Solutions for Homework Assignment 1, Math 502, Spring 2017

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1: Let $Y = L^1(\mu)$ where μ is counting measure on $\mathbb{N}$. Let X be the subspace $X = \{f \in Y \ : \sum_{n\in\mathbb{N}} n|f(n)| < \infty\}$, equipped with the norm inherited from Y.

(a) Show that X is a proper dense subspace of Y, and hence is not complete.

(b) Define $T : X \to Y$ by $Tf(n) = nf(n)$. Show that T is closed but not bounded.

(c) Let $S = T^{-1}$. Show that S is bounded and surjective but not open.

SOLUTION: (a) A sequence f belongs to X if and only if $f \in L^1(\nu)$, where ν is the weighted counting measure on $\mathbb{N}$ given by $\nu(A) = \sum_{n\in A} n$, in which case it also belongs to Y since $\mu \leq \nu$. Hence Y is a subspace of X. (L^1 is always a vector space.) Moreover, let f_0 be the sequence given by $f_0(n) = n^{-2}$. Then $f \in L^1(\mu)$ but $f \notin L^1(\nu)$ so X is a proper subspace of Y.

The set of sequences that have only finitely many many non-zero terms is dense in Y and is included in X. Hence X is dense in Y. This is because if $f \in Y$, $\lim_{N\to\infty} \sum_{n\geq N} |f(n)| = 0$, and therefore defining

$$f_N(n) = \begin{cases} f(n) & n < N \\ 0 & n \geq N \ , \end{cases}$$

each $f_N \in X$ and $\lim_{N\to\infty} \|f - f_N\|_Y = 0$.

(b) For $N \in \mathbb{N}$, define g_N by

$$g_N(n) = \begin{cases} 1 & n = N \\ 0 & n \neq N \ . \end{cases} \tag{0.1}$$

Then $\|g_N\|_Y = 1$ and $\|Tg_N\|_Y = n$, so there is no finite constant C such that $\|Tf\|_Y \leq C\|f\|_Y$ for all $f \in Y$. Hence T is unbounded. To see that T is closed, let $\{f_n\}$ be a convergent sequence in Y, and suppose that $\lim_{n\to\infty} f_n = f$ in Y and that $\lim_{n\to\infty} Tf_n = g$ in Y. Since for all $f \in Y$, and all $n \in \mathbb{N}$, $|f(n)| \leq \|f\|_Y$, convergence in Y implies point-wise convergence. Therefore, for all $m \in N$,

$$g(m) = \lim_{n\to\infty} Tf_n(m) = \lim_{n\to\infty} mf_n(m) = m\left(\lim_{n\to\infty} f_n(m)\right) = mf(m) = Tf(m) \ . \tag{0.2}$$

Thus, $Tf = g$, and the graph of T is closed.

(c) Note that S is given by $Sf(n) = n^{-1}f(n)$, and so $\|Sf\|_Y \leq \|f\|_Y$ showing that $\|S\| \leq 1$. (In fact, considering g_1 defined as in (0.1), $Sg_1 = g_1$, and hence $\|S\| \geq 1$. Altogether, we obtain $\|S\| = 1$.) If $f \in X$, $Tf \in Y$, and thus $f = S(Tf)$ which shows that S maps Y onto X, and it is clearly one-to-one since $Sf = 0$ implies $f = 0$. If S were open, $T = S^{-1}$ would be continuous, and therefore bounded, but by part **(b)**, it is not.

2: Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space X such that $\|\cdot\|_1 \leq \|\cdot\|_2$. Show that if X is complete with respect to both norms, then the norms are equivalent.

