Homework Assignment 4, Math 502, Spring 2017

Eric A. Carlen[1]
Rutgers University

February 28, 2017

1. Let $\mathcal{H} = L^2([0,2\pi],\mathcal{B},\mu)$ where μ is normalized Lebesgue measure. Let T be the linear map on $\mathcal{H}$ defined by

$$Tf(x) = \frac{i}{2}\left(\int_0^x f(t)\mathrm{d}\mu(t) - \int_x^{2\pi} f(t)\mathrm{d}\mu(t)\right) .$$

(a) Show that T is a Hilbert-Schmidt operator (and therefore compact), and that T is self adjoint.

(b) Show that if f is an eigenfunction of T, then f is continuously differentiable on $(0,2\pi)$, and use this to find all of the eigenvalues and eigenfunctions of T.

(c) The eigenfunctions will be closely related to (but not the same as) the orthonormal sequence $\{u_n\}_{n\in\mathbb{Z}}$ where $u_n(x) = e^{inx}$. Use the result of **(b)** to give another proof of the completeness of $\{u_n\}_{n\in\mathbb{Z}}$.

2. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Let $\mathscr{W}$ be the weakly open sets in $\mathcal{H}$, and let $\mathscr{O}$ be the norm-open sets in $\mathcal{H}$. A linear transformation T from $\mathcal{H}$ to $\mathcal{H}$ is *weak-weak* continuous in case for all $W \in \mathscr{W}$, $T^{-1}(W) \in \mathscr{W}$. A linear transformation T from $\mathcal{H}$ to $\mathcal{H}$ is *strong-weak* continuous in case for all $W \in \mathscr{W}$, $T^{-1}(W) \in \mathscr{O}$. A linear transformation T from $\mathcal{H}$ to $\mathcal{H}$ is *weak-strong* continuous in case for all $U \in \mathscr{O}$, $T^{-1}(U) \in \mathscr{W}$. A linear transformation T from $\mathcal{H}$ to $\mathcal{H}$ is *strong-strong* continuous in case for all $U \in \mathscr{O}$, $T^{-1}(U) \in \mathscr{O}$.

Show that a linear transformation T on $\mathcal{H}$ is weak-weak continuous if and only if it is strong-strong continuous if and only if it is strong-weak continuous. In other words, show that an operator has any of these types of continuity if and only if it is bounded, so these are not really three different types of continuity. Show that a linear transformation T on $\mathcal{H}$ is weak-strong continuous if and only if it is finite rank.

3. (a) Let $\mathcal{H}$ be a Hilbert space. Let $T \in \mathscr{B}(\mathcal{H})$, and suppose that $\mathcal{K}$ is a closed subspace of $\mathcal{H}$ in the range of T. Let P be the orthogonal projection onto $\mathcal{K}$. Show that $(PT)|_{\ker(PT)^\perp}$ is an injective bounded linear transformation from $\ker(PT)^\perp$ onto $\mathcal{K}$ with a bounded inverse.

(b) Show that if $T \in \mathscr{B}(\mathcal{H})$ is such that its range contains a closed infinite dimensional subspace, then T is not compact.

4. Let $\epsilon \in (0,1)$. A sequence $\{u_n\}_{n\in\mathbb{N}}$ of unit vectors in a Hilbert space $\mathcal{H}$ is *ϵ-almost orthonormal* in case

$$\sum_{m\neq m} |\langle u_m, u_n\rangle|^2 \leq \epsilon^2 .$$

(a) Suppose that $\{u_n\}_{n\in\mathbb{N}}$ is ϵ-almost orthonormal. Show that for all $\{\alpha_n\}_{n\in\mathbb{N}} \in \ell^2$,

$$\leq (1-\epsilon)\left(\sum_{n=1}^{\infty}|\alpha_n|^2\right)^{1/2} \leq \left\|\sum_{n=1}^{\infty}\alpha_n u_n\right\| \leq (1+\epsilon)\left(\sum_{n=1}^{\infty}|\alpha_n|^2\right)^{1/2} .$$

(b) Part **(a)** shows that $\{u_n\}_{n\in\mathbb{N}}$ is linearly independent. Let $\{v_n\}_{n\in\mathbb{N}}$ be the orthonormal sequence produced from $\{u_n\}_{n\in\mathbb{N}}$ by the Gram-Schmidt algorithm. Show that $\lim_{n\to\infty}\|u_n - v_n\| = 0$.

(c) Show that if $\{u_n\}_{n\in\mathbb{N}}$ is any sequence of unit vectors in $\mathcal{H}$ that converges weakly to zero, then for all $\epsilon \in (0,1)$ there is a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ that is ϵ-almost orthonormal.

(d) Show that $T \in \mathscr{B}(H)$ is compact if and only if for each orthonormal sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$, $\lim_{n\to\infty}\|Tu_n\| = 0$.

(e) Show that $T \in \mathscr{B}(H)$ is compact if and only if every closed subspace in the range of T is finite dimensional.