Solutions for Homework Assignment 2, Math 502, Spring 2017

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1: Show that a linear functional L on a normed space X is bounded if and only if its nullspace is closed.

SOLUTION: Let $V = \{x \ : \ L(x) = 0\} = L^{-1}\{0\}$. If L is bounded, and hence continuous, then V is closed since $\{0\}$ is closed.

For the converse, we may assume that L is not identically zero, or else the claim is trivial. In this case, there is some $x_0 \in X$ such that $L(x_0) = 1$. Since $x_0 \notin V$, if V is closed, there exists an $r > 0$ such that $B(r, x_0) \cap V = \emptyset$. Suppose that there exists $y \in B(1, 0)$ such that $|L(y)| > 1/r$. Replacing y by $z := e^{i\theta}y$ for some $\theta \in [0, 2\pi)$, we obtain a vector $z \in B(1, 0)$ such that $L(z) = |L(y)| > 1/r$. But then for all $t \in [-r, r]$, $x_0 + tz \in B(r, x_0)$, and $L(x_0 + tz) = 1 + t|L(y)|$ which is zero for $t = -1/|L(y)| \in (-r, r)$. This contradiction shows that there cannot exist any $y \in B(1, 0)$ such that $|L(y)| > 1/r$. Therefore, $\|L\| < 1/r$.

2: Let $(X, \|\cdot\|)$ be a normed vector space, and let Y be a proper, closed subspace of X. Show that for all $\alpha \in (0, 1)$, there exists $u \in X$, $\|u\| = 1$, such that $\alpha \leq \inf\{\|u - y\| \ : \ y \in Y\}$.

SOLUTION: Since Y is proper, there exists some $x_0 \notin Y$, and then since Y is closed, there exists some $r > 0$ such that $B(r, x_0) \cap Y = \emptyset$. Therefore, $d := \inf\{\|u - y\| \ : \ y \in Y\} \geq r > 0$. By the definition of d, for all $\alpha \in (0, 1)$, there exists $y_0 \in Y$ such that $\|x_0 - y_0\| < d/\alpha$. Then, by the definition of d,

$$\alpha\|x_0 - y_0\| \leq d \leq \|x_0 - y_0 - y\| \quad \text{for all} \quad y \in Y \ .$$

That is, $\alpha \leq \left\| \frac{x_0 - y_0}{\|x_0 - y_0\|} - \frac{y}{\|x_0 - y_0\|} \right\|$ for all $y \in Y$. Let $u = \|x_0 - y_0\|^{-1}(x_0 - y_0)$. N

3: Let $(X, \|\cdot\|)$ be an infinite dimensional normed vector space. Show that X contains an infinite sequence $\{u_j\}_{j\in\mathbb{N}}$ such that $\|u_j\| = 1$ for all j and $\|u_j - u_k\| \geq 1/2$ for all $j \neq k$, and hence that X is not locally compact.

SOLUTION: The sequence $\{u_n\}_{n\in\mathbb{N}}$ is constructed inductively as follows: Pick some $u_1 \in X$ with $\|u_1\| = 1$. For the induction, suppose that we have found vectors $\{u_1, \dots, u_n\}$ such that $\|u_j\| = 1$ for each j and if $j \neq k$, then $\|u_j - u_k\| \geq 1/2$. Let $V_n := \mathrm{span}(\{u_1, \dots, u_n\})$ which is is closed since it is finite dimensional proper since X is infinite dimensional. By Riesz's Lemma, we may choose u_{n+1} with $\|u_{n+1}\| = 1$ snd $\|u_{n+1} - y\| \geq 1/2$ for all $y \in Y$. In particular we have $\|u_j - u_j\| \geq 1/2$ for all $1 \leq j < k \leq n + 1$. No subsequence of $\{u_n\}_{n\in\mathbb{N}}$ can be a Cauchy sequences, and hence no subsequence can converge.

In a metric space compactness and sequential compactness are the same thing, and so the closed unit ball in X is not compact because one cannot extract a convergence sequence from every infinite sequence.

All balls in X are homeomorphic, and so none are compact. Since every neighborhood of every point in X contains a closed ball, X is not locally compact.

4: Prove the polarization identity, and show that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces and $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a surjective linear isometry, then T is unitary.

SOLUTION: For all $f, g \in \mathcal{H}$, we compute

$$\begin{aligned}
\|f+g\|^2 &= \langle f+g, f+g\rangle = \|f\|^2 + \|g\|^2 + 2\Re\langle f, g\rangle \\
-\|f-g\|^2 &= \langle f+g, f+g\rangle = -\|f\|^2 - \|g\|^2 + 2\Re\langle f, g\rangle \\
-i\|f+ig\|^2 &= i\langle f+ig, f+ig\rangle = -i\|f\|^2 - i\|g\|^2 + i2\Im\langle f, g\rangle \\
i\|f-ig\|^2 &= i\langle f+ig, f+ig\rangle = i\|f\|^2 + i\|g\|^2 + i2\Im\langle f, g\rangle
\end{aligned}$$

(Here we use the convention that $\langle , \rangle$ is conjugate linear on the left.) Summing on both sides and dividing by 4 we obtain

$$\langle f, g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2 - i\|f+ig\|^2 + i\|f-ig\|^2\right) \ .$$

Next, any linear isometry is injective, and so T is invertible. Since T is a linear isometry, none of the terms on the right in the polarization identity are changes when f and g are replaced by Tf and Tg respectively. Hence $\langle Tf, Tg\rangle = \langle f, g\rangle$ for all $f, g \in \mathcal{H}_1$ Thus, T is unitary.

5: Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Show that every orthonormal sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ converges weakly to zero, and that the unit sphere in $\mathcal{H}$ is weakly dense in the unit ball in $\mathcal{H}$.

SOLUTION: For all $f \in \mathcal{H}$, $\sum_{n=1}^{\infty} |\langle f, u_n\rangle|^2 \le \|f\|^2$ by Bessel's inequality, and hence $\lim_{n\to\infty}\langle f, u_n\rangle = 0$. Since f is arbitrary, this shows that $\{u_n\}_{n\in\mathbb{N}}$ converges weakly to zero.

Next, let $g \in \mathcal{H}$, $0 \le \|g\| \le 1$. Let $\{u_n\}_{n\in\mathbb{N}}$ be any orthonormal sequence such that u_1 is a multiple of g. Such a sequence is obtained by the Gram-Schmidt procedure starting form g. Now define

$$f_n = g + (1 - \|g\|^2)^{1/2} u_{n+1}$$

for all $n \in \mathbb{N}$. Evidently $\|f_n\|^2 = \|g\|^2 + (1 - \|g\|^2) = 1$ for all n, and by the first part, for all $f \in H$, $\lim_{n\to\infty}\langle f, f_n\rangle = \langle f, g\rangle$ which shows that $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to g.