Notes on Compact Operators on Hilbert Space for Math 502

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Contents

0.1 Norms of self-adjoint operators on Hilbert space

Let $T \in \mathscr{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space. Since for all $f \in \mathcal{H}$, there is a unit vector v such that $\langle v, f\rangle = \|f\|$, and since $|\langle v, f\rangle| \leq \|f\|$ for all unit vectors v, $\|Tu\| = \sup\{|\langle v, Tu\rangle| \ : \|v\| = 1\}$. Therefore,

$$\|T\| = \sup\{|\langle v, Tu\rangle| \ : \|u\| = 1 \ , \|v\| = 1\} \ . \tag{0.1}$$

When T is self-adjoint; i.e., when $T = T^*$, there is a simpler formula:

0.1 LEMMA. *Let $T \in \mathscr{B}(\mathcal{H})$, $T = T^*$. Then*

$$\|T\| = \sup\{|\langle u, Tu\rangle| \ : \|u\| = 1 \ \} \ . \tag{0.2}$$

Proof. Temporarily define $C_T := \sup\{|\langle u, Tu\rangle| \ : \|u\| = 1 \ \}$. Let u, v be unit vectors in $\mathcal{H}$ such that $v \neq \pm u$. Since T is self-adjoint,

$$\begin{aligned}
\langle u+v, T(u+v)\rangle &= \langle u, Tu\rangle + \langle v, Tv\rangle + 2\Re(\langle v, Tu\rangle)\\
\langle u-v, T(u-v)\rangle &= \langle u, Tu\rangle + \langle v, Tv\rangle - 2\Re(\langle v, Tu\rangle) \ .
\end{aligned}$$

Therefore,

$$\Re(\langle v, Tu\rangle) = \frac{1}{4}\left(\langle u+v, T(u+v)\rangle - \langle u-v, T(u-v)\rangle\right) \ .$$

Defining $f = u+v$ and $g = u-v$, neither of which is zero, we obtain

$$\begin{aligned}
\Re(\langle v, Tu\rangle) &= \frac{1}{4}\left(\|f\|^2\frac{\langle f, Tf\rangle}{\|f\|^2} - \|g\|^2\frac{\langle g, Tg\rangle}{\|g\|^2}\right)\\
&\leq \frac{1}{4}(\|f\|^2 + \|g\|^2)C_T = C_T \ .
\end{aligned}$$

Replacing v by $e^{i\theta}v$ and varying θ, we obtain that $|\langle v, Tu\rangle| \leq C_T$ for all unit vectors u and v, the excluded case $v = \pm u$ being trivial. Now (0.2) follows from (0.1). □

If $T \in \mathscr{B}(\mathcal{H})$, then $\|T^*\| = \|T\|$, whcih follows from (0.1) since $|\langle v, T^*u\rangle| = |\langle u, Tv\rangle|$. Also, for all unit vectors u and all $S, T \in \mathscr{B}(\mathcal{H})$, $\|STu\| \leq \|S\|\|Tu\| \leq \|S\|\|T\|$ so that $\|ST\| \leq \|S\|\|T\|$. In particular, $\|T^2\| \leq \|T\|^2$, and then by a simple induction, $\|T^n\| \leq \|T\|^n$ for all $n \in \mathbb{N}$. The inequality can be strict: For example, if T is nilpotent of order n, so that $T^n = 0$, but $T \neq 0$, then $0 = \|T^n\| < \|T\|^n$.

0.2 THEOREM. *For all $T \in \mathscr{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, $\|T^*T\| = \|T\|^2$.*

Proof. Since T^*T is self-adjoint, $\|T^*T\| = \sup\{|\langle u, T^*Tu\rangle| \ : \|u\| = 1\ \}$. However, $\langle u, T^*Tu\rangle = \langle Tu, Tu\rangle = \|Tu\|^2$, so that

$$\|T^*T\| = \sup\{\|Tu\|^2 \ : \|u\| = 1\ \} = \|T\|^2\ .$$

□

In particular, if T is self-adjoint, then $\|T^2\| = \|T\|^2$, and then it follows that $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$. The identity $\|T^*T\| = \|I\|^2$ is known as the C^* *algebra identity* because of its crucial role in the Gelfand-Naimark theory of C^* algebras.

0.2 Compact operators

0.3 DEFINITION (Compact operator)**.** An operator $T \in \mathscr{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, is *compact* in case whenever $\{f_n\}_{n\in\mathbb{N}}$ is a weakly convergent sequence in $\mathcal{H}$, $\{Tf_n\}_{n\in\mathbb{N}}$ is a strongly convergent sequence in $\mathcal{H}$.

0.4 EXAMPLE. *Let $(\Omega, \mathscr{M}, \mu)$ be a measure space, and let $\mathcal{H} = L^2(\Omega, \mathscr{M}, \mu))$. Let K be a square integrable function on the product space $(\Omega \times \Omega, \mathscr{M} \otimes \mathscr{M}, \mu \otimes \mu)$ and define*

$$\|K\|^2 = \int_{\Omega\times\Omega} |K|^2 \mathrm{d}\mu \otimes \mu\ .$$

Then for each $h \in \mathcal{H}$, and each $x \in \Omega$,

$$\left|\int_\Omega K(x,y)h(y)\mathrm{d}\mu(y)\right| \leq \left(\int_\Omega |K(x,y)|^2 \mathrm{d}\mu(y)\right)^{1/2} \|h\|\ . \tag{0.3}$$

By Fubini's Theorem, the right hand side is a square integrable function of x, and hence for all $f \in \mathcal{H}$, the function Kf defined by

$$Kf(x) := \int_\Omega K(x,y)f(y)\mathrm{d}\mu(y) \tag{0.4}$$

belongs to $\mathcal{H}$, and moreover, $\|Kf\| \leq \|K\|\|f\|$. Therefore, the map $f \mapsto Kf$ is a bounded linear transformation on $\mathcal{H}$.

