \|T_0 x\|_Y| \le \|(T - T_0)x\|_Y \le \|T - T_0\|\|x\| \le \epsilon C \ .$$

It follows that $\|T_0 x\|_Y \le \|Tx\|_Y + \epsilon C$ for all $x \in K$. Hence for all $k \in \mathbb{N}$, if we choose ϵ (and then T_0) so that $\epsilon < 2^{-k-1}/C$, $2^{-k-1} B_{p_{T_0}} \cap K \subset 2^{-k} B_{p_T} \cap K$. By Lemma 1.9, $\mathscr{O} \subset \mathscr{O}_0$. □

2 Banach spaces

2.1 Banach spaces of bounded linear transformations

2.1 DEFINITION (Banach space)**.** A *Banach space* is a normed vector space $(X, \|\cdot\|)$ that is complete in its norm topology.

2.2 THEOREM. *Let $(X, \|\cdot\|_X)$ be a normed space, and let $(Y, \|\cdot\|_Y)$ be a Banach space. Then $\mathscr{B}(X, Y)$, equipped with the operator norm is Banach space.*

Proof. Let $\{T_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathscr{B}(X, Y)$. Then for each $x \in X$, $\{T_n x\}_{n \in \mathbb{N}}$ is a Cauchy sequence in Y. Since Y is complete, there exists $y \in Y$ such that $y = \lim_{n\to\infty} T_n x$. Define a function T mapping X to Y by $Tx = y$.

To see that T is linear, fix $x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. Then

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \lim_{n\to\infty} T_n(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \lim_{n\to\infty} T_n x_1 + \alpha_2 \lim_{n\to\infty} T_n x_2 = \alpha_1 T_n x_1 + \alpha_2 T x_2 \ .$$

To see that T is bounded, first observe that since $|\|T_n\| - \|T_m\|| \leq \|T_n - T_m\|$, $\{\|T_n\|\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and hence $a := \lim_{n\to\infty} \|T_n\|$ exists.

Now consider any $x \in X$ with $\|x\|_X < 1$. By the triangle inequality,

$$\|Tx\|_Y \leq \|T_n x\|_Y + \|(T - T_n)x\|_Y \leq \|T_n\| + \|(T - T_n)x\|_Y \ , \tag{2.1}$$

and since $\lim_{n\to\infty} \|(T - T_n)x\|_Y = 0$, $\|Tx\|_Y \leq \lim_{n\to\infty} \|T_n\|$. This shows that T is bounded, and in fact, $\|T\| \leq \lim_{n\to\infty} \|T_n\|$. Now the same reasoning that led to (2.1) shows that $\|T_n x\| \leq \|T\| + \|(T - T_n)x\|_Y$ for all $x \in X$ with $\|x\|_X < 1$. Therefore,

$$\lim_{n\to\infty} \|T_n x\| \leq \|T\| \ . \tag{2.2}$$

Altogether, $\|T\| = \lim_{n\to\infty} \|T_n\|$. Now considering the Cauchy sequence $\{T_n - T\}_{n\in\mathbb{N}}$, we conclude that $\lim_{n\to\infty} \|T_n - T\| = 0$. which shows that T is the operator norm limit of the Cauchy sequence $\{T_n\}_{n\in\mathbb{N}}$. □

An important spacial case of Theorem 2.2 is that in which $(Y, \|\cdot\|) = (\mathbb{C}, |\cdot|)$, which is of course the simplest example of a Banach space.

2.3 DEFINITION (Dual space)**.** Let $(X, \|\cdot\|)$ be a normed space. Let X^* denote the set of bounded linear transformations from $(X, \|\cdot\|)$ to $(\mathbb{C}, |\cdot|)$ and let $\|\cdot\|_*$ denote the operator norm on X^*. Then $(X^*, \|\cdot\|_*)$ is a Banach space called the *dual space* to $(X, \|\cdot\|)$. The norm $\|\cdot\|_*$ is called the *dual norm*. Elements of X_* are referred to as *bounded linear functionals on* X. Applying the same construction to $(X^*, \|\cdot\|_*)$ we obtain the *second dual* $(X^{**}, \|\cdot\|_{**})$, which is also a Banach space.

