Midterm Solutions, Math 502, Spring 2017

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Instructions Solve 3 of the 5 problems. Circle the number of the problems you want to be graded here:

1 2 3 4 5

1. Let X be a Banach space. A set $A \subset X$ is *weakly bounded* in case for all $L \in X^*$, $\{|L(x)| \ : \ x \in A\}$ is a bounded subset of $[0,\infty)$. A is *norm bounded* in case $\{\|x\| \ : \ x \in A\}$ is a bounded subset of $[0,\infty)$. Show that A is weakly bounded if and only if it is norm bounded.

SOLUTION If $\|x\| \leq C$ for all $x \in A$, then $|L(x)| \leq \|L\|\|x\| \leq C\|L\|$ for all $L \in X^*$ and $x \in A$. Therefore, strong boundedness implies weak boundedness.

On the other hand, for each $x \in A$, let ϕ_x denote the evaluation functional $L \mapsto L(x)$ on L^*. Then if A is weakly bounded, for all $L \in X^*$,

$$\sup_{x\in A} |\{|\phi_x(L)|\} = \sup_{x\in A} |\{|L(x)|\} < \infty \ .$$

By the Uniform Boundedness Principle, $\sup_{x\in A} |\{\|\phi_x\|\} < \infty$, but, as a consequence of the Hahn-Banach Theorem, for each x, $\|\phi_x\| = \|x\|$. Thus, weak boundedness implies strong boundedness.

2. Let $\mathcal{H}$ be a Hilbert space, and $T \in \mathscr{B}(\mathcal{H})$. Let B be the norm closed unit ball in $\mathcal{H}$.

(a) Show that $T(B)$ is weakly closed in $\mathcal{H}$.

(b) Let S be a linear transformation from $\mathcal{H}$ to $\mathcal{H}$. Show that $\{Sf_n\}_{n\in\mathbb{N}}$ converges in norm whenever $\{f_n\}_{n\in\mathbb{N}}$ converges weakly iff and only if $S(B)$ is compact in $\mathcal{H}$ for the norm topology.

SOLUTION (a) Since B is norm closed and convex, as a consequence of the Separation Theorem, B is weakly closed. Since B is also bounded, by Alaoglu's theorem, B is weakly compact. Since bounded operators are continuous from the weak to the weak topology, the image of every weakly compact set under T is weakly compact, and then (since the weak topology is Hausdorff) weakly closed. In particular, $T(B)$ is weakly closed. (Which trivially implies that B is is also strongly closed.)

For **(b)**, assume first that $\{Sf_n\}_{n\in\mathbb{N}}$ converges in norm whenever $\{f_n\}_{n\in\mathbb{N}}$ converges weakly. Since $S(B)$ is closed in the norm topology by part **(a)**, and since the norm topology is metric,

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it suffices to show that from every sequence $\{g_n\}_{n\in\mathbb{N}} \in S(B)$, we can select a norm convergent subsequence. Since each $g_n \in S(B)$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $Sf_n = g_n$ for all n. As a consequence of Alaoglu's Theorem, we can select a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges weakly to some $f \in B$. But then $\{g_n\}_{n\in\mathbb{N}} = \{Sf_n\}_{n\in\mathbb{N}}$ converges in norm to Sf.

Conversely, suppose that $S(B)$ is compact in $\mathcal{H}$ for the norm topology. Let $\{f_n\}_{n\in\mathbb{N}}$ be any weakly convergent sequence in $\mathcal{H}$ with weak limit f. As a consequence of the Uniform Boundedness Principle, $\sup_{n\in\mathbb{N}}\{\|f_n\|\} < \infty$, and rescaling the sequence, we may assume without loss of generality that $\{f_n\}_{n\in\mathbb{N}} \subset B$, and hence $\{Sf_n\}_{n\in\mathbb{N}} \subset S(B)$. Since $S(B)$ is sequentially compact (the norm topology being metric), we may select a subsequence $\{Sf_{n_k}\}_{k\in\mathbb{N}}$ that converges strongly to some $h \in \mathcal{H}$. Then for all $g \in \mathcal{H}$,

$$\langle g, h\rangle = \lim_{k\to\infty} \langle g, Sf_{n_k}\rangle = \lim_{k\to\infty} \langle S^*g, f_{n_k}\rangle = \langle S^*g, f\rangle = \langle g, Sf\rangle \ .$$

Since g is arbitrary, $h = Sf$.

3. Let X be a Banach space. Recall that X is *reflexive* in case the isometric embedding $x \mapsto \phi_x$ of X into X^{**}, where $\phi_x(L) = L(x)$ for all $L \in X^*$, is surjective. Show that X is reflexive if and only if X^* is reflexive.

SOLUTION Let $f \in X^{***}$. The embedding $x \mapsto \phi_x$ is continuous (even isometric), and hence $x \mapsto f(\phi_x)$ is the composition of continuous functions, so that for some $L_f \in X^*$, for all $x \in X$,

$$f(\phi_x) = L_f(x) = \phi_x(L_f) \ . \qquad (*)$$

Suppose that X is reflexive. Then every $\phi \in X^{**}$ has the form ϕ_x for some $x \in X$, and then $(*)$ becomes

$$f(\phi) = \phi(L_f) \qquad (**)\ .$$

Since f is an arbitrary element of X^{***}, this shows that every element of X^{***} is an evaluation functional on X^{**}. Hence X^* is reflexive.

