Test One Solutions, Math 292, Spring 2018

March 6, 2018

1. Find the general solution of the differential equations:
(a) $x' = \frac{2}{t}x + t^{-3}x^2 - t$
(b) $x'' = 1 + (x')^2$

SOLUTION (a) We look for a solution of the form $x_1(t) = Ct^\alpha$. We find $\alpha = 2$ and $c^2 = 1$. Hence we have two solutions, namely $\pm t^2$. We choose $x_1(t) = t^2$.

Now define $y = x - t^2$. We find

$$\begin{aligned} y' = x' - 2t &= \frac{2}{t}(y + t^2) + t^{-3}(t + t^2)^2 - 3t \\ &= \frac{4}{t}y + \frac{1}{t^3}y^2 \ . \end{aligned}$$

This is a Bernoulli equation. We substitute $z = 1/y$, and find

$$z' = -\frac{1}{y^2}\left(\frac{4}{t}y + \frac{1}{t^3}y^2\right) = -\frac{4}{t}z - \frac{1}{t^3} \ .$$

Multiplying through by t^4, we find $(t^4z)' = -t$. Integrating, $z = \frac{C}{t^4} - \frac{1}{2t^2}$. Finally, the general solution is $x = z^{-1} + t^2$, which is

$$x(t) = \left(\frac{C}{t^4} - \frac{1}{2t^2}\right)^{-1} + t^2 \ .$$

Notice that for $c = 0$ we get our second particular solution, $-t^2$.

(b) Let $y = x'$. The equation becomes $y' = 1 + y^2$ which is separable. Integrating both sides of $\frac{y'}{1+y^2} = 1$, we find $\arctan(y) = t + C_1$, and so

$$y = \tan(t + C_1) = -\frac{\mathrm{d}}{\mathrm{d}t}\ln(\cos(t + C_1)) \ .$$

Integrating once more,

$$x(t) = -\ln(\cos(t + C_1)) + C_2 \ .$$

2. (a) Let $v(x) = (1 - x^4)^{1/2}$ Consider the solution of $x'(t) = v(x(t))$ with $x(0) = 0$. Does this solution exist for all t and remain within the interval $(-1, 1)$ for all t? Justify your answer.

(b) Let $v(x) = (1-x^4)^2$ Consider the solution of $x'(t) = v(x(t))$ with $x(0) = 0$. Does this solution exist for all t and remain within the interval $(-1,1)$ for all t? Justify your answer.

(c) Let $v(x)$ be a continuously differentiable function on $\mathbb{R}$ such that $v(-1) = v(1) = 0$ and such that $v(x) > 0$ for all $x \in (-1,1)$. Let Ψ_t be the flow transformation on $(-1,1)$ associated to v. Compute, for all $x_0\ in(-1,1)$,

$$\lim_{t\to\infty} \Psi_t(x_0) \quad \text{and} \quad \lim_{t\to\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x}\Psi_t\right)(x_0) .$$

Justify your answers.

SOLUTION For **(a)**, we use Barrow's formula to compute the time T it takes for the solution to reach $x = 1$. This is

$$T = \int_0^1 \frac{1}{\sqrt{1-z^4}}\mathrm{d}z = \int_0^1 \frac{1}{\sqrt{1-z^2}}\frac{1}{\sqrt{1+z^2}}\mathrm{d}z \le \int_0^1 \frac{1}{\sqrt{1-z^2}}\mathrm{d}z = \frac{\pi}{2} .$$

Hence in this case, no, the solution exits in a finite time.

(b) In this case $v(x)$ is a polynomial, which naturally has a bounded derivative on any bounded interval. Hence v is Lipschitz, and the solution exists and remains in $(-1,1)$ for all t.

(c) We are given that the function $v(x)$ has a bounded derivative on $[-1,1]$. Hence for any $x_0 \in (-1,1)$, the solution $x(t)$ with $x(0) = x_0$ exists for all t, and stays in the interval for all t. Since $v(x) > 0$ on $(-,1,1)$, $x'(t) = v(x(t)) > 0$ for all t. Hence $x(t)$ is a bounded increasing function, and $\lim_{t\to\infty} x(t)$ exists. It cannot be anything other than 1 since if $\lim_{t\to\infty} v(x(t)) > 0$ is not compatible with $x(t) < 1$ for all t. Hence $\lim_{t\to\infty} x(t) = 1$. Thus,

$$\lim_{t\to\infty} \Psi_t(x_0) = 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}x}\Psi_t(x_0) = \lim_{t\to\infty} \frac{v(x(t))}{v(x_0)} = 0 .$$

More informally, since $\lim_{t\to\infty} \Psi_t(x_0) = 1$, independent of x_0, the large t limit of the flow is not at all sensitive to changes in the initial data.

3. Let

$$A = \begin{bmatrix} 0 & -3/4 \\ 4 & -2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix} .$$

(a) Compute e^{tA} and e^{tB}.

(b) solve $\mathbf{x}'(t) = B\mathbf{x}(t)$ with $\mathbf{x}(0) = (1,2)$.

