Solutions for Homework Assignment 4, Math 502, Spring 2017

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1. Let $\mathcal{H} = L^2([0, 2\pi], \mathcal{B}, \mu)$ where μ is normalized Lebesgue measure. Let T be the linear map on $\mathcal{H}$ defined by

$$Tf(x) = \frac{i}{2}\left(\int_0^x f(t)\mathrm{d}\mu(t) - \int_x^{2\pi} f(t)\mathrm{d}\mu(t)\right) .$$

(a) Show that T is a Hilbert-Schmidt operator (and therefore compact), and that T is self adjoint.

(b) Show that if f is an eigenfunction of T, then f is continuously differentiable on $(0, 2\pi)$, and use this to find all of the eigenvalues and eigenfunctions of T.

(c) The eigenfunctions will be closely related to (but not the same as) the orthonormal sequence $\{u_n\}_{n\in\mathbb{Z}}$ where $u_n(x) = e^{inx}$. Use the result of **(b)** to give another proof of the completeness of $\{u_n\}_{n\in\mathbb{Z}}$.

SOLUTION Define $K(x,t) = \begin{cases} i/2 & t < x \\ 0 & x = t \\ -i/2 & t > x \end{cases}$. Then $Tf(x) = \int_0^{2\pi} K(x,t)f(t)\mathrm{d}t$, and then T is Hilbert-Schmidt since $\int_0^{2\pi}\int_0^{2\pi} |K(x,t)|^2\mathrm{d}x\mathrm{d}t = \frac{1}{4} < \infty$. Also notie that $K(t,x) = \overline{K(x,t)}$. Therefore, for all $f, g \in \mathcal{H}$,

$$\langle f, Tg\rangle = \int_0^{2\pi}\int_0^{2\pi} \overline{f(x)}K(x,t)g(t)\mathrm{d}x\mathrm{d}t = \int_0^{2\pi}\int_0^{2\pi} \overline{K(t,x)f(x)}g(t)\mathrm{d}x\mathrm{d}t = \langle Tf, g\rangle \ .$$

Thus, $T = T^*$. For **(b)**, note that for all $f \in \mathcal{H}$, and all $x < y \in [0, 2\pi]$,

$$|Tf(x) - Tf(y)| \leq \int_x^y |f(t)|\mathrm{d}t \leq \sqrt{|y-x|}\|f\|$$

Therefore, every function in the range of f is continuous, and is even Hölder continuous of order $1/2$. Since all eigenfunctions of T are in the range of T, if f is an eigenfunction of T, with $Tf = \lambda f$, f is continuous, and by the Fundamental Theorem of Calculus, Tf is differentiable with $\lambda f'(x) = (Tf)'(x) = if(x)$. This shows that if $Tf = \lambda f$, and $\lambda = 0$, then $f = 0$. Hence 0 is not an eigenvalue of T. Hence if f is an eigenfunction, $f'(x) = (i/\lambda)f(x)$, and hence $f(x)$ is a multiple of $e^{ix/\lambda}$.

By what we have proved above, every function in the range of T is continuous, and so for all $f \in \mathcal{H}$, $Tf(0)$ and $Tf(2\pi)$ are well defined. From the formula defining f, $Tf(0) = -Tf(2\pi)$. Hence if f is an eigenfunction of T with eigenvalue λ, $f(0) = -f(2\pi)$. Therefore $e^{2\pi/\lambda} = -1$, and so $1/\lambda = 1/2 + k$, $k \in \mathbb{Z}$.

Therefore, for $j, k \in \mathbb{Z}$, define $v_k(x) = e^{i(1/2+k)x}$. It is easy to check that v_k is indeed an eigenvector with eigenvalue $\lambda_k = 2/(1+2k)$. Hence the eigenvalues of T are the numbers $\lambda_k := 2/(1+2k)$, $k \in \mathbb{Z}$, and the eigenfunctions are the non-zero multiples of the functions v_k, $k \in \mathbb{Z}$.

For **(c)**, note that we have shown $\ker(T) = \{0\}$, and by the Hilbert-Schmidt Theorem, there is an orthonormal basis of $\ker(T)^\perp = \mathcal{H}$ consisting of eigenfunctions of T. Since eigenvectors corresponding to distinct eigenvalues of a self adjoint operator are orthogonal, $\{v_k\}_{k\in\mathbb{Z}}$ is a complete orthonormal set in $\mathcal{H}$.