In fact, K is compact. To see this, let $\{f_n\}$ be a sequence that converges weakly to $f \in \mathcal{H}$. By Fubini's Theorem, for almost every $x \in X$, $y \mapsto K(x,y)$ is square integrable, and thus we may write $Kf = \langle K(x,\cdot), f\rangle$, and then by the weak convergence, $Kf(x) = \lim_{n\to\infty} Kf_n(x)$ for almost every x. Moreover, since weakly convergence sequences are uniformly bounded, there exists $C < \infty$

such that $\|f_n\| \le C$ for all n, and then $\|f\| \le C$ as well since $\|f\| \le \liminf_{n\to\infty} \|f_n\|$. Therefore, by (0.3)

$$|K(f - n_n)(x)|^2 \le 4C^2 \left(\int_\Omega |K(x,y)|^2 \mathrm{d}\mu(y) \right) .$$

Then by the Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty} \|Kf - Kf_n\|^2 = 0$. Thus, $\{Kf_n\}_{n\in\mathbb{N}}$ converges strongly to f. This proves that K is a compact operator.

The class of compact operators considered in this example is called the class of Hilbert-Schmidt integral operators

An even simpler example is given by the class of *finite rank operators* on a Hilbert space $\mathcal{H}$. First, we fix a useful notational convention. Given two vectors $f, g \in \mathcal{H}$, let $|g\rangle\langle f|$ denote the operator on $\mathcal{H}$ given by

$$|g\rangle\langle f|h = \langle f, h\rangle g \quad \text{for all } h \in \mathcal{H} . \tag{0.5}$$

An operator $T \in \mathscr{B}(\mathcal{H})$ has finite rank if its range, $\mathrm{ran}(T)$, is a finite dimensional subspace of $\mathcal{H}$, or equivalently, if the orthogonal complement of its null-space, $\ker(T)^\perp$, is finite dimensional. If T is finite rank and $\{f_1, \dots, f_m\}$ is an orthonormal basis for $\ker(T)^\perp$, then we may write T in the form

$$T = \sum_{j=1}^m |g_j\rangle\langle f_j| , \tag{0.6}$$

where for each j, $g_j = Tf_j$. Conversely, every operator of the form (0.6), even without the assumption that $\{f_1, \dots, f_m\}$ is orthonormal, is finite rank. It is very simple to show that every finite rank operators is compact; this is left to the reader.

0.5 THEOREM. *Let $T \in \mathscr{C}(\mathcal{H})$. Then $T^* \in \mathscr{C}(\mathcal{H})$, and for all $S \in \mathscr{B}(H)$, ST and TS belong to $\mathscr{C}(H)$. Finally, $\mathscr{C}(\mathcal{H})$ is an operator norm closed subspace of $\mathscr{B}(\mathcal{H})$.*

Proof. To show that $T^* \in \mathscr{C}(\mathcal{H})$ whenever $T \in \mathscr{C}(H)$, let $T \in \mathscr{H}$ and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ that converges weakly to $f \in \mathcal{H}$. We must show that $\lim_{n\to\infty} \|T^*(f_n - f)\| \neq 0$. If this is not the case, then for some $\epsilon > 0$, there is a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $\|T^*(f_{n_k} - f)\| \ge \epsilon$ for all k. Passing to this subsequence, $\|T^*(f_k - f)\| \ge \epsilon$ for all k. Then there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ of unit vectors such that

$$|\langle Tu_k, f_k - f\rangle| = |\langle u_k, T^*(f_k - f)\rangle| = \|T^*(f_k - f)\| \ge \epsilon$$

for all k. The norm closed unit ball B in $\mathcal{H}$ is weakly sequentially compact, and hence there exists a (further) subsequence $\{u_{k_\ell}\}_{\ell\in\mathbb{N}}$ that converges weakly to some $u \in B$. Since T is compact, $\lim_{\ell\to\infty} \|T(u_{k_\ell} - u)\| = 0$. Therefore, for all sufficiently large ℓ, $\|T(u_{k_\ell} - u)\| < \epsilon/2$, and then for all such ℓ,

$$|\langle Tu, f_{k_\ell} - f\rangle| \ge |\langle Tu_{k_\ell}, f_{k_\ell} - f\rangle| - \frac{1}{2}\epsilon \ge \frac{1}{2}\epsilon .$$

However this is impossible since $\{f_n\}_{n\in N}$ converges weakly to f. This contradiction shows that T^* is compact.

It is evident that since S takes norm convergent sequences to norm convergent sequences, then ST is compact for all $S \in \mathscr{B}(\mathcal{H})$ and $T \in \mathscr{C}(\mathcal{H})$. Since $TS = (S^*T^*)^*$, the first part of the proof shows that TS is compact for all $S \in \mathscr{B}(\mathcal{H})$ and $T \in \mathscr{C}(\mathcal{H})$.