There is a natural embedding of X into X^{**}: For each $x \in X$, define the function ϕ_x on X^* by

$$\phi_x(L) = L(x) \quad \text{for all } L \in X^* \ . \tag{2.3}$$

It is evident that ϕ_x is a linear functional from X^* to $\mathbb{C}$, and for all $x \in X$, $L \in X^*$, $|\phi_x(L)| = |L(x)| \leq \|L\|_* \|x\|$. Therefore ϕ_x is bounded, so that $\phi_x \in X^{**}$, and in fact,

$$\|\phi_x\|_{**} \leq \|x\| \ .$$

Thus, the map $x \mapsto \phi_x$ is a contractive linear map from $(X, \|\cdot\|)$ into $(X^{**}, \|\cdot\|_{**})$, which is a Banach space. The results of the next section show that this contractive map is actually an isometry, so that every normed vector spaces is isometrically embedded into a Banach space in a canonical manner. We may identify the closure of the image as the *completion* of $(X, \|\cdot\|)$.

2.2 The real Hahn-Banach extension theorem

In this section we consider real vector spaces X. Every complex vector space X is also a vector space over $\mathbb{R}$, only now, for each non-zero $x \in X$, x and ix are linearly independent, and in the next section we shall extend the results of this section to complex vector spaces through this connection.

2.4 LEMMA (Helly's Lemma). *Let $(X, \|\cdot\|)$ be a real normed vector space, and let V be a subspace of X. Let $x \in X$, $x \notin V$, and let $W = \mathrm{span}(\{x\} \cup V)$, so that V is a subspace of W of co-dimension 1. Let L be a linear functional on V such that*

$$|L(y)| \le C\|y\| \quad \text{for all } y \in V\ . \tag{2.4}$$

Then there exists an extension $\widetilde{L}$ of L as a linear functional to W such that with this same C, $|L(w)| \le C\|w\|$ for all $w \in W$.

Proof. Let $y_1, y_2 \in V$. By the triangle inequality,

$$L(y_1) - L(y_2) = L(y_1 - y_2) \le C\|y_1 - y_2\| = C\|(y_1 + x) - (y_2 + x)\| \le C\|y_1 + x\| + C\|y_2 + x\|\ .$$

Rearranging terms, to bring all y_2 terms to the left, and all y_1 terms to the right,

$$-L(y_2) - C\|y_2 + x\| \le -L(y_1) + C\|y_1 + x\|\ .$$

Since y_1, y_2 are arbitrary,

$$a := \sup_{y \in V}\{\ -L(y) - C\|y + x\|\} \le \inf_{y \in V}\{\ -L(y) + C\|y + x\|\} =: b\ .$$

Pick any real number $\tilde{\lambda}$ satisfying $\tilde{\lambda} \in [a, b]$. Define $\widetilde{L}(y + tx) = L(y) + t\tilde{\lambda}$. Then for $t > 0$,

$$\widetilde{L}(y + tx) = t[L(y/t) + \tilde{\lambda}] \le t[L(y/t) - L(y/t) + C\|y/t + x\|] = C\|y + tx\|\ ,$$

while for $t < 0$,

$$\widetilde{L}(y + tx) = t[L(y/t) + \tilde{\lambda}] \ge t[L(y/t) - L(y/t) - C\|y/t + x\|] = |t|C\|y/t + x|| = C\|y + tx\|\ ,$$

Hence, for all $t \in \mathbb{R}$, and all $y \in V$, $|\widetilde{L}(y + tx)| \le C\|y + tx\|$. □

2.5 THEOREM (The Real Hahn-Banach Extension Theorem). *Let $(X, \|\cdot\|)$ be a real normed vector space, and let Y be a subspace of X. Let L be a linear functional on Y such that for some $C < \infty$, $|L(y)| \le C\|y\|$ for all $y \in Y$. Then there exists a linear functional $\widetilde{L}$ on all of X such that for this same C, $|\widetilde{L}(x)| \le C\|x||$ for all $x \in X$ and such that $\widetilde{L}y = Ly$ for all $Y \in Y$.*

