# Homework Assignment 7, Math 502, Spring 2017

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**1.** Let $t > 0$, and let $\mathrm{d}\nu_t = \gamma_t(x)\mathrm{d}x$. Define the *variance-t Hermite polynomials*, $\{h_n^{(t)}\}_{n\geqslant 0}$ to be the orthonormal sequence one obtains by applying the Gram-Schmidt algorithm to the sequence $\{x^n\}_{n\geqslant 0}$. Find formulas for

$$\frac{\mathrm{d}}{\mathrm{d}x}h_n^{(t)}(x) \qquad \text{and} \qquad xh_n^{(t)}(x)$$

as linear combinations of the variance-$t$ Hermite polynomials

**2.** Define $F := \{f \in L^1 \ : \ \widehat{f} \in L^1\}$.

**(a)** Show that for all $p \in [1,\infty]$, $f \in L^p$ and $\|f\|_p \leqslant \|f\|_1^{1/p}\|\widehat{f}\|_1^{1-1/p}$, and that $F$ is dense in $L^p$ for all $p \in [1,\infty)$. *Hint: for $f \in L^1 \cap L^2$, consider $\gamma_t * f$.*

**(b)** For $f \in L^1$, define

$$\check{f}(x) := \int_{\mathbb{R}} e^{i2\pi kx} f(k)\mathrm{d}k \ .$$

Show that for all $f \in F$, $\check{\widehat{f}}(x) = f(x)$.

**3.** Let $f \in L^2$. Let $V$ denote the norm closure of the span of translates of $f$, i.e., the functions $\tau_a f$, $a \in \mathbb{R}$. Show that $V = L^2$ if and only the set $\{k \in \mathbb{R} \ : \ \widehat{f}(k) = 0\}$ is a set of measure zero.

**4.** Let $A$ and $B$ be be bounded, closed intervals in $\mathbb{R}$. Define a function $K(x,y)$ on $\mathbb{R}^2$ by

$$K(x,y) = \int_{\mathbb{R}} e^{i2\pi k(y-x)} 1_B(k) 1_A(y)\mathrm{d}k \ .$$

**(a)** Show $f \mapsto \int_{\mathbb{R}} K(x,y)f(y)\mathrm{d}y =: T_K f$ defines a Hilbert-Schmidt operator on $L^2(\mathbb{R})$.

**(b)** Use the result of part **(a)** to show that the only function $f \in L^2$ such that $f$ has support in $A$ and $\widehat{f}$ has support in $B$ is the zero function.