Solutions for Practice Test Two, Math 292, 2018

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1. (a) Find the general solution of

$$x^2u''(x) - xu'(x) - 3u(x) = 0$$

for $x > 0$.
(b) Find the general solution of

$$x^2u''(x) - xu'(x) - 3u(x) = x - 3$$

(c) Find the solution of

$$x^2u''(x) - xu'(x) - 3u(x) = x - 3$$

with $u(1) = u(2) = 0$.

SOLUTION (a) We look for solutions of the form $u(x) = x^\alpha$. We find that $u_1(x) = x^3$ and $u_2(x) = x^{-1}$ are two linearly indpendent solutions. Thus the general solutions of the homogeneous equation is

$$u(x) = c_1x^3 + c_2\frac{1}{x} \ .$$

(b) We must compute

$$M(x) = \begin{bmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{bmatrix} \quad \text{and} \quad N(x,y) = \begin{bmatrix} u_1(y) & u_2(y) \\ u_1(x) & u_2(x) \end{bmatrix} \ ,$$

and them a particular solutions is

$$u_p(x) = \int_{x_0}^x \frac{\det(N(x,y))}{\det(M(y))} r(y)\mathrm{d}s \ .$$

We find $det(M(y)) = -4y$, $\det(N(x,y)) = y^3/x - x^3/y$, $r(y) = (y-3)/y^2$ Taking $x_0 = 1$,

$$\begin{aligned} u_p(x) &= \frac{1}{4}\int_1^x \left(x^3\frac{y-3}{y^4} - \frac{1}{x}(y-3) \right) \mathrm{d}y \\ &= \frac{1}{4}\left(x^3\left(\frac{1}{y^3} - \frac{1}{2y^2}\right) + \frac{1}{x}\left(3y - \frac{y^2}{2}\right)\right)\Bigg|_1^x \\ &= 1 - \frac{x}{4} - \frac{1}{8}\left(x^3 + \frac{5}{x}\right) \ . \end{aligned}$$

The last term in parentheses is a linear combination of solutions of the homogeneous equations and may be ignored. Hence we simplify to

$$u_p(x) = 1 - \frac{x}{4} ,$$

which one readily checks is a solution. The general solution we seek then is

$$u(x) = c_1 x^3 + c_2 \frac{1}{x} + 1 - \frac{x}{4} .$$

(c) Using the general solution found above, we compute

$$u(1) = c_1 + c_2 + \frac{3}{4} \qquad \text{and} \qquad u(2) = 8c_1 + \frac{c_2}{2} + \frac{1}{2} .$$

This is the same as

$$\begin{bmatrix} 4 & 4 \\ 16 & 1 \end{bmatrix} (c_1, c_2) = (-3, -1) .$$

We find $c_1 = -1/60$ and $c_2 = -11/15$. Hence

$$u(x) = -\frac{1}{60} x^3 + = -\frac{11}{15} \frac{1}{x} + 1 - \frac{x}{4} .$$

2. (a) Find functions p and q such that if we define $\mathcal{L}u = (pu')' + qu$, u satisfies $xu'' - (1+x)u' + u = 0$ if and only if $\mathcal{L}u = 0$ everywhere on $(0, \infty)$.

(b) Are there any solutions of $\mathcal{L}u = 0$ that have more than one zero on $(0, \infty)$? Justify your answer: First find an equation of the form

$$y'' + V(x)y(x)$$

so that every solution of this equation on $(0, \infty)$ is a nonzero multiple of a solution of $\mathcal{L}u = 0$, and apply the Strum oscillation theorems.

(c) Find the Green's function for $\mathcal{L}u = f$ subject to $u(1) = u(2) = 0$, and find the solution of

$$\mathcal{L}u = e^x .$$

SOLUTION In standard for the homogeous equations is $u'' + Pu' + Qu = 0$ with $P = -x/(1+x)$ and $Q = 1/x$. Hence

$$p = e^{\int P} = \frac{e^{-x}}{x} .$$

Therefore,

$$xe^x \left[\left(\frac{e^{-x}}{x} u' \right)' + \frac{e^{-x}}{x^2} u \right] = u'' - \left(1 + \frac{1}{x} \right) u' + \frac{1}{x} u .$$

Hence we take

$$\mathcal{L}u = \left(\frac{e^{-x}}{x} u' \right)' + \frac{e^{-x}}{x^2} u .$$

For **(b)**, since $P = -\left(1 + \frac{1}{x}\right)$ and $Q = \frac{1}{x}$,

$$V(x) = Q - \frac{1}{2}P' - \frac{1}{4}P^2 = \frac{1}{x} - \frac{1}{4}\left(\left(1 - \frac{1}{x}\right)^2 + \frac{2}{x^2}\right) = \frac{1}{2x} - \frac{1}{4} - \frac{3}{4x^2} < 0 \ .$$

Hence there are no such solutions, because we have a theorem asserting that when V is negative, there is at most one zero.