Proof. Consider the set of pairs (V, L_V) of subspaces V of X and bounded linear functionals L_V on V partially ordered so that $(V, L_V) \prec (W, L_W)$ in case $V \subset W$ and $L_W\big|_V = L_V$. By Zorn's Lemma, since every linearly ordered chain containing (Y, L) has a maximal element, the set of all such pairs has a maximal element (V, L_V). By Lemma 2.4, $Y = X$, or else (Y, L) would not be maximal. □

2.3 The complex Hahn-Banach extension theorem

In this section $(X, \|\cdot\|)$ will denote a complex normed vector space, and $(X_{\mathbb{R}}, \|\cdot\|)$ will denote the real normed vector space obtained by regarding X as a real vector space, and equipping it with the same norm. As usual, X^* denotes the set of bounded (complex) linear functionals on $(X, \|\cdot\|)$, and $X^*_{\mathbb{R}}$ denotes the set of bounded (real) linear functional on $(X_{\mathbb{R}}, \|\cdot\|)$.

2.6 LEMMA (Murray's Lemma). *Let $(X, \|\cdot\|)$ be a complex normed vector space. Let p be a real linear functional on X. That is $p : X \to \mathbb{R}$ and for all $x_1, x_2 \in X$ and $t_1, t_2 \in \mathbb{R}$, $p(t_1x_1 + t_2x_2) = t_1p(x_1) + t_2p(x_2)$. Define the functional L_p on X by*

$$L_p(x) = p(x) - ip(ix) \ . \tag{2.5}$$

Then L_p is complex linear on X, and for all $x \in X$, $p(x) = \Re(L_p(x))$, and $\|L_p\| = \|p\|$. That is,

$$\sup\{|L_p(x)| \ : \ \|x\| < 1 \ \} = \sup\{|p(x)| \ : \ \|x\| < 1 \ \} \ . \tag{2.6}$$

Finally, if $p = \Re \circ L$, $L \in X^*$, then $L = L_p$. Thus the map $L \mapsto \Re \circ L$ is a real linear isometric isomorphism between X^* and $X^*_{\mathbb{R}}$.

Proof. Evidently, for all $x_1, x_2 \in X$ and $t_1, t_2 \in \mathbb{R}$, $L_p(t_1x_1 + t_2x_2) = t_1L_pp(x_1) + t_2L_p(x_2)$. Therefore, it suffices to show that for all $x, y \in X$, $L_p(x + iy) = L_p(x) + iL_p(y)$, and this is a simple calculation. The fact that $p(x) = \Re(L_p(x))$ is evident. It follows that $|p(x)| \leq |L_p(x)| \leq \|L_p\|\|x\|$ for all $x \in X$, and therefore that $\|p\| \leq \|L_p\|$. Conversely, for all $x \in X$, there is some $\theta \in [0, 2\pi)$ such that $e^{i\theta}L_p(x) > 0$, and then

$$|L_p(x)| = e^{i\theta}L_p(x) = L_p(e^{i\theta}x) = \Re(L_p(e^{i\theta}x)) = p(e^{i\theta}x) \leq \|p\|\|e^{i\theta}x\| = \|p\|\|x\| \ .$$

This shows that $\|L_p\| \leq \|p\|$. altogether, we have $\|L_p\| = \|p\|$. Finally, if $p = \Re \circ L$, then $L(x)$ and $L_p(x)$ are complex linear functionals that have the same real parts for all $x \in X$. Hence $L = L_p$. □

2.7 THEOREM (The Complex Hahn-Banach Extension Theorem). *Let $(X, \|\cdot\|)$ be a complex normed vector space, and let Y be a subspace of X. Let L be a linear functional on Y such that for some $C < \infty$, $|L(y)| \leq C\|y\|$ for all $y \in Y$. Then there exists a linear functional $\widetilde{L}$ on all of X such that for this same C, $|\widetilde{L}(x)| \leq C\|x\|$ for all $x \in X$ and such that $\widetilde{L}y = Ly$ for all $y \in Y$.*

