Homework Assignment 6, Math 502, Spring 2017

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1. Let ϕ be the function defined on $\mathbb{R}$ by $\phi(s) = \begin{cases} \frac{1}{2}s^2 & |s| < 1 \\ |s| - 1/2 & s \geq 1 \ . \end{cases}$.

(a) Show that ϕ is convex and continuous, and is an Orlicz function.

(b) Compute ϕ^* and ϕ^{**}, with the latter computation giving a direct verification of the Fenchel-Moreau Theorem in this case.

(c) Let $(L_\phi, \|\cdot\|_\phi)$ be the associated Orlicz space for the measure space $(\mathbb{R}, \mathcal{B}, \mathrm{d}x)$, where $\mathrm{d}x$ denotes Lebesgue measure. Consider the functions $f_\beta = |x|^{-\beta}$, $\beta \in (1/2, 1)$ Show that none of these functions belong to L^p for any $1 \leq p \leq 2$, but that each of them belongs to L_ϕ.

(d) Show moreover that if $f \in L_\phi$ there exist functions $g \in L^1$ and $h \in L^2$ such that $f = g + h$, and show that $L^1 \subset L_\phi$ and $L^2 \subset L_\phi$. What is the relation between L^ϕ and the Banach space considered in Folland's Exercise 4, chapter 6 for $p = 1$ and $r = 2$?

SOLUTION (a) The function ϕ is continuously differentiable with $\phi'(s) = s$ for $|s| \leq !$, $\phi'(s) = 1$ for $s \geq 1$ and $\phi'(s) = -1$ for $s < -1$. Since the derivative is non-decreasing, ϕ is convex. It follows that $\partial\phi(\mathbb{R}) = [-1, 1]$, and that for each $y \in [-1, 1]$, $t \in \partial\phi(t)$. Since $\phi'(1) = 1$, $\{1\} = \partial\phi(1)$, and evidently $\lim_{t\to\infty} \phi(t) = \infty$. Hence ϕ is an Orlicz function.

For **(b)**, by the cases of equality in Young's inequality, $\phi(s) + \phi^*(t) = xy$ when $s = t$ (so that $t \in \partial\phi(s)$). Hence for $y \in [-1, 1]$, $\phi^*(t) = t^2 - t^2/2 = t^2/2$. For $|t| > 1$, $\phi^*(t) = \infty$. Thus,

$$\phi^*(t) = \begin{cases} \frac{1}{2}t^2 & |t| \leq 1 \\ \infty & |t| > 1 \ . \end{cases} \ .$$

The same reasoning with cases of equality in Young's inequality shows that $\phi^{**}(s) = \phi(s)$, and indeed, this is how one proves the Fenchel-Moreau Theorem.

For **(c)**, no power of $|x|$ is intolerable on $(\mathbb{R}, \mathcal{B}, \mathrm{d}x)$, so $f_\beta \notin L^p$ for any p. Now note that for $t > 0$,

$$\int_{\mathbb{R}} \phi\left(\frac{f_\beta(x)}{t}\right) \mathrm{d}x \leq \int_{[-t,t]^c} \frac{1}{2} \frac{|x|^{-2\beta}}{t^2} \mathrm{d}x + \int_{[-t,t]} \frac{|x|^{-\beta}}{t} \mathrm{d}x = \frac{1}{2\beta - 1} t^{-2\beta-1} + \frac{2}{1-\beta} t^{-\beta} \ .$$

Evidently, when $\beta \in (1/2, 1)$, this can be made smaller than $\phi(1) = 1/2$ by choosing t large enough.

For **(d)**, recall that

$$\int_{\mathbb{R}} \phi\left(\frac{|f(x)|}{\|f\|_\phi}\right) \mathrm{d}x \le \phi(1) = \frac{1}{2}\ .$$

Define $A = \{x\ :\ |f(x)| \ge \|f\|_\phi\}$. Then

$$\begin{aligned} \frac{1}{2} &\ge \int_A \phi\left(\frac{|f(x)|}{\|f\|_\phi}\right) \mathrm{d}x + \int_{A^c} \phi\left(\frac{|f(x)|}{\|f\|_\phi}\right) \mathrm{d}x \\ &\ge \int_A \frac{|f(x)|}{\|f\|_\phi} \mathrm{d}x + \int_{A^c} \frac{1}{2}\left(\frac{|f(x)|}{\|f\|_\phi}\right)^2 \mathrm{d}x \end{aligned}$$

It is evident from this that if we define $g := f1_A$ and $h := f1_{A^c}$ that $g \in L^1$, $h \in L^2$ and $g = g + h$.

