Homework Assignment 5, Math 502, Spring 2017

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1. Let ϕ be the function defined on $\mathbb{R}$ by $\phi = \begin{cases} \infty & x < 0 \\ x \ln x & x \geq 0 \ . \end{cases}$.

(a) Show that ϕ is convex and lower-semicontinuous.

(b) Compute ϕ^* and ϕ^{**}, with the latter computation giving a direct verification of the Fenchel-Moreau Theorem in this case.

2. Let $1 \leq p < q < \infty$. Let ϕ be the function defined on $\mathbb{R}$ by

$$\phi(x) = \max\left\{ \frac{1}{p}|x|^p \ , \ \frac{1}{q}|x|^q \right\} \ .$$

(a) Show that ϕ is convex and continuous. Compute $\partial\phi(1)$ and show that ϕ is an Orlicz function. Also compute $\partial\phi(x_0)$ at $x_0 = (q/p)^{1/(q-p)}$.

(b) Compute ϕ^* .

(c) Show that $L_\phi = L^p \cap L^q$ and find explicit constant C_1 and C_1 such that for all $f \in L^p \cap L^q$,

$$C_0(\|f\|_p + \|f\|_q) \leq \|f\|_\phi \leq C_1(\|f\|_p + \|f\|_q) \ . \qquad (*)$$

Hint for the last part: Let $f \in L_\phi$, the Orlicz space given by ϕ for the measure space $(\Omega, \mathcal{M}, \mu)$. Let A be the set $\{x \in \Omega \ : \ |f(x)| \geq \|f\|_\phi\}$. Show that $f1_A \in L^q(\Omega, \mathcal{M}, \mu)$ and that $f1_{\Omega\backslash A} \in L^p(\Omega, \mathcal{M}, \mu) \cap L^\infty(\Omega, \mathcal{M}, \mu)$. Conversely, show that if $f = g + h$ where $g \in L^q(\Omega, \mathcal{M}, \mu)$ and $h \in L^p(\Omega, \mathcal{M}, \mu) \cap L^\infty(\Omega, \mathcal{M}, \mu)$, then $f \in L_\phi$.

3. Fix a measure space $(\Omega, \mathcal{M}, \mu)$, and for $1 \leq p \leq \infty$, let L^p denote $L^p(\Omega, \mathcal{M}, \mu)$. Show that for all $1 \leq p < r \leq \infty$, $f \mapsto \|f\|_p + \|f\|_r$ is a norm on $L^p \cap L^r$, making $L^p \cap L^r$ a Banach space.

(b) Show that for all q with $p < q < r$, $L^p \cap L^r \subset L^q$ and the inclusion map is continuous for the norm from part **(a)**.

4. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space with $\mu(\Omega) = 1$; i.e., a probability space.

(a) Show that for all $1 \leq p < q \leq \infty$, if $f \in L^q$, then $f \in L^p$ and $\|f\|_p \leq \|f\|_q$.

(b) Show that for all $f \in L^q$, $q > 1$, $\log(\|f\|_q) \geq \int_\Omega \log|f| \mathrm{d}\mu$.

5. Let $\Omega = \mathbb{N}$, $\mathcal{M} = 2^{\mathbb{N}}$, and let μ be counting measure so that for $1 \leq p \leq \infty$, $(\Omega, \mathcal{M}, \mu) = \ell^p$.

(a) Show that for all $1 \leq q < p \leq \infty$, if $f \in \ell^q$, then $f \in \ell^p$ and $\|f\|_p \leq \|f\|_q$.

(b) Show that ℓ_p is separable for $1 \leq p < \infty$, but not for $p = \infty$.