Now let $\{T_n\}_{n\in\mathbb{N}}$ be a norm convergent sequence in $\mathscr{C}(H)$ and let T be the limit in $\mathscr{B}(\mathcal{H})$. We must show that $T \in \mathscr{C}(\mathcal{H})$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a weakly convergent sequence with limit f. Then there is a finite constant C such that $\|f_k\| \le C$ for all k, and also $\|f\| \le C$.

Pick $\epsilon > 0$, and then pick N so that $\|T_n - T\| < \epsilon$ whenever $n \ge N$. Then for all k, ℓ,

$$\|Tf_k - Tf_\ell\| \le \|(T - T_N)f_k\| + \|T_N(f_k - f_\ell)\| + \|(T - T_N)f_k\| \le C\epsilon + \|\|T_N(f_k - f_\ell)\|\| + C\epsilon \ .$$

Since T_N is compact, there is a finite M so that whenever $k, \ell \ge M$, $\|\|T_N(f_k - f_\ell)\|\| < \epsilon$. Altogether,

$$k, \ell \ge M \quad \Rightarrow \quad \|Tf_k - Tf_\ell\| \le (2C + 1)\epsilon \ .$$

Since $\epsilon > 0$ is arbitrary, this shows that $\{Tf_k\}$ is a Cauchy sequence. Let g denote the limit. Then for all $h \in \mathcal{H}$,

$$\langle h, g\rangle = \lim_{n\to\infty} \langle h, Tf_n\rangle = \lim_{n\to\infty} \langle T^*h, f_n\rangle = \langle T^*h, f\rangle = \langle h, Tf\rangle \ ,$$

and therefore $Tf = g = \lim_{n\to\infty} Tf_n$. □

A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of $T \in \mathscr{B}(\mathcal{H})$ in case there is a non-zero vector $f \in \mathcal{H}$ such that $Tf = \lambda f$, and in this case, f is called an *eigenvector* of T with the eigenvalue λ.

If λ is an eigenvalue of T, the corresponding *eigenspace* $\mathcal{H}_\lambda$ is the subspace of $\mathcal{H}$ spanned by all of the eigenvectors of T with eigenvalue λ. That is,

$$\mathcal{H}_\lambda = \ker((\lambda I - T) \ ,$$

which shows that $\mathcal{H}_\lambda$ is always closed.

0.6 THEOREM. *Let T be a compact operator on a Hilbert space $\mathcal{H}$. Then for each $r > 0$, there are at most finitely many $\lambda \in \mathbb{C}$ such that λ is an eigenvalue of T and $|\lambda| \ge r$. Moreover, if λ is an non-zero eigenvalue of T, then* $\dim(\ker(\lambda I - T)) < \infty$.

Proof. If for any non-zero λ, $\dim(\ker(\lambda I - T)) = \infty$, then there exists an orthonormal sequence $\{u_n\}_{n\in\mathbb{N}}$ of eigenvectors of T, $Tu_n = \lambda_n u_n$, such that $\inf_{n\in\mathbb{N}}\{|\lambda_n|\} > 0$. Then $\{u_n\}_{n\in\mathbb{N}}$ converges weakly to zero, but $\{Tu_n\}_{n\in\mathbb{N}}$ does not converge strongly. If T is seld-adjoint, so that eigenvectors with distinct eigenvalues are necessarily orthognal, essentially the same argument can be used for the second part.

To prove the second part in general, suppose that there are infnitely many eigenvlaies $\{\lambda_n\}_{n\in\mathbb{N}}$ with $|\lambda_n| \ge r$ for all n. For each n, let u_n be a unit vector with $Tu_n = \lambda_n u_n$. Passing to a subsequence, we may suppose that u_n converges weakly to $u \in B$ as $n \to \infty$. Define $\mathcal{K}_n = \text{span}(\{u_1, \dots, u_n\})$. Since any set of eigenvectors with distinct eigenvalues is linesrly independent, $\mathcal{K}_{n+1} \cap \mathcal{K}_n^\perp \ne \{0\}$ for any n. Define $v_1 = u_1$, and for all $n \in \mathbb{N}$, choose any unit vector $v_{n+1} \in \mathcal{K}_{n+1} \cap \mathcal{K}_n^\perp$. Since v_n converges weakly to 0 as $n \to \infty$, Then $\lim_{n\to\infty} \|Tv_n\| = 0$. Now note that $(\lambda_n I - T)v_n \in \mathcal{K}_{n-1}$ for each $n \ge 2$. Hence v_n is orthongal to $(\lambda_n I - T)v_n$, and hence

$$\|Tv_n\|^2 = \|(T - \lambda_n)v_n + \lambda_n v_n\|^2 = \|(T - \lambda_n)v_n\|^2 + \|\lambda_n v_n\|^2 \ge r \ .$$

This contradiction shows that there do not exists infinitely many eigenvlaues λ with $|\lambda| \ge r$. □

0.3 The Hilbert-Schmidt Spectral Theorem

In an infinite dimensional Hilbert space, bounded operators, even bounded self-adjoint operators, need not have any eigenvalues at all. For example, let $\mathcal{H} := L^2([0,1], \mathscr{B}, \mu)$ where μ is Lebesgue measure. Define $Tt(t) = tf(t)$. Then it is easily checked that $\|T\| = 1$ and $T = T^*$. However, if $Tf = \lambda f$, then $(t-\lambda)f(t) = 0$ for almost every t, and this is impossible unless $f = 0$. However, for compact self-adjoint operators, the situation is different.

