Homework Assignment 6, Math 502, Spring 2017

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April 6, 2017

1. Let ϕ be the function defined on $\mathbb{R}$ by $\phi(s) = \begin{cases} \frac{1}{2}s^2 & |s| < 1 \\ |s| - 1/2 & s \geq 1 \ . \end{cases}$.

(a) Show that ϕ is convex and continuous, and is an Orlicz function.

(b) Compute ϕ^* and ϕ^{**}, with the latter computation giving a direct verification of the Fenchel-Moreau Theorem in this case.

(c) Let $(L_\phi, \|\cdot\|_\phi)$ be the associated Orlicz space for the measure space $(\mathbb{R}, \mathcal{B}, \mathrm{d}x)$, where $\mathrm{d}x$ denotes Lebesgue measure. Consider the functions $f_\beta = |x|^{-\beta}$, $\beta \in (1/2, 1)$ Show that none of these functions belong to L^p for any $1 \leq p \leq 2$, but that each of them belongs to L_ϕ.

(d) Show moreover that if $f \in L_\phi$ there exist functions $g \in L^1$ and $h \in L^2$ such that $f = g + h$, and show that $L^1 \subset L_\phi$ and $L^2 \subset L_\phi$. What is the relation between L^ϕ and the Banach space considered in Folland's Exercise 4, chapter 6 for $p = 1$ and $r = 2$?

2. Let X be a non-reflexive Banach space. Let $\phi \in X^{**}$ be such the ϕ is not in the image of X under the canonical embedding of X into X^{**}. Let $H := \ker(\phi)$, which is a closed subspace of X^*.

(a) Show that the function $x \mapsto |||x|||$ defined by

$$|||x||| = \sup\{|L(x)| \ : \ L \in H \quad \textit{and} \quad \|L\| = 1\}$$

is a norm on X such that for all $x \in X$, $|||x||| \leq \|x\|$.

(b) Show that $(X, |||\cdot|||)$ is a Banach space.

(c) Show that H is a *norming subspace* of X^*, meaning that there exists a $c \in (0,1)$ such that for all $x \in X$,

$$\sup\{|L(x)| \ : \ L \in H \quad \textit{and} \quad \|L\| = 1\} \ \geq c\|x\| \ .$$

3. Let X be a Banach space. Let Y be a closed subspace. and let K be a norm-compact subset of X. Show that $Y + K$ is closed.

4. Let $(X, \|\cdot\|)$ be a uniformly convex and uniformly smooth Banach space. Let K be non-empty norm-closed convex subset of X. Use the uniform convexity and smoothness to show that for all $x \notin K$, there exists $L \in X^*$ and $a < b$ in $\mathbb{R}$ such that for all $y \in K$,

$$\Re(L(x)) = b > a \geq \Re(L(y)) \ .$$

Hint: Consider Theorem 3.5.4 from the Hilbert space chapter in the text version of the notes.