Proof. Define $p = \Re \circ L$. By Lemma 2.6, $\|p\| = \|L\|$, where the norms are computed on Y. By Theorem 2.5, there exists a linear functional $\widetilde{p}$ on $X_{\mathbb{R}}$ such that $\widetilde{p}(y) = p(y)$ for all $y \in Y$, and such that $\|\widetilde{p}\| = \|p\|$. Then $L_{\widetilde{p}}$ is a complex linear functional on X such that for all $y \in Y$,

$$\Re(L_{\widetilde{p}}(y)) = \widetilde{p}(y) = p(y) = \Re(L(y)) \ .$$

Replacing y by iy, yields equality of the imaginary parts as well. Therefore, $L_{\widetilde{p}}(y) = L(y)$ for all $y \in Y$, and again by Lemma 2.6, $\|L_{\widetilde{p}}\| = \|\widetilde{p}\|$. Altogether, $\|L_{\widetilde{p}}\| = \|L\|$. □

2.8 THEOREM. *Let $(X, \|\cdot\|)$ be a non-trivial normed vector space. For all $x \in X$, there exists $L_x \in X^*$ such that $\|L_x\| = 1$ and $L_x(x) = \|x\|$.*

Proof. Let $x \in X$, $x \neq 0$, and let Y be the one-dimensional subspace X spanned by x. The general element of Y has the form αx, $\alpha \in \mathbb{C}$. Define a linear functional L on Y by $L(\alpha x) = \alpha\|x\|$. Evidently $\|L\| = 1$ and $L(x) = \|x\|$. Let L_x denote the norm-preserving extension of L to all of X that is provided by the Hahn-Banach Theorem. For $x = 0$, the claim is true, since by the first part, unit vectors L in X^* exist, and any such unit vector will do. □

2.9 THEOREM. *Let $(X, \|\cdot\|)$ be a normed vector space. For $x \in X$, let ϕ_x denote the element of X^{**} given by $\phi_x(L) = L(x)$ for all $L \in X^*$. The map $x \mapsto \phi_x$ is an isometric imbedding of X into X^{**}.*

Proof. We have already observed that $x \mapsto \phi_x$ is linear and that $\|\phi_x\| \le \|x\|$. Let L_x be an element of X^* such that $\|L_x\|_* = 1$ and $L_x(x) = \|x\|$. Then $\phi_x(L_x) = L_x(x) = \|x\|$, and hence $\|\phi_x\| \ge \|x\|$. Altogether, we have the isometry $\|\phi_x\| = \|x\|$. □

2.10 DEFINITION. A Banach space $(X, \|\cdot\|)$ is *reflexive* in case every element of X^{**} is of the form ϕ_x for some $x \in X$ where ϕ_x is the element of X^{**} given by $\phi_x(L) = L(x)$ for all $L \in X^*$. In other words, X is reflexive if the natural isometry that embeds X in X^{**} is surjective.

Theorem 2.9 provides a natural way to *complete* a normed vector space that is not a Banach space: Identify it with its image in the Banach space X^{**} under the isometric map $x \mapsto \phi_x$. The closure of the image in X^{**} is a Banach space in which the isometric image of X is dense.

A number of applications of the Hahn-Banach Theorem will be made it what follows, but a simple alternative proof of Lemma 1.25, Riesz's Lemma, is worthwhile to give at this point. Recall that Lemma 1.25 says that if $(X, \|\cdot\|)$ be a normed vector space, and Y be a proper, closed subspace of X, then for all $\alpha \in (0,1)$, there exists $u \in X$, $\|u\| = 1$, such that

$$\alpha \le \inf\{\|u - y\| \ : \ y \in Y\} \ .$$

Second Proof of Lemma 1.25. Let $x \in X$, $x \notin Y$. Define L on $\mathrm{span}(Y \cup \{x\})$ by $L(y + \alpha x) = \alpha$ for all $y \in Y$, $\alpha \in \mathbb{C}$. Then $\ker(L) = Y$ which is closed in $\mathrm{span}(Y \cup \{x\})$, and hence is bounded on $\mathrm{span}(Y \cup \{x\})$. By the Hahn-Banach Theorem, there is a non-zero element $\widetilde{L}$ of X^* that is zero on all of Y.