0.7 THEOREM. *Let T be a self-adjoint compact operator on a Hilbert space $\mathcal{H}$. Then either $\|T\|$ or $-\|T\|$ is an eigenvalue of T (or both are).*

Proof. By Lemma 0.1, $\|T\| = \sup\{|\langle u, Tu\rangle| \ : \|u\| = 1\ \}$. Therefore, either

$$\|T\| = \sup\{\langle u, Tu\rangle \ : \|u\| = 1\ \} \quad \text{or} \quad \|T\| = \sup\{-\langle u, Tu\rangle \ : \|u\| = 1\ \}\ , \tag{0.7}$$

or both. Suppose first that $\|T\| = \sup\{\langle u, Tu\rangle \ : \|u\| = 1\ \}$. We may assume that $T \neq 0$ to avoid trivialities.

It follows that there exists a sequence of unit vectors $\{u_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} |\langle u, Tu\rangle| = \|T\|$. Since every bounded sequence contains a weakly convergent subsequence, we may select a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ that converges weakly to some $u \in \mathcal{H}$ with $\|u\| \leq 1$.

We now show that in fact $\|u\| = 1$ and $\langle u, Tu\rangle = \|T\|$. Note that

$$\langle u, Tu\rangle - \langle u_{n_k}, Tu_{n_k}\rangle = \langle u - u_{n_k}, Tu\rangle + \langle u_{n_k}, T(u - u_{n_k})\rangle\ .$$

Since $\{u - u_{n_k}\}_{k\in\mathbb{N}}$ converges weakly to 0, $\lim_{k\to\infty}\langle u - u_{n_k}, Tu\rangle = 0$. Moreover, since T is compact, $\{T(u - u_{n_k})\}_{k\in\mathbb{N}}$ converges strongly to zero and hence

$$\limsup_{k\to\infty} |\langle u_{n_k}, T(u - u_{n_k})\rangle| \leq \limsup_{k\to\infty} \|T(u - u_{n_k})\| = 0\ .$$

Therefore

$$\langle u, Tu\rangle - \|T\| = \lim_{k\to\infty} \left(\langle u, Tu\rangle - \langle u_{n_k}, Tu_{n_k}\rangle\right) = 0\ .$$

Hence $\|T\| = \langle u, Tu\rangle$. By the Cauchy-Schwarz inequality, $\|T\| = |\langle u, Tu\rangle| \leq \|T\|\|u\|^2$. Since $\|u\| \leq 1$, we must have $\|u\| = 1$.

Now that we have found a unit vector u such that $\langle u, Tu\rangle = \|T\| \geq \langle v, Tv\rangle$ for all unit vectors $v \in \mathcal{H}$, let h be any unit vector, and define the function φ on $(-1/2, 1/2)$ by

$$\varphi(t) = \frac{\langle u + th, T(u + th)\rangle}{\|u + th\|^2}\ , \tag{0.8}$$

and note that $\varphi(0) \geq \varphi(t)$ for all $t \in (-1/2, 1/2)$. Since T is self adjoint, and since $\langle u, Tu\rangle = \|T\|$,

$$\begin{aligned}
\langle u + th, T(u + th)\rangle &= \langle u, Tu\rangle + t\langle h, Tu\rangle + t\langle u, Th\rangle + t^2\langle h, Th\rangle \\
&= \|T\| + \langle u, Tu\rangle + t\langle h, Tu\rangle + t\langle Tu, h\rangle + t^2\langle h, Th\rangle \\
&= \|T\| + \langle u, Tu\rangle + 2t\Re(\langle Tu, h\rangle) + t^2\langle h, Th\rangle\ .
\end{aligned}$$

Therefore,

$$\varphi(t) = \frac{\|T\| + 2t\Re(\langle Tu, h\rangle + t^2\langle h, Th\rangle}{1 + t2\Re(\langle u, h\rangle) + t^2}\ ,$$

Computing the derivative, we find $0 = \Re(\langle Tu, h\rangle - \|T\|\Re(\langle u, h\rangle)$. Replacing h by ih, we see that also $\Im(\langle Tu, h\rangle = \|T\|\Im(\langle u, h\rangle$, and altogether that $\langle Tu - \|T\|u, h\rangle = 0$ for all unit vectors h, and hence for all vectors h. Taking $h = Tu - \|T\|u$, we conclude that $Tu = \|T\|u$. Thus, u is an eigenvector of T with eigenvalue $\|T\|$.

Now suppose that the second alternative in (0.7) is valid. This is the same as $\|T\| = -\inf\{\langle u, Tu\rangle \ : \ \|u\| = 1\}$. The same reasoning proves the existence of a unit vector u such that $-\|T\| = \langle u, Tu\rangle$. Defining $\varphi(t)$ exactly as in (0.8), we have this time that $\varphi(0) \leq \varphi(t)$ for all $t \in (-1/2, 1/2)$. Again, this means $\varphi'(0) = 0$, and computing the derivative as above we find that $Tu = -\|T\|u$. □

The following simple lemmas will be frequently useful.