Next, for $f \in L^1$, $t > 0$, define $B := \{x\ :\ |f(x)| > t\}$.

$$\int_{\mathbb{R}} \phi\left(\frac{|f(x)|}{t}\right) \mathrm{d}x \le \int_{B^c} \frac{1}{2}\frac{|f(x)|^2}{t^2} \mathrm{d}x + \int_B \frac{|f(x)|}{t} \mathrm{d}x \le \frac{1}{t}\|f\|_1$$

since on B^c, $|f|^2/t^2 \le |f|/t$. Likewise, for $f \in L^2$,

$$\int_{\mathbb{R}} \phi\left(\frac{|f(x)|}{t}\right) \mathrm{d}x \le \int_{B^c} \frac{1}{2}\frac{|f(x)|^2}{t^2} \mathrm{d}x + \int_B \frac{|f(x)|}{t} \mathrm{d}x \le \frac{1}{t^2}\|f\|_2^2$$

since on B, $|f|^2/t^2 \ge |f|/t$. altogether, we see that when for $f \in L^1$, $\|f\|_\phi \le 2\|f\|_1$ and for $f \in L^2$, $\|f\|_\phi \le \sqrt{2}\|f\|_2$. This shows that the norm $\|\cdot\|_{L^1+L^2}$ on $L^1 + L^2$ from Folland's Exercise 4, chapter 6 satisfies $\|\cdot\|_\phi \le 2\|\cdot\|_{L^1+L^2}$, and we have seen that the functions in L_ϕ are precisely the functions in $L^1 + L^2$. Since $L^1 + L^2$ is complete in both of these norms, it is a consequence of the open mapping theorem that the two norms are equivalent.

2. Let X be a non-reflexive Banach space. Let $\phi \in X^{**}$ be such the ϕ is not in the image of X under the canonical embedding of X into X^{**}. Let $H := \ker(\phi)$, which is a closed subspace of X^*.

(a) Show that the function $x \mapsto |||x|||$ defined by

$$|||x||| = \sup\{|L(x)|\ :\ L \in H \quad \textit{and} \quad \|L\| = 1\}$$

is a norm on X such that for all $x \in X$, $|||x||| \le \|x\|$.

(b) Show that $(X, |||\cdot|||)$ is a Banach space.

(c) Show that H is a *norming subspace* of X^*, meaning that there exists a $c \in (0,1)$ such that for all $x \in X$,

$$\sup\{|L(x)|\ :\ L \in H \quad \textit{and} \quad \|L\| = 1\}\ \ge c\|x\|\ .$$

SOLUTION We may assume that $\|\phi\|_{**} = 1$. Since $|||\cdot|||$ is defined as the supremum of a family of convex and homogeneous functions, it is evident that $|||\cdot|||$ is convex and homogeneous. It remains to show that $|||x||| > 0$ for $x \ne 0$.

To show this, we begin with a general observation. Let $(Y =, \|\cdot\|)$ be a Banach space. Let $L, M \in Y^*$ with $\{L, M\}$ linearly independent. Then $\ker(L) \not\subset \ker(M)$ and $\ker(M) \not\subset \ker(L)$. To prove this, it suffices by symmetry to show that $\ker(M) \not\subset \ker(L)$.

Since $L \neq 0$, there is some y_0 such that $L(y_0) = 1$. Then for all $y \in Y$, $y = (y - L(y)y_0) + L(y)y_0$, and $(y - L(y)y_0) \in \ker(L)$. Then if $\ker(M) \subset \ker(L)$, it follows that

$$M(y) = M(y - L(y)y_0) + M(L(y)y_0) = M(y_0)L(y) \ .$$

Since y is arbitrary, this means that $M = L(y_0)L$, and this contradicts the linear independence of $\{L, M\}$. Hence $\ker(M) \not\subset \ker(L)$.