SOLUTION: (Note: This statement is weaker than the Theorem of Tao that was proved in class. The solution below is a simplification of the proof given of Tao's Theorem.) It suffices to show that for some finite constant C, $\|\cdot\|_2 \leq C\|\cdot\|_1$. Let $T : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ be the identity map $Tx = x$, and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$ in $(X, \|\cdot\|_1)$ and $\lim_{n\to\infty} Tx_n = y$ in $(X, \|\cdot\|_2)$. Since T is the identity, and since $\|\cdot\|_1 \leq \|\cdot\|_2$,

$$0 = \lim_{n\to\infty} \|x_n - y\|_2 \geq \lim_{n\to\infty} \|x_n - y\|_1 \ .$$

Hence, in $(X, \|\cdot\|_1)$, $x = \lim_{n\to\infty} x_n = y$, showing that the graph of T is closed. By the Closed Graph Theorem, it is bounded, and hence there is a finite constant C such that for all $x \in T$, $\|x\|_2 = \|Tx\|_2 \leq C\|x\|_1$, as was to be shown.

3: Let X and Y be Banach spaces. Let $T : X \to Y$ be a linear map such that $f \circ T \in X^*$ for all $f \in Y^*$. Show that T is bounded.

FIRST SOLUTION: By the Closed Graph Theorem, it suffice to show that the graph of T is closed. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence n X such that $\lim_{n\to\infty} x_n = x$ in X and $\lim_{n\to\infty} Tx_n = y$ in Y. Then since (by definition) each $f \in X^*$ is continuous on Y, $\lim_{n\to\infty} Tx_n = f(y)$, and since by hypothesis $f \circ T$ is continuous on X, $\lim_{n\to\infty} f(Tx_n) = f(Tx)$. Since Y^* separates points $y = Tx$, and hence the graph of T is closed as was to be shown.

SECOND SOLUTION: By definition, $\|f(T(x))| \leq \|f\|_{Y^*}\|T(x)\|_Y < \infty$ for all $x \in X$, and all $f \in Y^*$. Therefore, define $\mathcal{A} = \{f \circ T \ : \ f \in Y^* \ , \ \|f\|_{Y^*} = 1\}$, which, by hypothesis, is set of continuous linear transformations form X to $\mathbb{C}$, which are both Banach space, and by the above, we have

$$\sup_{f\circ T\in\mathcal{A}} \{|f(T(x))|\} \leq \|T(x)\|_Y\} < \infty \ .$$

By the Uniform Boundedness Principle,

$$\sup_{f\circ T\in\mathcal{A}} \{\|f \circ T\|_{Y^*}\} < \infty \ .$$

this means that there is a finite constant C such that for all x with $\|x\|_X \leq 1$, and all $y \in Y^*$ with $\|y\|_{Y^*} \leq 1$, $f(T(x))| \leq C$. Since for all $y \in Y$, $\|y\|_Y = \sup\{|f(y)| \ : f \in Y^* \ , \ \|f\|_{Y^*} = 1\}$, this means that $\|T(x)\|_Y \leq C$ for all x with $\|x\|_X \leq 1$, and this means that $\|T\| \leq C$.

4: Let X and T be Banach spaces, and let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of bounded linear transformations from X to Y such that for each $x \in X$. $\{Tx_n\}_{n\in\mathbb{N}}$ is a convergent sequence in Y. For all $x \in X$, define $Tx \ \lim_{n\to\infty} T_n x$. Show that T is a bonded linear transformation form X to Y.

SOLUTION: Point-wise limits of linear transformations are linear, so T is linear. Nest, since the norm function is continuous, for each $x \in X$, $\lim_{n\to\infty} \|T_n x\|_Y = \|Tx\|_Y < \infty$. Since convergent sequences in $\mathbb{R}$ are bounded, for each $x \in X$, $\sup_{n\in\mathbb{N}} \|T_n(x)\|_Y < \infty$. Then by the Uniform Boundedness Principle,

$$C := \sup_{n\in\mathbb{N}} \|T_n\|_Y < \infty \ .$$

Then for any $x \in X$, and any $n \in \mathbb{N}$, $\|T_n x\|_Y \leq C\|x\|_X$. Again by continuity of the norm function,

$$\|T(x)\|_Y = \lim_{n\to\infty} \|T_n(x)\|_Y \leq C\|x\|$$

showing that $\|T\| \leq C$.