Now note that for all $k \in \mathbb{Z}$, $u_k(x) = e^{-ix/2}v_k(x)$ for all x. For $f \in H$, define $Uf(x) = e^{-ix/2}f(x)$ then clearly U is unitary, and so $\{Uv_j\}_{k\in\mathbb{Z}}$ is a complete orthonormal set in $\mathcal{H}$.

2. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Let $\mathscr{W}$ be the weakly open sets in $\mathcal{H}$, and let $\mathscr{O}$ be the norm-open sets in $\mathcal{H}$. A linear transformation T from $\mathcal{H}$ to $\mathcal{H}$ is *weak-weak* continuous in case for all $W \in \mathscr{W}$, $T^{-1}(W) \in \mathscr{W}$. A linear transformation T from $\mathcal{H}$ to $\mathcal{H}$ is *strong-weak* continuous in case for all $W \in \mathscr{W}$, $T^{-1}(W) \in \mathscr{O}$. A linear transformation T from $\mathcal{H}$ to $\mathcal{H}$ is *weak-strong* continuous in case for all $U \in \mathscr{O}$, $T^{-1}(U) \in \mathscr{W}$. A linear transformation T from $\mathcal{H}$ to $\mathcal{H}$ is *strong-strong* continuous in case for all $U \in \mathscr{O}$, $T^{-1}(U) \in \mathscr{O}$.

Show that a linear transformation T on $\mathcal{H}$ is weak-weak continuous if and only if it is strong-strong continuous if and only if it is strong-weak continuous if and only if it is weak -weak continuous In other words, show that an operator has any of these types of continuity if and only if it is bounded, so these are not really three different types of continuity. Show that a linear transformation T on $\mathcal{H}$ is weak-strong continuous if and only if it is finite rank.

SOLUTION We know that a linear transformation T on $\mathcal{H}$ is strong-strong continuous if and only if T is bounded. Therefore we must show that the following are equivalent:

(1) T is bounded.

(2) T is weak-weak continuous.

(3) T is strong-weak continuous.

Suppose that T is bounded. Then so is T^*. For all $f, g \in \mathcal{H}$, $\langle f, Tg\rangle = \langle T^*f, g\rangle$. Let $\mathcal{F} := \{f_1, \dots, f_n\}$ be any finite subset of $\mathcal{H}$, and let $\epsilon > 0$. Then

$$Tg \in V_{\mathcal{F},\epsilon} \quad \iff \quad |\langle T^*f_j, g\rangle| < \epsilon \quad \text{for} \quad j = 1, \dots, n\ .$$

$Tg \in V_{\mathcal{F},\epsilon} \iff g \in V_{\mathcal{G},\epsilon}$, where $\mathcal{G} = \{T^*f_1, \dots, T^*f_n\}$. Thus, when T is bounded $T^{-1}(V_{\mathcal{F},\epsilon}) = V_{\mathcal{G},\epsilon}$, and hence T is weak-weak continuous because it pulls back sets in a neighborhood base at 0 into sets in this neighborhood base.

This shows that *(1)* implies *(2)*, and since the strong topology is stronger than the weak topology, if T is weak-weak continuous, it is also strong-weak continuous, and hence *(2)* implies *(3)*.

We now show that *(3)* implies *(1)* Suppose that T is not bounded. Then there is a sequence $\{u_n\}_{n\in\mathbb{N}}$ of unit vectors such that $\|Tu_n\| \geq n^2$ for all n. Evidently, $\lim_{n\to\infty} \left\|\frac{1}{n}u_n\right\| = 0$, and hence if T is strong-weak continuous, $\{T(n^{-1}u_n)\}_{n\in\mathbb{N}}$ converges weakly to zero. But weakly convergent sequences are bounded, so that $\{\|T(n^{-1}u_n)\|\}_{n\in\mathbb{N}}$ must be bounded. But $\|T(n^{-1}u_n)\| \geq n$, so this is impossible. Hence is T is not bounded, T is not strong-weak continuous, and *(3)* implies *(1)*.

