Solutions for Test 2, Math 292 Spring 2018

April 28, 2018

1. (a) Find the general solution of the homogeneous differential equation

$$x^2u''(x) - 2xu'(x) + 2u(x) = 0$$

on $x > 0$.

(b) Find the general solution to the equation

$$x^2u''(x) - 2xu'(x) + 2u(x) = x^3$$

on $x > 0$.

(c) Find functions $p(x)$ and $q(x)$ such that with $\mathcal{L}u(x) = (pu')' + qu$, $\mathcal{L}u(x) = 0$ on $x > 0$ if and only if $x^2u''(x) - 2xu'(x) + 2u(x) = 0$. Then find the Green's function for $\mathcal{L}$ subject to the boundary conditions $u(1) = u(2) = 0$, and use it to find the solution of

$$\mathcal{L}u(x) = \frac{1}{x} \ .$$

SOLUTION (a) We seek a solution of the form $u(x) = x^\alpha$. The α must satsify $\alpha^2 - 3\alpha + 2 = 0$, so we get the two solutions $u_1(x) = x$ and $u_2(x) = x^2$. The general solution then is

$$u(x) = c_1x + c_2x^2 \ .$$

(b) We compute $M(x) = \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}$. Therefore, $\det(M(x)) = x^2$. Likewise, $N(x,y) = \begin{bmatrix} y & y^2 \\ x & x^2 \end{bmatrix}$. Therefore, $\det(N(x,y)) = yx^2 - xy^2$. Finally $r(y) = y$. Hence a particular solutions is given by

$$u_p(x) = \int_1^x \frac{yx^2 - xy^2}{y^2} y\mathrm{d}y = \left(x^2y - \frac{xy^2}{2}\right)\Big|_1^x$$

We can ignore the terms that come from evaluating the integral at $y = 1$ since this yields a linear combination of $u_1(x)$ and $u_2(x)$. Thus, $x^3/2$ is a particular solutuion, and the general solution is

$$u(x) = c_1x + c_2x^2 + \frac{1}{2}x^3 \ .$$

(c) In standard form our equation is $u'' + Pu' + Qu$ with $P = -2/x$ and $Q = 2/x^2$. We find $p(x) = x^{-2}$ and so $q(x) = 2x^{-4}$. We redfince u_1 and u_2 to be linear combinations of x and x^2 such that $u_1(1) = 0$ and $u_2(2) = 0$. An easy way to do this is to take

$$u_1(x) = x - x^2 \quad \text{and} \quad u_2(x) = 2x - x^2 \ .$$

Then we have $p(1) = 1$, $u_1'(1) = -1$ and $u_2(1) = 1$ Then $c = 1$ and

$$G(x,y) = \frac{1}{C}\begin{cases} u_1(x)u_2(y) & y \geq x \\ u_1(y)u_2(x) & x \geq y \end{cases} = \begin{cases} (x-x^2)(2y-y^2) & y \geq x \\ (y-y^2)(2x-x^2) & x \geq y \end{cases} \tag{0.1}$$

The solution then is

$$\begin{aligned} u(c) &= (2x-x^2)\int_1^x (y-y^2)\frac{1}{y}\mathrm{d}y + (x-x^2)\int_x^2 (2y-y^2)\frac{1}{y}\mathrm{d}y \\ &= x - \frac{3}{2}x^2 + \frac{1}{2}x^3 \ . \end{aligned}$$

2: The equation

$$(\ln(x) - 1)u''(x) - \frac{1}{x}u'(x) + \frac{1}{x^2}u(x) = 0$$

is solved by $u(x) = x$.

(a) Find the general solution of this equation on $x > e$.

(b) Find the solution of

$$(\ln(x) - 1)u''(x) - \frac{1}{x}u'(x) + \frac{1}{x^2}u(x) = \frac{(\ln(x)-1)^2}{x}$$

that satisfies $u(e^2) = 0$ and $u'(e^2) = 1$.