For the converse, suppose that X^* is reflexive. If X is not reflexive, the image of X under the isometric embedding $x \mapsto \phi_x$ is a proper closed subspace V of X^{**}, and then by the Hahn-Banach Theorem, there exists a non-zero $f \in X^{***}$ that vanishes on V. Since X^* is reflexive, there exists $L_f \in X^*$ such that $(**)$ is true for all $\phi \in X^{**}$. But then

$$0 = f(\phi_x) = L_f(\phi_x) = L_f(x)$$

for all x, so that $L_f = 0$, and hence $f = 0$. This contradiction shows that V cannot be a proper subspace of X^{**}.

4. (a) Let m denote Lebesgue measure on $[0, 2\pi]$. Show that if E is any Borel set in $[0, 2\pi]$ then

$$\lim_{j\to\infty} \int_E \cos(jx)\mathrm{d}m = \lim_{j\to\infty} \int_E \sin(jx)\mathrm{d}m = 0 \ .$$

(b) Consider any increasing sequence $\{n_k\}$ of the natural numbers. Define E to be the set of all x for which

$$\lim_{k\to\infty} \sin(n_k x) \quad \text{exists} \ .$$

Use part **(a)** and the identity $2\sin^2 x = 1 - \cos(2x)$ to how that $m(E) = 0$.

SOLUTION For $n \in \mathbb{N}$, define $u_{2n} = (2\pi)^{-1/2}\cos(nx)$ and $u_{2n-1} = (2\pi)^{-1/2}\sin(nx)$. Then $\{u_n\}_{n\in\mathbb{N}}$ is orthonormal, and hence for any $f \in \mathcal{H} = L^2([0,2\pi],\mathscr{B},m)$, $\lim_{n\to\infty}\int_{[0,2\pi]} f u_m \mathrm{d}m = 0$ In particular, for all Borel $E \subset [0,2\pi]$,

$$\lim_{n\to\infty}\int_E u_m \mathrm{d}m = 0 \ ,$$

which takes care of **(a)**. Now let E be the set on which $\lim_{k\to\infty}\sin(n_k x)$ exists, and define

$$f(x) = \begin{cases} \lim_{k\to\infty}\sin(n_k x) & x \in E \\ 0 & x \notin E \ . \end{cases}$$

By the identity $2\sin^2 x = 1-\cos(2x)$, part **(a)**, and the Lebesgue Dominated Convergence Theorem with dominating function 1,

$$2\int_{[0,2\pi]} f^2 \mathrm{d}m = m(E) - \lim_{k\to\infty}\int_E \cos(2n_k x)\mathrm{d}x = m(E) \ .$$

Hence if $D := \{x \in E \ : \ f(x) \geq 2^{-1/2}\}$, and $m(E) > 0$, then $m(D) > 0$. But then by part **(a)**, and the Lebesgue Dominated Convergence Theorem with dominating function 1,

$$2^{-1/2}m(D) \leq \lim_{k\to\infty}\int_D \sin(2n_k x)\mathrm{d}x = 0 \ ,$$

which is a contradiction. Hence, $m(E) > 0$ is impossible.

5. Let $\mathscr{J}$ be an operator norm-closed two sided ideal in $\mathscr{B}(\mathcal{H})$, $\mathcal{H}$ separable. (That is, $\mathscr{J}$ is a closed subalgebra of $\mathscr{B}(\mathcal{H})$ and for all $S \in \mathscr{J}$ and all $T \in \mathscr{B}(\mathcal{H})$, $ST, TS \in \mathscr{J}(\mathcal{H})$. Show that if $\mathscr{J}$ contains a single non-compact operator S, then $\mathscr{J} = \mathscr{B}(\mathcal{H})$. (*Hint:* We know that if S is not compact, there exists an infinite dimensional closed subspace in its range.)

SOLUTION We will show that $I \in \mathscr{J}$, where I is the identity. Then for all $T \in \mathscr{B}(\mathcal{H})$, $T = TI \in \mathscr{J}$. Let S be a non-compact element of $\mathscr{J}$. Let V be a closed infinite dimensional subspaces in its range. Let P be the orthogonal projection onto $\mathcal{K}$, Then $PS \in \mathscr{J}$. Let $\mathcal{L} = (\ker(PS))^\perp$, which is a closed subspace of $\mathcal{H}$. Let Q be the orthogonal projection onto $\mathcal{L}$. Then $R := PS|_{\mathcal{L}}$ is a bounded injective map from $\mathcal{L}$ onto $\mathcal{K}$

Now let $\{u_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, let $\{v_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{K}$, and let $\{w_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{L}$. Define

$$V := \sum_{n\in\mathbb{N}} |u_n\rangle\langle v_n| \quad \text{and} \quad W := \sum_{n\in\mathbb{N}} |w_n\rangle\langle u_n| \ .$$

Then V is an isometry from $\mathcal{K}$ onto $\mathcal{H}$, and W is an isometry from $\mathcal{H}$ onto $\mathcal{L}$. Then $VPSW$ is a bounded injective map from $\mathcal{H}$ onto $\mathcal{H}$, and hence by the Open Mapping Theorem, it has a bounded inverse. It also belongs to $\mathscr{J}$ since $S \in \mathscr{J}$. Thus $\mathscr{J}$ contains an element with a bounded inverse, and thus $I \in \mathscr{J}$.

If $\mathcal{H}$ is not separable, the set of all operators on $\mathscr{B}(H)$ whose ranges have a countable dimension is a proper two sided ideal that contains non-compact operators. It is of little interest compared to the ideal of all compact operators.