If T is weak-strong continuous, then for some $\epsilon > 0$ and some $\mathcal{F} := \{f_1, \ldots, f_n\}$,

$$V_{\mathcal{F},\epsilon} \subset T^{-1}(B(1,0)) \ .$$

We may suppose without loss of generality, applying the Gram-Schmidt algorithm, and adjusting ϵ as needed, that $\mathcal{F}$ is orthonormal. If $x \in \mathcal{F}^\perp$, $tx \in V_{\mathcal{F},\epsilon}$ for all t, and hence $\|T(tx)\| = |t|\|Tx\| < 1$ for all t. It follows that $Tx = 0$. Let P be the orthogonal projection onto the span of $\mathcal{F}$. Then for all x, $Tx = TPx$. Since $\mathcal{F}$ is orthonormal, $Px = \sum_{j=1}^n \langle f_j, x\rangle f_j$, and hence for all $x \in H$,

$$Tx = \sum_{j=1}^n \langle f_j, x\rangle Tf_j = \left(\sum_{j=1}^n |Tf_j\rangle\langle f_j|\right) x \ .$$

Hence T is finite rank.

3. **(a)** Let $\mathcal{H}$ be a Hilbert space. Let $T \in \mathscr{B}(\mathcal{H})$, and suppose that $\mathcal{K}$ is a closed subspace of $\mathcal{H}$ in the range of T. Let P be the orthogonal projection onto $\mathcal{K}$. Show that $(PT)|_{\ker(PT)^\perp}$ is an injective bounded linear transformation from $\ker(PT)^\perp$ onto $\mathcal{K}$ with a bounded inverse.

(b) Show that if $T \in \mathscr{B}(\mathcal{H})$ is such that its range contains a closed infinite dimensional subspace, then T is not compact.

SOLUTION since $\mathcal{K}$ is contained in the range of T, for all $y \in \mathcal{K}$, there is some $x \in \mathcal{H}$ such that $Tx = y$, and then of course $y = PTx$. Let Q be the orthogonal projection onto $\ker(PT)^\perp$. Then for all $x \in \mathcal{H}$, $x = Qx + Q^\perp x$ and so $PTx = PTQx + PTQ^\perp x = PTQx$. Hence the range of PT, namely $\mathcal{K}$, is the same as the range of PT restricted to $\ker(PT)^\perp$. Since PI is injective on $\ker(PT)^\perp$, PT us a continuous linear isomorphism from $\ker(PT)^\perp$ to $\mathcal{K}$. By the Open Mapping Theorem it has a bounded inverse.

Suppose that $\mathcal{K}$ is an infinite dimensional subspace in the range of $T \in \mathscr{B}(\mathcal{H})$. Let $\{u_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $\mathcal{K}$. Define $v_n = (PT)^{-1}u_n$. Then by the first part, $\{v_n\}_{n\in\mathbb{N}}$ is a bounded sequence. Since $\mathcal{H}$ is weakly sequentially compact, there is a weakly convergent subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$. But $\{Tv_{n_k}\}_{k\in\mathbb{N}}$ is orthonormal, and hence cannot be norm convergent. Hence T is not compact.

4. Let $\epsilon \in (0,1)$. A sequence $\{u_n\}_{n\in\mathbb{N}}$ of unit vectors in a Hilbert space $\mathcal{H}$ is ϵ-*almost orthonormal* in case

$$\sum_{m\neq n} |\langle u_m, u_n\rangle|^2 \leq \epsilon^2 \ .$$

(a) Suppose that $\{u_n\}_{n\in\mathbb{N}}$ is ϵ-almost orthonormal. Show that for all $\{\alpha_n\}_{n\in\mathbb{N}} \in \ell^2$,

$$(1-\epsilon)\left(\sum_{n=1}^\infty |\alpha_n|^2\right)^{1/2} \leq \left\|\sum_{n=1}^\infty \alpha_n u_n\right\| \leq (1+\epsilon)\left(\sum_{n=1}^\infty |\alpha_n|^2\right)^{1/2} \ .$$

(b) Part **(a)** shows that $\{u_n\}_{n\in\mathbb{N}}$ is linearly independent. Let $\{v_n\}_{n\in\mathbb{N}}$ be the orthonormal sequence produced from $\{u_n\}_{n\in\mathbb{N}}$ by the Gram-Schmidt algorithm. Show that $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

(c) Show that if $\{u_n\}_{n\in\mathbb{N}}$ is any sequence of unit vectors in $\mathcal{H}$ that converges weakly to zero, then for all $\epsilon \in (0,1)$ there is a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ that is ϵ-almost orthonormal.