(c) We easily find that one solutions is $w_1(x) = e^x$, and a second solution is $w_2(x) = -(x+1)$. If you find, say, w_1 by inspection you can get w_2 using $u_2 = vw_1$ where

$$v = \int \frac{1}{u_1^2} e^{-\int P} \ .$$

Note that

$$\frac{2}{e}w_1(1) + w_2(1) = 0 \qquad \text{and} \qquad \frac{3}{e^2}w_1(2) + w_2(2) = 0 \ .$$

Hence we define

$$u_1(x) = 2e^{x-1} - (x+1) \qquad \text{and} \qquad u_2(x) = 3e^{x-2} - (x+1) \ .$$

Then the Green's Function is given by

$$G(x,y) = \frac{1}{C}\begin{cases} u_1(x)u_2(y) & y \geq x \\ u_1(y)u_2(x) & x \geq y \ , \end{cases} \tag{0.1}$$

and C itself is simply given by $C = -u_2(1)u_1'(1)p(1) = (2 - 3e^{-1})e^{-1}$. The solution then is

$$\begin{aligned} u(x) &= \frac{u_2(x)}{C}\int_1^x (2e^{y-1} - (y+1))e^y \mathrm{d}y + \frac{u_1(x)}{C}\int_x^2 (3e^{y-2} - (y+1))e^y \mathrm{d}y \\ &= \frac{1}{(2-3e^{-1})e^{-1}}\left(\frac{3}{2}(1-x)e^{2x-2} + (x-1)e^{2x-1} - e^{x+1} + \frac{e^2}{2}(1+x)\right) \\ &= \frac{1}{(2e-3)}\left(\frac{3}{2}(1-x)e^{2x} + (x-1)e^{2x+1} - e^{x+2} + \frac{e^4}{2}(1+x)\right) \ . \end{aligned}$$

One can check that this is indeed the solution. It is probably easiest to consider the equation $\mathcal{L}u = e^x$ in the equivalent form

$$xu'' - (1+x)u' + u = x^2e^{2x} \ .$$

3. Let $\mathcal{L}$ be the Sturm-Liouville operator defined by

$$\mathcal{L}u(x) = (1+x^2)((1+x^2)u'(x))' \ .$$

(a) Find all eigenvalues and eigenfunctions $\mathcal{L}u(x) = \lambda u(x)$ subject to $u(0) = 0$ and $u(1) = 0$. *Hint: recall that the derivative of* $\arctan(x)$ *is* $(1+x^2)^{-1}$*, and consider the function* v *defined by* $u(x) = v(\arctan(x))$*, and compute* $\mathcal{L}u(x)$ *in terms of* v*.*

(b) Solve the wave equation

$$\frac{\partial^2}{\partial t^2}h(x,t) = \mathcal{L}h(x,t)$$

subject to $h(0,t) = h(1,t) = 0$ for all t, and subject to

$$h(x,0) = 0 \quad \text{and} \quad \frac{\partial}{\partial t}h(x,0) = \frac{x - x^3}{(1+x^2)^2} \ .$$

Hint: If the function on the right looks unfamiliar after you have completed part **(a)**, *compute* $\mathcal{L}$ *applied to this function.*

(c) Consider the wave equation

$$\frac{\partial^2}{\partial t^2}h(x,t) = \mathcal{L}h(x,t)$$

subject to $h(0,t) = h(1,t) = 0$ for all t, and subject to

$$h(x,0) = x - x^3 \quad \text{and} \quad \frac{\partial}{\partial t}h(x,0) = 0 \ .$$

Find integrals giving numbers β_k so that the solution $h(x,t)$ is given by

$$h(x,t) = \sum_{k=1}^{\infty} \cos(\sqrt{|\lambda_k|}t)u_k(x)$$

where $\{\lambda_k\}$ and $\{u_k\}$ are the eigenvectors and eigenvalue sequence found in part **(a)**.

SOLUTION Let $u(x) = v(\arctan(x))$, and then $u'(x) = (1+x^2)^{-1}v'(\arctan(x))$. Therefore, $(1+x^2)u'(x) = v'(\arctan(x))$. In this way we find

$$\mathcal{L}u(x) = v''(\arctan(x)) \ .$$

The eigenvalue problem $\mathcal{L}u = \lambda u$ is then the same as

$$v''(y) = \lambda v(y)$$

for $y = \arctan(x)$, and boundary conditions $v(0) = v(\pi/4) = 0$.

The general solution is $v(y) = \alpha\sin(\sqrt{-\lambda}y) + \beta\cos(\sqrt{-\lambda}y)$, and from the boundary conditions, $\beta = 0$ and $\sqrt{-\lambda}\pi/4 = k\pi$. Thus, $\lambda_k = -(4k)^2$ and $v_k(y) = \sin(4ky)$, and thus $u_k(x) = \sin(4k\arctan(x))$. Here $k = 1, 2, 3 \ldots$.