(c) Solve

$$\mathbf{x}'(t) = B\mathbf{x}(t) + (1,1) \qquad \text{with} \qquad \mathbf{x}(0) = (0,1) .$$

SOLUTION For **(a)** We compute $\det(A - tI) = t^2 + 2t + 3$ which has the roots $-1 \pm i\sqrt{2}$. Since

$$A - (-1 + i\sqrt{2})I = \begin{bmatrix} 1 - i\sqrt{2} & -3/4 \\ 4 & -1 - i\sqrt{2} \end{bmatrix} ,$$

$\mathbf{v} = (1 + i\sqrt{2}, 4)$ is an eigenvector with eigenvalue $-1 + i\sqrt{2}$. Hence we have the complex solution

$$\mathbf{z}(t) = e^{t(-1+i\sqrt{2})}(1 + i\sqrt{2}, 4) = e^{-t}(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t))[(1,4) + (i\sqrt{2}, 0)] .$$

This gives us two real solutions:

$$\mathbf{x}_1(t) = e^{-t}\cos(\sqrt{2}t)(1,4) - \sin(\sqrt{2}t)(\sqrt{2},0) \quad \text{and} \quad \mathbf{x}_1(t) = e^{-t}\sin(\sqrt{2}t)(1,4) + \cos(\sqrt{2}t)(\sqrt{2},0) \ .$$

Then

$$e^{tA} = [\mathbf{x}_1(t), \mathbf{x}_2(t)][\mathbf{x}_1(0), \mathbf{x}_2(0)]^{-1} = e^{-t}\begin{bmatrix} \cos(\sqrt{2}t) + 2^{-1/2}\sin(\sqrt{2}t) & -\frac{3}{8}2^{1/2}\sin(\sqrt{2}t) \\ 2^{3/2}\sin(\sqrt{2}t) & \cos(\sqrt{2}t) + 2^{-1/2}\sin(\sqrt{2}t) \end{bmatrix} .$$

We next compute $\det(B - tI) = t^2 - 8t + 16$ which has the repeated root 4. Since

$$B - 4I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} ,$$

There eigenvectors with eigenvalue 4 are the non-zero multiples of $\mathbf{v} = (1,1)$. We have

$$e^{tB} = e^{4t}e^{t(B-4I)} = e^{4t}(I + t(B - 4I)) = e^{4t}\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} .$$

For part **(b)**, the solution is

$$\mathbf{x}(t) = e^{tB}\mathbf{x}(0) = e^{4t}\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}(1,2) = e^{4t}(1+t, 2+t) \ .$$

For part **(c)**, we use Duhamel's formula. Note that $(1,1)$ as an eigenvector of B so that

$$e^{(t-s)B}(1,1) = e^{4t}e^{-4s}(1,1)$$

and hence the solution is

$$e^{4t}(1, 1+t) + \left(\int_0^t e^{-4s}\mathrm{d}s\right)e^{4t}(1,1) = e^{4t}(1, 1+t) + \frac{1}{4}\left(e^{4t} - 1\right)(1,1) \ .$$

4. Consider the vector field

$$\mathbf{v}(x,y) = ((x-1)(y-1), 4x - y^2) \ .$$

Find all equilibrium points, and determine which are Lyapunov stable, which are asymptotically stable, and which are unstable. Sketch the flow curves in the vicinity of each critical point.

SOLUTION At any equilibrium point, we must have either $x = 1$ or $y = 1$. If $x = 1$, we must also have $y^2 = 4$, so $y = \pm 2$. If $y = 1$, we must also have $4x = 1$, so $x = 1/4$. Hence we have three equilibrium points:

$$\mathbf{x}_1 = (1,2) \quad \mathbf{x}_2 = (1,-2) \quad \text{and} \quad \mathbf{x}_3 = (1/4, 1) \ .$$

To linearize, we compute

$$[D_{\mathbf{v}}(\mathbf{x})] = \begin{bmatrix} y-1 & x-1 \\ 4 & -2y \end{bmatrix} .$$

Then

$$[D_{\mathbf{v}}(\mathbf{x}_1)] = \begin{bmatrix} 1 & 0 \\ 4 & -4 \end{bmatrix} .$$

The eigenvalues are evidently 1 and -1. Hence this equilibrium point is unstable.

Likewise,

$$[D_{\mathbf{v}}(\mathbf{x}_2)] = \begin{bmatrix} -3 & 0 \\ 4 & 4 \end{bmatrix} .$$

The eigenvalues are evidently -3 and 4. Hence this equilibrium point is unstable.

Finally,

$$[D_{\mathbf{v}}(\mathbf{x}_3)] = \begin{bmatrix} 0 & -3/4 \\ 4 & -2 \end{bmatrix} .$$

As we have seen in a previous problem, the eigenvalues are $-1 \pm i\sqrt{2}$. Hence this equilibrium point is asymptotically stable.

The first two sketches look like saddle point contour plots, with arrows going in on one line, and out on another. The third sketch should show an inward spiral, turning counter-clockwise.