(d) Show that $T \in \mathscr{B}(H)$ is compact if and only if for each orthonormal sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$, $\lim_{n\to\infty} \|Tu_n\| = 0$.

(e) Show that $T \in \mathscr{B}(H)$ is compact if and only if every closed subspace in the range of T is finite dimensional.

SOLUTION We compute

$$\left\|\sum_{n=1}^{\infty} \alpha_n u_n\right\|^2 = \sum_{m,n=1}^{\infty} \overline{\alpha_m}\alpha_n \langle u_m, u_n\rangle = \sum_{n=1}^{\infty} |\alpha_n|^2 + \sum_{m\neq n}^{\infty} \overline{\alpha_m}\alpha_n \langle u_m, u_n\rangle \ .$$

Therefore, using the Cauchy-Schwarz inequality on the double sum,

$$\begin{aligned}\left|\left\|\sum_{n=1}^{\infty} \alpha_n u_n\right\|^2 - \sum_{n=1}^{\infty} |\alpha_n|^2\right| &= \left|\sum_{m\neq n}^{\infty} \overline{\alpha_m}\alpha_n \langle u_m, u_n\rangle\right| \\ &\leq \left(\sum_{m,n=1}^{\infty} |\alpha_m|^2|\alpha_n|^2\right)^{1/2} \left(\sum_{m\neq n}^{\infty} |\langle u_m, u_n\rangle|^2\right)^{1/2} \\ &\leq \epsilon \sum_{n=1}^{\infty} |\alpha_n|^2 \ .\end{aligned}$$

This shows that

$$(1-\epsilon)\sum_{n=1}^{\infty} |\alpha_n|^2 \leq \left\|\sum_{n=1}^{\infty} \alpha_n u_n\right\|^2 \leq (1+\epsilon)|\alpha_n|^2 \ .$$

Now use the facts that $1+\epsilon \leq (1+\epsilon)^2$ since $\epsilon > 0$, and $1-\epsilon > 1-2\epsilon+\epsilon^2 = (1-\epsilon)^2$ since $\epsilon \in (0,1)$.

For **(b)**, let P_n be the orthogonal projection onto the span of $\{u_1,\dots,u_n\}$, which is the same as the span of $\{v_1,\dots,v_n\}$. By definition, $v_n := \|P_{n-1}^\perp u_n\|^{-1} P_{n-1}^\perp u_n$ so that $P_{n-1}^\perp u_n = \|P_{n-1}^\perp u_n\| v_n$ and

$$\|u_n - v_n\| \leq \|u_n - P_{n-1}^\perp u_n\| + \left\|\|P_{n-1}^\perp u_n\| v_n - v_n\right\| = \|P_{n-1}u_n\| + |\|P_{n-1}^\perp u_n\| - 1| \ .$$

By the Pythagorean Theorem, $\|P_{n-1}^\perp u_n\| = \sqrt{1 - \|P_{n-1}u_n\|^2}$. Altogether,

$$\|u_n - v_n\| \leq \|P_{n-1}u_n\| + 1 - \sqrt{1 - \|P_{n-1}u_n\|^2} \ .$$

Hence to show that $\lim_{n\to\infty} \|u_n - v_n\| = 0$, it suffices to show that $\lim_{n\to\infty} \|P_{n-1}u_n\| = 0$.

Since $P_{n-1}u_n$ is in the span of $\{u_1,\dots,u_{n-1}\}$, there are numbers $\alpha_1,\dots,\alpha_{n-1}$ such that $P_{n-1}u_n = \sum_{j=1}^{n-1} u_j$ and hence

$$\sum_{j=1}^{n-1} |\alpha_j|^2 \leq (1-\epsilon)^{-2}\|P_{n-1}u_n\|^2 \leq (1-\epsilon)^{-2} \ . \tag{0.1}$$