By the definition of $\|\widetilde{L}\|_*$, for all $\alpha \in (0,1)$, there is a unit vector $u \in X$ such that $\widetilde{L}(u)$ is real and $\widetilde{L}(u) \ge \alpha\|\widetilde{L}\|_*$. Since for all $y \in Y$,

$$\alpha\|\widetilde{L}\|_* \le \widetilde{L}(u) = \widetilde{L}(u - y) \le \|\widetilde{L}\|_*\|u - y\| \ ,$$

$\inf\{\|u - y\| \ : \ y \in Y\} \ge \alpha$. □

2.4 The weak and weak-$*$ topologies

2.11 DEFINITION. Let $(X, \|\cdot\|)$ be a normed space, and let $(X^*, \|\cdot\|_*)$ be its dual. For $x \in X$, let $\phi_x \in X^{**}$ be given by $\phi_x(L) = L(x)$ for all $L \in X^*$. The *weak topology on* X is the weakest topology on X under which each $L \in X^*$ is continuous. The *weak-$*$ topology on* X^* is the weakest topology on X under which each of the functionals ϕ_x, $x \in X$, is continuous

By Theorem 1.27, the weak topology makes X a topological vectors space, and the weak-$*$ topology makes X^* a topological vector space. By Theorem 1.28, if X^* is separable, then the relative weak topology is metrizable on bounded subsets of X, and if X is separable, then the relative weak-$*$ topology is metrizable on bounded subsets of X^*

One reason it is often useful to consider the weak-$*$ topology on X^* is on account of the following Theorem:

2.12 THEOREM (Alaoglu's Theorem)**.** *Let $(X, \|\cdot\|)$ be a normed space, and let B denote the closed unit ball in X^*. Then B is compact in the weak-$*$ topology.*

Proof. For each $x \in X$, define $D_x = \{z \in \mathbb{C} \ \ |z| \leq \|x\|\}$ which is a compact subset of $\mathbb{C}$. By Tychonov's Theorem, $\mathscr{D} := \prod_{x\in X} D_x$, which consists of all complex values functions ϕ on X such that for each $x \in X$, $\phi(x)\ inD_x$, is compact in the product topology. Let $\mathscr{L}$ denote the subset of $\mathscr{D}$ consisting of linear functions. It is easy to that Ł is a closed subset of $\mathscr{D}$. (The same argument that we applied in the Hilbert space setting can be made here.) Hence $\mathscr{L}$ is compact, and the elements of $\mathscr{L}$ are precisely the elements of B. Moreover, the product topology on $\mathscr{D}$ is precisely the weakest topology that makes all of the evaluation maps $\phi \mapsto \phi(x)$ continuous, but this is precisely the weak-$*$ topology on B. □

If $(X, \|\cdot\|)$ is reflexive, then the closed unit ball in X is weakly compact since we may then identify the weak topology on X with the weak-$*$ topology on X^{**}. But when $(X, \|\cdot\|)$ is reflexive, the unit ball in X need not be weakly compact.

2.5 The uniform boundedness principle and related theorems

2.13 THEOREM. *Let $(X, \|\cdot\|_X)$ be a Banach space, and let $(Y, \|\cdot\|_Y)$ be a normed space. Let $\mathscr{T} \subset \mathscr{B}(X, Y)$. For each T in $\mathscr{T}$, let $\|T\|$ denote the operator norm of T. Suppose that for all $x \in X$,*

$$\sup_{T\in\mathscr{T}} \{\|Tx\|_Y\} < \infty \ . \tag{2.7}$$

Then

$$\sup_{T\in\mathscr{T}} \{\|T\|\} < \infty \ . \tag{2.8}$$

Proof. For $n \in \mathbb{N}$, define $E_n = \bigcap_{T\in\mathscr{T}} \{x \in X \ : \|Tx\|_Y \leq n\} = \{x \in X \ : \ \sup_{T\in\mathscr{T}} \{\|Tx\|_Y\} \leq n\}$. Since for each T, $\{x \in X \ : \|Tx\|_Y \leq n\}$ is closed, E_n is closed, and by (2.7), $\bigcup_{n\in\mathbb{N}} E_n = X$. By Baire's Theorem, for some n, the interior of E_n is non-empty.. Hence for some $n \in \mathbb{N}$, $r > 0$ and $x_0 \in X$, $B(r, x_0) \subset E_n$. That is, if $\|x\|_X \leq 1$, and $0 \leq s < r$, $x_0 + sx \in E_n$. Therefore, for all $T \in \mathscr{T}$,

$$s\|Tx\|_Y = \|T(sx - x_0) + Tx_0\|_Y \leq n + \|Tx_0\| \ .$$

Define $C = \sup_{T\in\mathscr{T}}\{\|Tx_0\|_Y\}$ which is finite by (2.7). Then $\|Tx\|_Y \leq (n + C)/s$ for all x with $\|x\|_X \leq 1$, and this means that $\|T\| \leq (n + C)/r$, showing that $\sup_{T\in\mathscr{T}}\{\|T\|\} \leq (n + C)/r$. □