To apply this to the problem at hand, we take $Y = X^*$, and then for any $x \in X$, $x \neq 0$, consider the set $\{\phi, \phi_x\}$ in $Y^* = X^{**}$. For all $\alpha \in \mathbb{C}$, $\alpha\phi_x = \phi_{\alpha x}$, which is in the image of X under the canonical embedding, and so by hypothesis $\phi \neq \alpha\phi_x$ for any $x \in C$. That is, $\{\phi, \phi_x\}$ is linearly independent. By the observation made above, there exists $L_x \in \ker(\phi)$ such that $L_x \notin \ker(\phi_x)$, and we may assume that $\|L_x\|_* = 1$. . Hence, by the definition of $\|\!| \cdot |\!\|$,

$$|\!|\!| x |\!|\!| \geq |L_x(x)| > 0 \ .$$

For the rest of of this problem, we need to “upgrade” this qualitative result to a quantitative result. Here is an additional pice of information that has not been used so far: The image V of X under the canonical embedding in X^{**} is a closed subspace of X^{**}, and hence for some $r > 0$, $B_{X^{**}}(r, \phi) \cap V = \emptyset$. That is, for all $z \in X$, $\|\phi - \phi_z\|_{**} \geq c$. Fixing $x \neq 0$, and replacing z by αx, so that $\phi_z = \alpha\phi_x$, we have that $\inf_{\alpha\mathbb{C}} \|\phi - \alpha\phi_x\| \geq c$, with $c > 0$ independent of x.

0.1 LEMMA. *Let $(Y, \|\cdot\|)$ be a Banach space. Let $L, M \in Y^*$ with $\|M\|_*, \|L\|_* = 1$, and suppose that for some $c > 0$,*

$$\inf_{\alpha\in\mathbb{C}} \|M - \alpha L\|_* \geq c \ . \tag{0.1}$$

Then

$$\sup\{|M(z)| \ : z \in \ker(L) \quad \text{and} \quad \|z\| \leq 1\} \geq \frac{c}{2} \ . \tag{0.2}$$

Proof. For any $\epsilon > 0$, we may choose $y_0 \in Y$ such that $\|y_0\| \leq 1 + \epsilon$ and $L(y_0) = 1$. Then for all $y \in Y$,

$$y = (y - L(y)y_0) + L(y)y_0 \ ,$$

and $y - L(y)y_0 \in \ker(L)$. Then $M(y) = M(y - L(y)y_0) + M(y_0)L(y)$, which yields

$$|M(y) - M(y_0)L(y)| = |M(y - L(y)y_0)| \ .$$

By (0.1), there is $y \in Y$, $\|y\| = 1$, such that $|M(y) - M(y_0)L(y)| \geq c - \epsilon$. Therefore, $|M(y - L(y)y_0)| \geq c - \epsilon$, and $\|y - L(y)y_0\| \leq 2 + \epsilon$. It follows that if we define $z = \|y - L(y)y_0\|^{-1}(y - L(y)y_0)$, then $\|z\| = 1$, $z \in \ker(L)$, and $M(z) \geq (c - \epsilon)/(2 + \epsilon)$. Since $\epsilon > 0$ is arbitrary, (0.2) is proved. □

We now apply the problem at hand in the case $Y = X^*$, and we take $\phi \in X^{**}$, $\|\phi\|_{**} = 1$ that is not in the range of the canonical injection of X into X^{**}, and we take $x \in X$, $\|x\| = 1$ so that $\|\phi_x\| = 1$. Then as explained above, for some $c > 0$ independent of x,

$$\inf\{\|\phi - \alpha\phi_x\|_{**} \ ; \alpha \in \mathbb{C}\} > c \ .$$

It follows from the lemma that there exists $M_x \in X^*$ such that $\|M_x\| = 1$, $M_x \in \ker(\phi_x)$ and $|\phi(M_x)| \geq c/3$.

Now for any $L \in X^*$ with $\|L\|_* = 1$, write

$$L = \left(L - \frac{\phi(L)}{\phi(M_x)}M_x\right) + \frac{\phi(L)}{\phi(M_x)}M_x \ .$$

Since $M_x \in \ker(\phi_x)$, $x \in \ker(M_x)$, and hence

$$L(x) = \left(L - \frac{\phi(L)}{\phi(M_x)}M_x\right)(x) \ .$$

However, $\left(L - \frac{\phi(L)}{\phi(M_x)}M_x\right) \in \ker(\phi)$ and $\left\|L - \frac{\phi(L)}{\phi(M_x)}M_x\right\| \le \|L\| + \frac{|\phi(L)|}{|\phi(M_x)|}\|M_x\| = 1 + \frac{3}{c}$. Therefore,

$$\sup\{|L(x)| \ : L \in X^* \ , \ \|L\|_* = 1\} \ \le \left(1 + \frac{2}{c}\right) \sup\{|L(x)| \ : L \in \ker(\phi) \ , \ \|L\|_* = 1\} \ .$$

That is, for all x with $\|x\| = 1$,

$$|\!|\!|x|\!|\!| \le \|x\| \le \left(1 + \frac{3}{c}\right) |\!|\!|x|\!|\!| \ ,$$

and by homogeneity, this is true in general. The completeness of X in the norm $|\!|\!| \cdot |\!|\!|$ follows immediately from the completeness of X in the norm $\|\cdot\|$.