But

$$\begin{aligned}\|P_{n-1}u_n\|^2 &= \langle P_{n-1}u_n, P_{n-1}u_n\rangle = \langle P_{n-1}u_n, u_n\rangle \\ &= \sum_{j=1}^{n-1}\overline{\alpha_j}\langle u_j, u_n\rangle \le \left(\sum_{\jmath=1}^{n-1}|\alpha_j|^2\right)^{1/2}\left(\sum_{\jmath=1}^{n-1}|\langle u_j, u_n\rangle|^2\right)^{1/2} \\ &\le \frac{\|P_{n-1}u_n\|}{1-\epsilon}\left(\sum_{j=1}^{n-1}|\langle u_j, u_n\rangle|^2\right)^{1/2}\end{aligned}$$

where we used (0.1) in the last line. It follows that $\|P_{n-1}u_n\| \le \dfrac{1}{1-\epsilon}\left(\sum_{j\neq n}|\langle u_j, u_n\rangle|^2\right)^{1/2}$. Since $\sum_{n=1}^{\infty}\left(\sum_{j\neq n}|\langle u_j, u_n\rangle|^2\right) = \epsilon^2 < \infty$, $\lim_{n\to\infty}\left(\sum_{j\neq n}|\langle u_j, u_n\rangle|^2\right) = 0$, and hence $\lim_{n\to\infty}\|P_{n-1}u_n\| = 0$.

For **(c)**, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of unit vectors that converges weakly to zero. For each m, let P_m be the orthogonal projection onto the span of $\{u_1,\dots,u_m\}$. Then each P_m is finite rank, and hence compact. Therefore, $\lim_{n\to\infty}\|P_m u_n\| = 0$ for each m. Fix $\epsilon > 0$, and choose a subsequence $\{u_{n_k}\}_{n\in\mathbb{N}}$ recursively as follows. Take $u_{n_1} = u_1$. Then with $\{u_{n_1},\dots,u_{n_{m-1}}\}$ selected, choose u_{n_m} so that $\|P_{n_{m-1}}u_{n_m}\|^2 \le 2^{-m-1}\epsilon^2/m$ for all $n \ge n_m$. Passing to this subsequence, and dropping the subscripts, we have that

$$\sum_{m\neq n}|\langle u_m, u_n\rangle|^2 = 2\sum_{n=2}^{\infty}\sum_{m=1}^{n-1}|\langle u_m, u_n\rangle|^2 \ .$$

For $m < n$, $|\langle u_m, u_n\rangle|^2 \le \|P_{n-1}u_n\|^2 < 2^{-n-1}\epsilon^2/n$, so that

$$\sum_{m\neq n}|\langle u_m, u_n\rangle|^2 \le \sum_{n=1}^{\infty}2^{-m}\epsilon^2 = \epsilon^2 \ .$$

For **(d)**, If T is compact then for any orthonormal sequence $\{u_n\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty}\|Tu_n\| = 0$ since $\{u_n\}_{n\in\mathbb{N}}$ converges weakly to zero.

On the other hand, suppose that $\lim_{n\to\infty}\|Tw_n\| = 0$ whenever $\{w_n\}_{n\in\mathbb{N}}$ is orthonormal. If T is not compact, then there is some sequence $\{f_n\}_{n\in\mathbb{N}}$ that converges weakly to some f, but $\{Tf_n\}_{n\in\mathbb{N}}$ does not converge strongly to Tf. Then $\{f_n - f\}_{n\in\mathbb{N}}$ converges weakly to zero, but $\{T(f_n - f)\}_{n\in\mathbb{N}}$ does not converge strongly to zero. Hence for some $\epsilon > 0$, $\|T(f_n - f)\| \ge \epsilon$ for infinitely many n. Passing to a subsequence, we may suppose that $\|T(f_n - f)\| \ge \epsilon$ for all n, while $\{f_n - f\}_{n\in\mathbb{N}}$ still converges weakly to zero. Since weakly convergent sequences are bounded in norm, for some finite C, $\|f_n - f\| \le C$ for all n. Also, $\epsilon \le \|T(f_n - f)\| \le \|T\|\|f_n - f\|$, and so

$$\frac{\epsilon}{\|T\|} \le \|f_n - f\| \le C$$

for all n. Thus, if we define $u_n = \|f_n - f\|^{-1}(f_n - f)$, $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of unit vectors that converse weakly to zero, and such that $\|Tu_n\| \ge \epsilon/C$ for all n. By part **(c)**, passing to a further subsequence, we may assume that $\{u_n\}_{n\in\mathbb{N}}$ is δ-almost orthonormal, and then by part **(b)**, there is an orthonormal sequence $\{v_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\|u_n - v_n\| = 0$.