0.8 LEMMA. *Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Suppose that $\mathscr{K}$ is a subspace of $\mathcal{H}$ such that $\mathscr{K}$ is invariant under T, meaning that $Tf \in \mathcal{K}$ for all $f \in \mathcal{K}$. Then $\mathscr{K}^\perp$ is also invariant under T.*

Proof. Let $f \in V$, and $g \in V^\perp$. Then $0 = \langle Tf, g\rangle = \langle f, Tg\rangle$ so that $Tg \in V^\perp$. □

In particular, if T is a self-adjoint operator on $\mathcal{H}$, and $\mathcal{K} = \ker(T)$, then $K^\perp$ is invariant under T, and being a closed subspace of $\mathcal{H}$, $K^\perp$ is a Hilbert space in its own right. If T is a compact self-adjoint operator, he restriction of T to $\mathcal{K}^\perp$, $T|_{\mathcal{K}^\perp}$, is an injective compact self-adjoint operator on $\mathcal{K}^\perp$.

Next, any eigenvalues of a self-adjoint operator on a Hilbert space $\mathcal{H}$ are necessarily real:

0.9 LEMMA. *Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Suppose that $f \neq 0$ and that $Tf = \lambda f$ for some $\lambda \in \mathbb{C}$. Then $\lambda \in \mathbb{R}$.*

Proof. Let $u = \|f\|^{-1}f$. Then $\lambda = \langle u, Tu\rangle = \langle Tu, u\rangle = \overline{\langle u, Tu\rangle} = \overline{\lambda}$. □

0.10 THEOREM (Hilbert-Schmidt Spectral Theorem)**.** *Let T be a self-adjoint compact operator on a Hilbert space $\mathcal{H}$. Let $\mathcal{K} = \ker(T)$. If $\dim(\mathcal{K}^\perp) =: m < \infty$, define the index set $\mathscr{J}$ to be $\{1, \dots, m\}$. Otherwise, define $\mathscr{J} := \mathbb{N}$. Then there exists an orthonormal basis $\{u_n\}_{n\in\mathscr{J}}$ for $\mathcal{K}^\perp$ consisting of eigenvectors of T with $Tu_j = \lambda_j$ for all $j \in \mathbb{N}$ such that λ_j is real, $|\lambda_k| \leq |\lambda_j|$ for all $j, k \in \mathscr{J}$ and $|\lambda_1| = \|T\|$. Moreover,*

$$T = \sum_{j\in\mathscr{J}} \lambda_j |u_j\rangle\langle u_j| \ , \tag{0.9}$$

where, in case $\mathscr{J} = \mathbb{N}$, the sum converges in the operator norm and $\lim_{n\to\infty} \lambda_n = 0$.

Proof. By restricting T to $(\ker(T))^\perp$, we may assume without loss of generality that $\ker(T) = \{0\}$, which we do in order to simplify the notation.

By Theorem 0.7, either $\|T\|$ or $-\|T\|$ is an eigenvalue of T, or both are. Let u_1 be an eigenvector of T with eigenvalue λ_1, where either $\lambda_1 = \|T\|$ or $\lambda_1 = -\|T\|$. If $\dim(\mathcal{H}) = 1$, we are done.

Otherwise, define $\mathcal{K}_1 = \mathrm{span}(\{u_1\})$. Then $\mathcal{K}_1$ is invariant under T, and then by Lemma 0.8, $\mathcal{K}_!^\perp$ is a invariant under T. Since the orthogonal complement of any set is closed, $\mathscr{K}_1^\perp$ is closed, and hence is a Hilbert space. Since $\mathcal{K}_1^\perp$ is invariant under T, the restriction of T to $\mathscr{K}_1$, $T\big|_{\mathcal{K}_1^\perp}$, is a self-adjoint compact operator on $\mathscr{K}_1^\perp$.

Therefore, we may apply Theorem 0.7 to $T\big|_{\mathcal{K}_1^\perp}$: Either $\left\|T\big|_{\mathcal{K}_1^\perp}\right\|$ or $-\left\|T\big|_{\mathcal{K}_1^\perp}\right\|$ is an eigenvalue of $T\big|_{\mathcal{K}_1^\perp}$. Choose u_2 to be an eigenvector of $T\big|_{\mathcal{K}_1^\perp}$ with eigenvalue $\lambda_2 = \pm\left\|T\big|_{\mathcal{K}_1^\perp}\right\|$. Evidently, $\left\|T\big|_{\mathcal{K}_1^\perp}\right\| \leq \|T\|$ and hence $|\lambda_2| \leq |\lambda_1|$. If $\dim(\mathcal{H}) = 1$, we are done. Otherwise we iterate.

Suppose we have found an orthonormal set $\{u_1, \dots, u_m\}$ consisting of eigenvectors of T with $Tu_j = \lambda u_j$ and $|\lambda_j| \leq |\lambda_k|$ for all $1 \leq j < k \leq m$. Suppose that $\dim(\mathcal{H}) > m$. Define $\mathcal{K}_m :=$ span$(\{u_1, \dots, u_n\})$, which, being spanned by eigenvectors, is evidently invariant under T. By Lemma 0.8, $\mathcal{K}_m^\perp$ is a closed subspace of $\mathcal{H}$ that is invariant under T.