To do proper sketches one needs the eigenvectors. We give the details for the case in which the eigenvalues are complex. (The other cases did not seem to be problematic.) An eigenvector of $[D_{\mathbf{v}}(\mathbf{x}_3)]$ with eigenvalue $-1 + i\sqrt{2}$ is $(3, 4(1 - i\sqrt{2})$. The real part of the corresponding solution is

$$e^{-t}(3\cos(\sqrt{2}t), 4\cos(\sqrt{2}t) + 4\sqrt{2}\sin(\sqrt{2}t)) = e^{-t}B(\cos(\sqrt{2}t), \sin(\sqrt{2}t))$$

where

$$B = \begin{bmatrix} 3 & 0 \\ 4 & 4\sqrt{2} \end{bmatrix} .$$

Then $B^T B = \begin{bmatrix} 25 & 16\sqrt{2} \\ 16\sqrt{2} & 32 \end{bmatrix}$. Computing the eigenvalues and eigenvectors, one sees that the major axis about 3 times as long as the minor axis, and it runs (approximately) along the line though $(0,0)$ and $(0.857, 1)$. Since $\det(B) > 0$, the direction of rotation in the spiral is counter-clockwise.

5. Let $M = \frac{1}{8}\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Consider the equation $M\mathbf{x}'' = -A\mathbf{x}$. Find an equivalent equation of the form $\mathbf{y}'' = -K\mathbf{y}$, and write down the general solution of this equation. Then solve $M\mathbf{x}'' = -A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{0}$ and $\mathbf{x}'(0) = (1, 0)$.

(b) Let $\mathbf{g} = (1, 1)$. Consider the differential equation

$$M\mathbf{x}''(t) + A\mathbf{x}(t) = \mathbf{g}\cos(\omega t)$$

for some positive number ω. For which values of ω, if any, is there resonance?

SOLUTION (a) M is doubly symmetric, and so an orthonormal basis of eigenvectors is given by $\{\mathbf{u}_1, \mathbf{u}_2\}$ where

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}(1, 1) \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}}(1, -1) .$$

Also, $M\mathbf{u}_1 = \mathbf{u}_1$, and $M\mathbf{u}_2 = \frac{1}{4}\mathbf{u}_2$. Therefore, the eigenvalues of $M^{-1/2}$ are 1, for the eigenvector $\mathbf{u}_1$, and 2, for the eigenvector $\mathbf{u}_2$, and

$$M^{-1/2} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} .$$

Then $K = M^{-1/2}AM^{-1/2} = \frac{1}{2}\begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix}$ and so $K\mathbf{u}_1 = 3\mathbf{u}_1$ and $K\mathbf{u}_2 = 4\mathbf{u}_2$. The two normal mode equations are $y_j'' = -\omega_j^2 y_j$, where $\omega_1 = \sqrt{3}$ and $\omega_2 = 2$ are the two natural frequencies. The solutions of these equations with $y_j(0) = 0$, and $y_j'(0) = 1$ are

$$y_1(t) = \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) \quad \text{and} \quad y_2(t) = \frac{1}{2}\sin(t/2)\ .$$

Notice that

$$(1,0) = \frac{1}{\sqrt{2}}(\mathbf{u}_1 + \mathbf{u}_2) \quad \text{and hence} \quad \mathbf{v}_0 := M^{-1/2}(1,0) = \frac{1}{\sqrt{2}}(\mathbf{u}_1 + 2\mathbf{u}_2)\ .$$

Thus the solution of $\mathbf{y}'' = -K\mathbf{y}$ with $\mathbf{y}(0) = \mathbf{0}$ and $\mathbf{y}'(0) = \mathbf{v}_0$ is

$$\mathbf{y}(t) = \frac{1}{\sqrt{2}}(y_1(t)\mathbf{u}_1 + 2y_2(t)\mathbf{u}_2)\ .$$

multiplying by $M^{1/2}$ gives us the solution $\mathbf{x}(t)$ that we seek:

$$\begin{aligned} \mathbf{x}(t) &= \frac{1}{\sqrt{2}}(y_1(t)\mathbf{u}_1 + y_2(t)\mathbf{u}_2) = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{3}}\sin(\sqrt{3}t)\mathbf{u}_1 + \frac{1}{2}\sin(t/2)\mathbf{u}_2) \\ &= \frac{1}{2}\left(\frac{1}{\sqrt{3}}\sin(\sqrt{3}t) + \frac{1}{2}\sin(2t)\ ,\ \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) - \frac{1}{2}\sin(2t)\right)\ . \end{aligned}$$

Notice that is was possible to avoid a lot of direct matrix multiplication calculations using eigenvector expansions. In fact, one can give an even simpler solution using the fact that M and A have the same eigenvectors; Just expand the solution in the common eigenbasis. The solution given above uses the general method that works even when M and A have different sets of eigenvectors.

(b) The forcing term is a multiple of $\mathbf{u}_1$, and so it drives only the mode with natural frequency $\omega_1 = \sqrt{3}$: There is resonance only at $\omega_1 = \sqrt{3}$.