Then $\|Tu_n\| \leq \|T(u_n - v_n)\| + \|Tv_n\| \leq \|T\|\|u_n - v_n\| + \|Tv_n\|$. The second term on the right approaches 0 as n increases by the assumption on T since $\{v_n\}_{n\in\mathbb{N}}$ is orthonormal. Therefore, $\lim_{n\to\infty} \|Tu_n\| = 0$. this contradiction shows that when f is such that $\lim_{n\to\infty} \|Tw_n\| = 0$ whenever $\{w_n\}_{n\in\mathbb{N}}$ is orthonormal, there is no sequence $\{f_n\}_{n\in\mathbb{N}}$ that converges weakly to f, but such that $\{Tf_n\}_{n\in\mathbb{N}}$ does not converge strongly to f.

For **(e)**, we have already seen that if T is compact, every closed subspace in the range of T is finite dimensional. Now suppose that T is not compact. We shall show that there is an infinite dimensional closed subspace in the range of T.

Since T is not compact, there is some orthonormal sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $\{Tu_n\}_{n\in\mathbb{N}}$ does not converge strongly to zero. Hence there is some $\delta > 0$ so that $\|Tu_n\| > \delta$ for infinitely many n. Passing to a subsequence, we may suppose that is is true for all n. Summarizing so far, if T is not compact, there is an orthonormal sequence $\{u_n\}_{n\in\mathbb{N}}$, and a number $\delta > 0$ so that $\lim_{n\to\infty} \|Tu_n\| = a$ and

$$\delta \leq \|Tu_n\| \leq \|T\| \tag{0.2}$$

for all n.

For any orthonormal sequence $\{v_n\}_{n\in\mathbb{N}}$, and any $S \in \mathscr{B}(\mathcal{H})$, and any $f \in \mathcal{H}$,

$$\sum_{n=1}^{\infty} |\langle Sv_n, f\rangle|^2 = \sum_{n=1}^{\infty} |\langle v_n, S^*f\rangle|^2 \leq \|S^*f\|^2$$

by Bessel's inequality. Hence $\lim_{n\to\infty}\langle Sv_n, f\rangle = 0$ for all $f \in \mathcal{H}$ so that $\{Sv_n\}_{n\in\mathbb{N}}$ converges weakly to zero. Thus, in the problem at hand, $\{Tu_n\}_{n\in\mathbb{N}}$ converges weakly to zero.

Since $\delta \leq \|Tu_n\| \leq \|T\|$ for all n, if we define $w_n = \|Tu_n\|^{-1}Tu_n$ for each n, then $\{w_n\}_{n\in\mathbb{N}}$ converges weakly to zero, and consists of unit vectors. Hence, we may select a subsequence that is $\frac{1}{2}$-almost orthonormal. Passing to such a subsidence, we may suppose that $\{w_n\}_{n\in\mathbb{N}}$ is $\frac{1}{2}$-almost orthonormal.

Now let $\mathcal{H}_0$ be the closed span of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$, so that $\{u_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}_0$. for all $x = \sum_{n=1}^{\infty} \alpha_n u_n \in \mathcal{H}_0$,

$$Tx = \sum_{n=1}^{\infty} \alpha_n u_n = \sum_{n=1}^{\infty} \alpha_n \|Tu_n\| w_n \ ,$$

and since $\{w_n\}_{n\in\mathbb{N}}$ is ϵ almost orthonormal, (using only the lower bound that comes along with this)

$$\frac{\delta}{2}\|x\| = \frac{\delta}{2}\left(\sum_{n=1}^{\infty} |\alpha_n|^2\right)^{1/2} \leq \frac{1}{2}\left(\sum_{n=1}^{\infty} |\alpha_n|^2 \|Tu_n\|^2\right)^{1/2} \leq \|Tx\| \ .$$

In particular, if $\{Tx_n\}$ is Cauchy in $\mathcal{H}$, $\{x_n\}$ is Cauchy in $\mathcal{H}_0$, so there is an $x \in \mathcal{H}_0$ such that $\lim_{n\to\infty} x_n = x$, and then $\lim_{n\to\infty} Tx_n = Tx$. In particular, the range of $T|_{\mathcal{H}_0}$ is closed in $\mathcal{H}$, and is an infinite dimensional subspace (containing the linearly independent set $\{w_n\}_{n\in\mathbb{N}}$) in the range of $\mathcal{H}$.