2.14 THEOREM (Open Mapping Theorem)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $T \in \mathscr{B}(X, Y)$ be surjective. Then T is an open mapping. In particular, when $T \in \mathscr{B}(X, Y)$ is injective as well as surjective, its inverse belongs to $\mathscr{B}(Y, X)$.*

Proof. Let $B_X(r, 0)$ denote the open unit ball of radius $r > 0$ in X, and let $B_Y(r, 0)$ denote the open unit ball of radius $r > 0$ in Y, Since $\{B_X(r, 0) \ \ r > 0\}$ is a neighborhood base at the origin for the topology on X, and $\{B_Y(r, 0) \ \ r > 0\}$ is a neighborhood base at the origin for the topology on Y, if for each $r > 0$, there is an $s > 0$ so that $B_Y(s, 0) \subset T(B_X(r, 0))$, then T maps open sets in X to open sets in Y. moreover, by homogeneity, to show that T has this property, it suffices to show that there is some $s > 0$ so that $B_Y(s, 0) \subset T(B_X(1, 0))$.

Since $X = \cup_{n\in\mathbb{N}} B_X(n,0)$, and since T is surjective,

$$Y = \bigcup_{n\in\mathbb{N}} \overline{T(B_X(n,0))} = \bigcup_{n\in\mathbb{N}} n\overline{T(B_X(1,0))}\ .$$

By Baire's Theorem, for some n, the interior of $n\overline{T(B_X(1,0))}$ is non-empty. Since these sets are a homeomorphic to one another, each of them has a non-empty interior, and hence $\overline{T(B_X(1,0))} \neq \emptyset$.

Thus, for some $y_0 \in Y$, and some $t > 0$, $B_Y(t, y_0) \subset \overline{T(B_X(1,0))}$. In other words, of $\|y-y_0\|_Y < t$, then $y - y_0 \in \overline{T(B_X(1,0))}$. In particular, $y_0 \in \overline{T(B_X(1,0))}$ Then since $\overline{T(B_X(1,0))}$ is convex,

$$\frac{1}{2}y = \frac{1}{2}(y - y_0) + \frac{1}{2}y_0 \in \overline{T(B_X(1,0))}\ .$$

This shows that $B_Y(t/2,0) \subset \overline{T(B_X(1,0))}$.

What we have proved so far shows that for all $y \in B_Y(t/2,0)$ and all $\epsilon > 0$, there is an $x \in B_X(1,0)$ such that $\|y - Tx\|_Y < \epsilon$. By homogeneity, we have that for all $r > 0$,

$$\text{for all } \epsilon > 0\ ,\ \|y\|_Y < r \quad \Rightarrow \quad \text{there exists } x \in B_X(2r/t,0) \text{ such that } \|y - Tx\|_Y < \epsilon\ . \tag{2.9}$$

Pick $y \in B_Y(t/4,0)$, and then pick $x_1 \in B_X(1/2,0)$ so that $\|y - Tx_1\|_Y \leq t/8$. Now apply (2.9) with $y - Tx_1$ in place of y, and choose $x_2 \in B_X(1/4,0)$ such that $\|(y_1 - Tx_1) - Tx_2\| \leq t/16$. Proceeding inductively, we construct an infinite sequence $\{x_n\}_{n\in\mathbb{N}}$ such that for all n, $x_n \in B_X(2^{-n-1},0)$ and $\|y - T(\sum_{j=1}^n x_j)\|_Y \leq t2^{-n-2}$. Note the $\sum_{j=1}^\infty x_j$ coverages to an element $x \in B_X(1,0)$, and then $\|y - Tx\|_Y = \lim_{n\to\infty} \|y - T(\sum_{j=1}^n x_j)\|_Y = 0$. Thus $y \in T(B_X(1,0))$ whenever $y \in B_Y(t/4,0)$ which means that $B_Y(t/4,0) \subset T(B_X(1,0))$.