SOLUTION (a) We have one solution $u_1(x) = x$. Putting the homogenous equation in standard form, $u'' + Pu' + Q = 0$, we have

$$P = \frac{-1/x}{\ln x - 1} \quad \text{and} \quad Q = \frac{1/x^2}{\ln x - 1} \ .$$

We seek a solution $u_2 = vu_1$, and know that we may take

$$v = \int \frac{1}{u_1^2}e^{-\int P}\mathrm{d}x = \int \frac{1}{x^2}\exp(\ln(\ln x + 1))\mathrm{d}x = \int \frac{1}{x^2}(\ln x - 1)\mathrm{d}x \ ,$$

Integrating by parts, we find

$$v(x) = -\frac{1}{x}\ln x \ ,$$

and so the second solution can be taken to be $u_2(x) = \ln x$. The general solution then is

$$u(x) = c_1 x + c_2 \ln x \ .$$

(b) We compute $M(x) = \begin{bmatrix} x & \ln x \\ 1 & x^{-1} \end{bmatrix}$. Therefore, $\det(M(x)) = 1 - \ln x$. Likewise, $N(x,y) = \begin{bmatrix} y & \ln y \\ x & \ln x \end{bmatrix}$. Therefore, $\det(N(x,y)) = y\ln x - x\ln y$. Finally $r(y) = (\ln y - 1)/y$. Hence a particular solutions is given by

$$u_p(x) = \int_{e^2}^x \frac{y\ln x - x\ln y}{(\ln y - 1)}\frac{\ln y - 1}{y}\mathrm{d}y = -\left(y\ln x - \frac{1}{2}(\ln y)^2\right)\Big|_1^x$$

We can ignore the terms that come from evaluating the integral at $x = 1$ since this yields a linear combination of $u_1(x)$ and $u_2(x)$. Thus, $-x\ln x + \frac{1}{2}x(\ln x)^2$ is a particular solutuion, and the general solution is

$$u(x) = c_1 x + c_2 \ln x - x\ln x + \frac{1}{2}x(\ln x)^2 \ .$$

We must now choose c_1 and c_2 so that $u(e^2) = 0$ and $u'(e^2) = 1$.

We compute

$$u(e^2) = c_1 e^2 + 2c_2 \qquad \text{and} \qquad u'(e^2) = c_1 + e^{-2}c_2 + 1 \ .$$

The first equation says that $c_2 = -\frac{e^2}{2}c_1$, and the second equation says $c_2 = -e^2 c_1$. Hence $c_1 = c_2 = 0$, and the solutions is simply

$$u(x) = x\ln x - \frac{1}{2}x(\ln x)^2 \ .$$

3. (a) One of the two equations

$$(\mathbf{1}) \quad y''(x) + \frac{1}{x}y(x) = f(x) \qquad\qquad (\mathbf{2}) \quad y''(x) - \frac{1}{x}y(x) = f(x)$$

has a solution satisfying $y(1) = y(L) = 0$ for every $L > 1$ and every continuous function f on $[1, L]$, and the other does not. Which one is which and why? Justify your answer.

(b) Let $\mathcal{L}u(x) = \frac{1}{1+x^2}((1 + x^2)u'(x))'$. Consider the eigenvalue equation $\mathcal{L}u(x) = \lambda u(x)$. Find a function $V_\lambda(x)$ such that for every solution of $\mathcal{L}u(x) = \lambda u(x)$, there is a solution of $y''(x) + V_\lambda(x)y(x) = 0$ that has the same zeros as u.

(c) Find a number c so that for $\lambda < c$, successive zeros of any solution of $\mathcal{L}u(x) = \lambda u(x)$ are separated by a distance of less than one. Find a number C so that for $\lambda > C$, successive zeros of any solution of $\mathcal{L}u(x) = \lambda u(x)$ are separated by a distance of more than one. Give upper and lower bounds on λ_1, the largest eigenvalue of $\mathcal{L}u(x)$ with boundary conditons $u(0) = u(1) = 0$. Justify all of your answers.

SOLUTION (a) These equations are both of the form $y'' + Vy = 0$. We know that when $V(x) < 0$, non-trivial solutions will have at most one zero. Hence for equation **(2)**, there is no non-trivial solution y with $y(1) = y(L) = 0$ for any $L > 1$. Thereofre,

$$y'' - \frac{1}{x}y = f$$

has a unique solution on $[1, L]$ for every conitnuous f on $[1, L]$, and every $L > 1$.