Therefore, we may apply Theorem 0.7 to $T\big|_{\mathcal{K}_m^\perp}$: Either $\left\|T\big|_{\mathcal{K}_m^\perp}\right\|$ or $-\left\|T\big|_{\mathcal{K}_m^\perp}\right\|$ is an eigenvalue of $T\big|_{\mathcal{K}_m^\perp}$. Choose u_{m+1} to be an eigenvector of $T\big|_{\mathcal{K}_m^\perp}$ with eigenvalue $\lambda_{m+1} = \pm\left\|T\big|_{\mathcal{K}_m^\perp}\right\|$. Then $Tu_{m+1} = \lambda_{m+1}$ so that u_{m+1} is also an eigenvector of T. Evidently,

$$\left\|T\big|_{\mathcal{K}_m^\perp}\right\| \leq \left\|T\big|_{\mathcal{K}_{m-1}^\perp}\right\|$$

and hence $|\lambda_{m+1}| \leq |\lambda_m|$. Thus, $\{u_1, \dots, u_{m+1}\}$ is an orthonormal set consisting of eigenvectors of T with $Tu_j = \lambda u_j$ and $|\lambda_j| \leq |\lambda_k|$ for all $1 \leq j < k \leq m+1$.

If $\dim(\mathcal{H}) := N < \infty$, the inductive construction terminates in N steps producing an orthonormal basis for $\mathcal{H}$. Otherwise it continues indefinitely producing an infinite orthonormal sequence $\{u_n\}_{n\in\mathbb{N}}$ with $Tu_n = \lambda_n u_n$ and $n \mapsto |\lambda_n|$ non-increasing on $\mathbb{N}$. By Theorem 0.6, for each $r > 0$, there can be only finitely many n such that $|\lambda_n| \geq r$. It follows that $\lim_{n\to\infty} \lambda_n = 0$.

We now claim that $\{u_n\}_{n\in\mathbb{N}}$ is complete. To see this, let $\mathcal{K}$ denote the orthogonal complement of the span of $\{u_n\}_{n\in\mathbb{N}}$, and note that since the span of $\{u_n\}_{n\in\mathbb{N}}$ is invariant under T, so is $\mathcal{K}$. Since $\ker(T) = \{0\}$, if $\mathcal{K} \neq \{0\}$, $\|T|_{\mathcal{K}}\| > 0$. We could then apply Theorem 0.7 to produce an unit vector u that is an eigenvector of T with $Tu = \pm\|T|_{\mathcal{K}}\|u$, and with $\langle u, u_n\rangle = 0$ for all n. But this is impossible since by the construction of $\{u_n\}_{n\in\mathbb{N}}$, for all unit vectors $u \in \mathcal{H}$ with $\langle u, u_j\rangle = 0$ for $j = 1, \dots, m$, $|\langle u, Tu\rangle| \leq |\lambda_m|$ and we have seen that $\lim_{n\to\infty} |\lambda_n| = 0$. Hence $\mathcal{K} = \{0\}$, which means that $\{u_n\}_{n\in\mathbb{N}}$ is complete.

Since $\{u_j\}_{j\in\mathscr{J}}$ is an orthonormal basis for $\mathcal{H}$, we have that for all $f \in H$, $f = \sum_{j\in\mathscr{J}} \langle u_j, f\rangle u_j$, and hence

$$Tf = T\left(\sum_{j\in\mathscr{J}} \langle u_j, f\rangle u_j\right) = \sum_{j\in\mathscr{J}} \langle u_j, f\rangle Tu_j = \sum_{j\in\mathscr{J}} \lambda_j |\langle u_j\rangle\langle u_j| f \ .$$

If $\mathscr{J}$ is finite, this proves (0.9).

For the case $\mathscr{J} = \mathbb{N}$, we now show that the sum in (0.9) converges in the operator norm. To do this, for each $n \in \mathbb{N}$, define $T_n = \displaystyle\sum_{j=1}^n \lambda_j |\langle u_j\rangle\langle u_j|$. Then all $n > m$ and unit vectors u,

$$|\langle u, (T_n - T_m)u\rangle| = \left|\sum_{j=m+1}^n \lambda_j |\langle u_j, u\rangle|^2\right| \leq |\lambda_{m+1}| \sum_{j=1}^\infty |\langle u_j, u\rangle|^2 = |\lambda_{m+1}| \ .$$

By Lemma 0.1, $\|T_n - T_m\| \leq |\lambda_{m+1}|$ and then since $\lim_{n\to\infty} \lambda_n = 0$, $\{T_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{B}(\mathcal{H})$. This sequence therefore converges in operator norm to a limit that must agree with T on the dense span of $\{u_j\}_{j\in N}$, and must therefore be T. This proves (0.9) in the infinite dimensional case. $\square$

There are several corollaries:

0.11 COROLLARY. *For all self-adjoint $T \in \mathscr{C}(\mathcal{H})$, there exists an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of $\mathcal{H}$.*

Proof. Let $\mathcal{K} := \ker(T)$, which is a closed subspace of $\mathcal{H}$, and therefore a Hilbert space in its own right. Combine any orthonormal basis for this space with the orthonormal basis of $\mathcal{K}^\perp$ that is provided by Theorem 0.10. □

0.12 COROLLARY. *$\mathscr{C}(\mathcal{H})$ is the operator norm closure of the set $\mathscr{F}(\mathcal{H})$ of finite rank operators on $\mathcal{H}$.*