3. Let X be a Banach space. Let Y be a closed subspace. and let K be a norm-compact subset of X. Show that $Y + K$ is closed.

SOLUTION Let $z \in X$, $z \notin Y + K$. We must find an $s > 0$ such that $B(s, z) \cap \{Y + K\} = \emptyset$. This is the case if and only if for all $w \in B(s, 0)$, $y \in Y$ and $x \in K$, $z + w \neq y + x$, which is the same as $z - y \neq x - w$. That is,

$$B(s, z) \cap \{Y + K\} = \emptyset \qquad \iff \qquad \{z + Y\} \cap \{B(s, 0) + K\} = \emptyset \ .$$

Since Y is closed, $z + Y$ is closed, and since $z \notin Y + K$, $\{z + Y\} \cap K = \emptyset$. Hence for each $x \in K$, there exists $r_x > 0$ such that $\{z + Y\} \cap B(2r_x, x) = \emptyset$. The totality of the sets $B(r_x, x)$, $x \in K$ is evidently an open cover of K. Since K is compact, there exists a finite set $\{x_1, \dots, x_n\} \subset K$ such that $K \subset \cup_{m=1}^n B(r_{x_m}, x_m)$. Now define $s = \min\{r_{x_1}, \dots, r_{x_m}\}$. Consider any $xinK$, Then for some m, $x \in B(r_{x_m}, x_m)$, and hence

$$B(s, x) \subset B(r_{x_m} + s, x_m) \subset B(2r_{x_m}, x_m) \ ,$$

and $\{z + Y\} \cap B(2r_{x_m}, x_m) = \emptyset$. Hence $\{z + Y\} \cap B(s, x) = \emptyset$, and since $x \in K$ is arbitrary, the proof is complete,.

4. Let $(X, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space. Let K be non-empty norm-closed convex subset of X. Use the uniform convexity and smoothness to show that for all $x \notin K$, there exists $L \in X^*$ and $a < b$ in $\mathbb{R}$ such that for all $y \in K$,

$$\Re(L(x)) = b > a \ge \Re(L(y)) \ .$$

Hint: Consider Theorem 3.5.4 from the Hilbert space chapter in the text version of the notes.

SOLUTION $K - x$ is norm-closed convex subset of X that does not contain 0. By the Projection Lemma for uniformly convex spaces, there exists a unique $y_0 \in K$ such that $\|x - y_0\| < \|x - y\|$ for all $y \in K$, $y \neq y_0$.

Let $y \in X - x$, $t \in [0, 1]$. Since $K - x$ is convex, $(1 - t)y_0 + ty \in K - x$. Therefore,

$$\|x - y_0 + t(y - y_0)\| \geq \|x - y_0\| \tag{0.3}$$

for all $t \in [0, 1]$. Since X is uniformly smooth, the norm function $\| \cdot \|$ is Frechet differentiable at all non-zero vectors, and hence at $x - y_0$, and if L_{x-y_0} is the unit vector in X^* such that $L_{x-y_0}(x - y_0) = \|x - y_0\|$ (which is unique by the uniform convexity of X),

$$\|x - y_0 + t(y - y_0)\| = \|x - y_0\| + t\Re(L_{x-y_0}(y - y_0)) + o(t) \ . \tag{0.4}$$

Combining this with (0.3), we have that $t\Re(L_{x-y_0}(y - y_0))) + o(t) \geq 0$ for all $t \in [0, 1]$. (Note that (0.4) is valid without restriction on the sign of t.) Dividing through by $t > 0$, and taking the limit $t \to 0$, we obtain

$$\Re(L_{x-y_0}(y - y_0)) \geq 0 \ .$$

Then since

$$\Re(L_{x-y_0}(y - y_0)) = \Re(L_{x-y_0}((y - x) + (x - y_0)) = \Re(L_{x-y_0}(y)) - \Re(L_{x-y_0}(x)) + \|x - y_0\| \ ,$$

if we define $L = -L_{x-y_0}$,

$$\Re(L(y)) \leq \Re(L(x)) - \|x - y_0\| \geq 0 \ ,$$

which gives us what we seek.