We also know that when $V \geq 0$ and $\int_1^\infty V(x)\mathrm{d}x = \infty$, which is the case for equation**(1)**, non-trivial solutions will have infnitely many zeros. Hence any non-trivla solution y with $y(1) = 0$ satsifies $y(L) = 0$ for infnitely many values of L, and for these vlaues of L,

$$y'' + \frac{1}{x}y = f$$

will not be solvable for every continuus f, and solutions, when they exist, will not be unique.

(b) Let $\mathcal{L}u(x) = u''(x) + \frac{2x}{1+x^2}u = 0$, so $\mathcal{L}u = \lambda u$ is the same as $u'' + Pu' + Qu$ with

$$P = \frac{2x}{1+x^2} \quad \text{and} \quad Q = -\lambda \ .$$

Then

$$V_\lambda(x) = Q - \frac{1}{2}P' - \frac{1}{4}P^2 = -\lambda - \frac{1}{1+x^2} \ .$$

(c) For any x_0, the function $w(x) = \sin(\pi(x - x_0))$ has a zero at x_0, followed by a zero at $x_0 + 1$, and it solves the equation

$$w'' + \pi^2 w = 0 \ .$$

Then by the Strum Comparisson Theorem, if $V_\lambda > \pi^2$ on $[x_0, x_0+1]$, then any solution of $y'' + V_\lambda y = 0$ that is zero at x_0, is zero again at some $x < x_0 + 1$. Likeise, if $V_\lambda < \pi^2$ on $[x_0, x_0 + 1]$, then any solution of $y'' + V_\lambda y = 0$ that is zero at x_0, is positive for all $x \in (x_0, x_0 + 1]$.

Now note that on $[0, 1]$,

$$-\lambda - 1 \le v_\lambda(x) \le -\lambda - \frac{1}{2} \ .$$

Hence λ_1, the value of λ such that there is a non-trivla solutions of $y'' + V_\lambda y = 0$ with $y(0) = y(1) = 0$ and $y(x) > 0$ for $x \in (0, 1)$ satsfies

$$-\pi^2 + \frac{1}{2} \le \lambda_1 \le -\pi^2 + 1 \ .$$

4: (a) Let $\mathcal{L}u(x) = u''(x)$. Find all eigenvalues and eigenfunctions of $\mathcal{L}$ with boundary conditions $u'(0) = 0$ and $u(1) = 0$.

(b) solve the equation

$$\frac{\partial}{\partial t}h(x,t) = \frac{\partial^2}{\partial^2 x}h(x,t)$$

for $x \in (0, 1)$, $t > 0$, subject to

$$\frac{\partial}{\partial x}h(0,t) = h(1,t) = 0$$

and

$$h(x, 0) = \cos(\pi x/2) - \cos(3\pi x/2) \ .$$

The eigenvalues λ of $\mathcal{L}$ are negative, so we can write the eigenvalue equation in the form

$$u''(x) = -\omega^2 u(x) \ ,$$

and the general solition is

$$u(x) = c_1 \sin(\omega x) + c_2 \cos(\omega x) \ .$$

From $u.(0) = 0$, we get $c_1 = 0$, and then from $u(1) = 0$, we get $\cos(\omega) = 0$ and so $\omega = (k + \frac{1}{2})\pi$ for some $k \in \mathbb{N}$. Therefore the eigenvalues and eigenfunctions (un-normalized) are

$$\lambda_k = -\left(k + \frac{1}{2}\right)^2 \pi^2 \quad \text{and} \quad u_k = \cos\left(\left(k + \frac{1}{2}\right)\pi x\right) \ .$$

Since h_0 is a linear combination of the eigenfunctions, namely $h_0 = u_1 - u_2$, $h(x, t)$ can be written as the same linear combination of the corresponding special solutions, namely

$$h(x,t) = e^{-t\pi^2/4}\cos(\pi x/2) - e^{-t9\pi^2/4}\cos(3\pi x/2) \ .$$