Proof. We have seen that $\mathscr{F}(\mathcal{H}) \subset \mathscr{C}(\mathcal{H})$, and that $\mathscr{C}(\mathcal{H})$ is closed so that $\overline{\mathscr{F}(\mathcal{H})} \subset \mathscr{C}(\mathcal{H})$. The formula (0.9) displays every self adjoint compact operator T as the operator norm limit of finite rank operators, and if T is compact, so are $T + T^*$ and $i(T^* - T)$. □

If $T \in \mathscr{C}(H)$, and $T = T^*$, and φ is a conitnuous function on defined on $[-\|T\|, \|T\|]$, then we define

$$\varphi(T) = \sum_{j\in\mathbb{N}} \varphi(\lambda_j)|u_j\rangle\langle u_j| \ .$$

0.4 The Fredholm Alternative

The Fundamental Theorem of Linear Algebra says that a linear transformation T between finite dimensional vector spaces V and W is invertible if and only if V and W have the same dimension *and* T is either injective or surjective. In other words, for linear maps between vector spaces of the same finite dimension, injectivity implies surjectivity and *vice-versa*.

In infnite dimensions, this is not true in general, but there is an important case in which is is true.

0.13 THEOREM. *Let T be a compact operator on a Hilbert space $\mathcal{H}$. Then $(I - T)$ is invertible if and only if $(I - T)$ is injective, and $(I - T)$ is invertible if and only if $(I - T)$ is surjective. Morover*

$$\dim(\ker(I - T)) = \dim((\mathrm{ran}(I - T))^\perp) \ . \tag{0.10}$$

0.14 Remark. The first part of the theorem can be expressed as saying that either $(I - T)$ is invertible, or else 1 is an eigenvalue of T. This is the *Fredholm alternative.* For $\lambda \neq 0$, $(\lambda I - T) = \lambda(I - \lambda^{-1}T)$, and hence $(\lambda I - T)$ is invertible if and ony if $(I - \lambda^{-1}T)$ is invertible, and $\lambda^{-1}T$ is comapct if and only if T is compact. Hence if T is compact, and $\lambda \neq 0$, either λ is an eigenvalue of T or else $\lambda I - T$ is invertible.

0.15 LEMMA. *Let T be a compact operator on a Hilbert space $\mathcal{H}$. Then either there exists $C > 0$ such that for all $f \in \mathcal{H}$,*

$$\|(I - T)f\| \geq C\|f\| \tag{0.11}$$

or else $\ker(I - T) \neq \{0\}$.

Proof. Suppose that there is no $C > 0$ such that (0.11) is valid for all $f \in \mathcal{H}$. The there exists a sequence of unit vectors $\{u_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \|(I-T)u_n\| = 0$. Since the closed unit ball B in $\mathcal{H}$ is weakly sequentially compact, by passing to a subsequence, we may assume that $\{u_n\}_{n\in\mathbb{N}}$ converges weakly to some $u \in B$, and then, since T is compact, that $\lim_{n\to\infty} \|Tu_n - Tu\| = 0$. Then since

$$|1 - \|Tu_n\|| = |\|u_n\| - \|Tu_n\|| \leq \|u_n - Tu_n\| = \|(I-T)u_n\| ,$$

$\lim_{n\to\infty} \|Tu_n\| = 1.$, and hence $\|Tu\| = 1$.

Since T commutes with $(I-T)$, and is bounded, the hypothesis that $\lim_{n\to\infty} \|(I-T)u_n\| = 0$ implies that

$$0 = \lim_{n\to\infty} \|T(I-T)u_n\| = \lim_{n\to\infty} \|(I-T)Tu_n\| = \|(I-T)Tu\| .$$

Since $\|Tu\| = 1$, this means that Tu is a non-zero vector in $\ker(I-T)$. □

0.16 LEMMA. *Let T be a compact operator on a Hilbert space $\mathcal{H}$. if $\ker(I-T) = \{0\}$, then $\operatorname{ran}(I-T)$ is closed.*

Proof. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in $\operatorname{ran}(I-T)$ that converges in norm to some $g \in \mathcal{H}$. We must show that $g \in \operatorname{ran}(I-T)$. Since $(I-T)$ is injective, for each $n \in \mathbb{N}$, there is a unique $f_n \in \mathcal{H}$ such that $(I-T)f_n = g_n$. Then by (0.11), $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$, and hence there exists $f \in \mathcal{H}$ such that $\lim_{n\to\infty} \|f_n - f\| = 0$. Then

$$g = \lim_{n\to\infty} (I-T)f_n = (I-T)f ,$$

showing that $g \in \operatorname{ran}(I-T)$. □

0.17 LEMMA. *Let $T \in \mathscr{B}(\mathcal{H})$ Then*

$$(\operatorname{ran}(T))^\perp = \ker(T^*) . \tag{0.12}$$

Proof. Let $f \in \ker(T^*)$. For all $g \in \mathcal{H}$, $0 = \langle T^*f, g\rangle = \langle f, Tg\rangle$, and hence $f \perp Tg$. Thus $\ker(T^*) \subset (\operatorname{ran}(T))^\perp$. Let $g \in (\operatorname{ran}(T))^\perp$. For all $f \in \mathcal{H}$, $0 = \langle T^*f, g\rangle = \langle f, Tg\rangle$, and hence $Tg = 0$. Thus, $(\operatorname{ran}(T))^\perp \subset \ker(T^*)$. □

Proof of Theorem 0.13. Suppose that $I - T$ is injective. By Lemma 0.16, $V_1 := \operatorname{ran}(I-T)$ is a closed subspace of $\mathscr{H}$, and hence a Hilbert space. Since T is a continuous vector space isomorphism of H onto V_1, it has a bounded inverse: By Lemma 0.15, the unique f such the $(I-T)f = g$; i.e., $(I-T)^{-1}g$, satsifies $\|(I-T)^{-1}g\| \leq C^{-1}\|g\|$, showing that $(I-T)^{-1} \in \mathscr{B}(\mathcal{H})$. (One could also invoke the Open Mapping Theorem.)