The final statement is clear since T^{-1} is continuous if and only if whenever U is open in X, $(T^{-1})^{-1}(U) = T(U)$ is open in Y. □

2.15 DEFINITION. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces, and let T be a linear transformation from X to Y, not necessarily bounded. The graph of T, $\Gamma(T)$, is the subset of $X \times Y$ given by

$$\Gamma(T) = \{(x, Tx)\ :\ x \in X\}\ .$$

The norm $\|(x,y)\| := \|x\|_X + \|y\|_Y$ makes $X \times Y$ into a normed vector space with the obvious rules for vector addition and scalar multiplication. A linear operator T from X to Y is said to be *closed* in case $\Gamma(T)$ is norm closed in $X \times Y$.

Beware the terminology, which is standard: while a map T is open if and only if the image under T of every open set is open, to say that T is closed does not mean that the image under T of every closed set is closed.

2.16 THEOREM (Closed Graph Theorem)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let T be a linear transformation from X to Y. If T is closed, then $T \in \mathscr{B}(X,Y)$.*

Proof. It is easy to see that $X \times Y$ is a Banach space in the norm $\|(x,y)\| = \|x\| + \|y\|$. Then since $\Gamma(T)$ is a norm closed subspace of $X \times Y$, it is a Banach space in this norm. The map $(x, Tx) \mapsto x$ is a linear bijection of X with $\Gamma(X)$, and since $\|x\|_X \leq \|x\|_X + \|Tx\|_Y$, it is bounded. (Its operator norm is no greater than 1.) By the Open Mapping Theorem, its inverse, which is the map $x \mapsto (x, Tx)$ is bounded. That is, there is a finite constant C such that $\|x\|_X + \|Tx\|_Y \leq C\|x\|_X$, which certainly implies that the operator norm of T is no greater than C. □

The following example of the use of the Closed Graph Theorem is due to Terry Tao.

2.17 THEOREM. *Let $(X, \mathscr{O})$ be a Hausdorff topological vector space, and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X such that these norm topologies are both at least as strong as $\mathscr{O}$, and such that X is complete in both norms. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.*

Proof. To say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms means that the identity transformation I is bounded from $(X, \|\cdot\|_1)$ to $(X, \|\cdot\|_2)$ and *vice-versa*. Note that $\Gamma(I) = \{(x, x) \ : \ x \in X\}$. Let (x, y) belong to the closure of $\Gamma(I)$. Then there is a sequence $\{x_n\}_{n\in N}$ such that $\lim_{n\to\infty} \|x_n - x\|_1 = 0$ and $\lim_{n\to\infty} \|x_n - y\|_2 = 0$. But then every open set $U \in \mathscr{O}$ that contains x contains x_n for all but finitely n, and every open set $V \in \mathscr{O}$ that contains y contains x_n for all but finitely n. If $U \cap V = \emptyset$, this is impossible, and since $\mathscr{O}$ is Hausdorff, it must be that $x = y$. Hence $(x, y) \in \Gamma(I)$. This $\Gamma(I)$ is closed, and hence I is bounded from $(X, \|\cdot\|_1)$ to $(X, \|\cdot\|_2)$. By symmetry, it is also bounded from $(X, \|\cdot\|_2)$ to $(X, \|\cdot\|_1)$. □

The utility of the closed graph theorem lies in the fact that if we seek to prove continuity of a linear transformation T between two normed space $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, we must show that whenever $\lim_{n\to\infty} x_n = x$ in X, then $\lim_{n\to\infty} Tx_n = Tx$ in Y. When the normed spaces are Banach spaces, the closed graph theorem reduces our burden to show that if $\lim_{n\to\infty} x_n = x$ in X and $\lim_{n\to\infty} Tx_n = y$, then $y = Tx$. That is we may assume that both sequences converge and need only identify the limit, as in the proof of Theorem 2.17. The do not need to prove that $\{Tx_n\}_{n\in\mathbb{N}}$ is convergent.

References

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