Suppose that V_1 is a proper subspace of $\mathcal{H}$. Then since $I - T$ is injecrtive, $V_2 := (I-T)V_1$ is a proper subspace of $V - 1 = (I-T)\mathcal{H}$, and it is closed since $(I-T)$ is a topological homeomorphism. We inductively define $V_{n+1} := (I-T)V_n$ for each $n \in \mathbb{N}$, and then, as above, we have that each V_{n+1} is a proper, closed subspace of V_n.

Then $V_n \cap V_{n+1}^\perp$ is a non-zero subspace for all $n \in \mathbb{N}$. Choose any unit vector $u_n \in V_n \cap V_{n+1}^\perp$. Since $V_n \subset V_m$ for all $n \geq m$, $\{u_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence. Since $u_n \in V_n$ and $Tu_n - u_n \in V_{n+1}$, u_n and $Tu_n - u_n$ are othogonal, and hence

$$\|Tu_n\|^2 = \|(Tu_n - u_n) + u_n\|^2 = \|Tu_n - u_n\|^2 + \|u_n\|^2 \geq 1 .$$

However, since $\{u_n\}_{n\in\mathbb{N}}$ is orthonormal, it converges weakly to 0, and then since T is compact $\lim_{n\to\infty} \|Tu_n\| = 0$. This contradiction shows that $\mathrm{ran}(I-T) = V_1 = \mathcal{H}$, and hence that $I-T$ is surjective onto $\mathcal{H}$ as well as injective, and we have already seen that $(I-T)^{-1} \in \mathscr{B}(\mathcal{H})$. This proves that $I-T$ is invertible if and only if it is injective.

Next, suppose thar $I-T$ is surjective. Then by Lemma 0.17, $!-T^* = (I-T)^*$ is injective, and since T^* is aslo compact, what we have proved above shows that $!-T^*$ is invertible in $\mathscr{B}(\mathcal{H})$. But then $I-T$ is invertible in $\mathscr{B}(\mathcal{H})$, with inverse $((I-T^*)^{-1})^*$.

Finally, suppose that $I-T$ is not invertible. Then $\ker(I-T)$ and $(\mathrm{ran}(I-T))^\perp = \ker(I-T^*)$ are eigenspaces of the compact operators T and T^*, and hence are fnite dimensional. Let $\{u_1,\dots,u_m\}$ be any orthonormal basis of $\ker(I-T)$, and let $\{v_1,\dots,v_n\}$ be any orthonormal basis of $(\mathrm{ran}(I-T))^\perp$. Let $p := \min\{m,n\}$, and define $F = \sum_{j=1}^{p} |v_j\rangle\langle u_j|$. Note that $(I-T+F)$ is injective if and only if if $p = m$, and $(I-T+F)$ is surjective if and only if if $p = n$. Then since $T-F$ is compact, by what we have proved above, $p = m = n$, and this proves (0.10). □

0.18 Remark. For $S \in \mathscr{B}(H)$, the *nullity of* S is defined by $\mathrm{nullity}(S) = \dim(\ker(S))$, and the *rank of* S is defined by $\mathrm{rank}(S) = \dim(\mathrm{ran}(S))$. When $\mathcal{H} = \mathbb{C}^n$, so that we may identity $\mathscr{B}(\mathcal{H})$ with the $n\times n$ matrices, we have the simple identity that for any subspace $\mathcal{K}$ of $\mathcal{H}$, $\dim(\mathcal{K}) + \dim(\mathcal{K}^\perp) = n$, and hence $\mathrm{rank}(S) = n - \dim((\mathrm{ran}(S))^\perp)$. Defining $T = I-S$, so that $S = I-T$, and noting that in finite dimensions every linear operator is compact, the identity (0.10) of Theorem 0.13 says that $\mathrm{nullity}(S) + \mathrm{rank}(S) = n$.

By Lemma 0.17, and what we have said above, and equivalent formulation is that

$$\mathrm{nullity}(S) = \mathrm{nullity}(S^*) \ . \tag{0.13}$$

This formulation has the advantage of not referring explicitly to the dimension, and as Theorem 0.13 shows, it remains true in infinite dimensions when $S = I-T$ with T compact. For $\lambda \neq 0$, write $S = \lambda^{-1}(\lambda I - T)$. Then of course $\mathrm{nullity}(\lambda I - T) = \mathrm{nullity}(\lambda^* I - T)$, and hence if T is compact operator, then λ is a eigenvalue of T if and only if λ^* is an eigenvalues of T^* and in that case, these eigenvalues have the same (finite) geometric multiplicity, just as in finite dimensions.