Next, since $\sin(\arctan(x)) = x/\sqrt{1+x^2}$ and $\cos(\arctan(x)) = 1/\sqrt{1+x^2}$, two applications of the double angle formulas give

$$u_1(x) = 4\frac{x - x^3}{(1+x^2)^2} \ .$$

The general solution $h(x,t)$ can be written as a linear combination of the special solutions $h_k(x,t)$:

$$h(x,t) = \sum_{k=1}^{\infty} \left[\alpha_k \sin(4kt)u_k(x) + \beta_k \cos(4kt)u_k(x)\right] \ .$$

Differentiating,

$$\frac{\partial}{\partial t}h(x,t) = \sum_{k=1}^{\infty}\left[4k\alpha_k \cos(4kt)u_k(x) - \beta_k 4k\sin(4kt)u_k(x)\right] .$$

Setting $t = 0$, we can siatisfy the initial conditions by taking $\beta_k = 0$ for all k, and then taking $\alpha_1 = 1/16$ and $\alpha_k = 0$ for all $k > 1$.

Finally for part **(c)**, we satisfy the initial conditions by taking $\alpha_k = 0$ for all k, and then taking

$$\beta_k = \frac{\langle x - x^3, u_k\rangle_\rho}{\|u_k\|_\rho^2}$$

where

$$\rho(x) = (1 + x^2)^{-1}$$

. (You can actually do the integrals abut are not required to).

4. Consider the equation $\sqrt{1+x^2}u'' + xu' = \lambda u$.

(a) .Write this equation in Sturm-Liouville form as an eigenvalue equation. That is, find positive functions $\rho(x)$ and $p(x)$ so that $\mathcal{L}u(x) = \lambda u(x)$ with

$$\sqrt{1+x^2}u'' + xu' = \frac{1}{\rho(x)}(p(x)u')' = \mathcal{L}u(x) .$$

(b) Find a function $V_\lambda(x)$ such that if u is any non-trivial solution of $\sqrt{1+x^2}u'' + xu' = \lambda u$, there is a solution $y(x)$ of

$$y''(x) + V_\lambda(x)y(x) = 0$$

that has its zeros in the same places as $u(x)$.

(c) Find a number κ_0 so that for $\lambda > \kappa_0$ you know that all solutions of $\mathcal{L}u = \lambda u$ have at most one zero in $(0,1)$. Justify your answer.

(d) Find a number κ_1 so that for $\lambda < \kappa_1$ you know that all solutions of $\mathcal{L}u = \lambda u$ with $u(0) = 0$ have a zero in $(0,1)$. Justify your answer.

(e) Let λ_1 be the largest eigenvalue of $\mathcal{L}$ for $u(0) = u(1) = 0$. Find numbers a and b so that you know that $a \le \lambda_1 \le b$. Justify your answer.

SOLUTION First, write the eqaution in standard form:

$$u'' + \frac{x}{\sqrt{1+x^2}}u' - \frac{\lambda}{\sqrt{1+x^2}}u .$$

Here,

$$P = \frac{x}{\sqrt{1+x^2}} = \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+x^2}$$

and so $p = e^{\int P} = e^{\sqrt{1+x^2}}$. Multiplying through we get

$$\left(e^{\sqrt{1+x^2}}u'\right)' - e^{\sqrt{1+x^2}}\frac{\lambda}{\sqrt{1+x^2}}u = 0 .$$

Undoing the two multiplications, we get

$$\sqrt{1+x^2}u'' + xu' = \lambda u = \sqrt{1+x^2}e^{-\sqrt{1+x^2}}\left(e^{\sqrt{1+x^2}}u'\right)' - \lambda u \ .$$

Hence we have

$$\mathcal{L}u = \sqrt{1+x^2}e^{-\sqrt{1+x^2}}\left(e^{\sqrt{1+x^2}}u'\right)' \ .$$

For **(b)** we have $P = x/\sqrt{1+x^2}$ and $Q = -\lambda/\sqrt{1+x^2}$. Then

$$V_\lambda = Q - \frac{1}{4}P^2 - \frac{1}{2}P' \ .$$

Computing we find

$$V_\lambda = -\frac{1}{4}\frac{x^2\sqrt{1+x^2} + 2 + 4\lambda(1+x^2)}{(1+x^2)^{3/2}} \ .$$

For **(c)**, we have a most one zero when V_λ is negative, This is the case exactly when

$$-x^2\sqrt{1+x^2} - 2 < 4\lambda(1+x^2)$$

on $(0,1)$. Simple calculations show that this is the case for $\lambda > 0.42$. (You can do slightly better). So we take $\kappa_0 = -0.42$.

For **(d)**, if $V_\lambda \geq \pi^2$ then any solution of $y'' + V_\lambda$ with $y(0) = 0$ will have a second zero in $(0,1)$. This is the case if for all $x \in (0,1)$

$$x^2\sqrt{1+x^2} + 2 + 4\lambda(1+x^2) < -4(1+x^2)^{3/2}\pi^2 \ .$$

which means for all $x \in (0,1)$,

$$\lambda < \frac{-4(1+x^2)^{3/2}\pi^2 - x^2\sqrt{1+x^2} - 2}{4(1+x^2)} \ .$$

A plot of the right hand side shows it is greater than -15 everywhere on $(0,1)$. Hence $\lambda < -15$ is too negative.

For **(e)**, the above gives $-15 < \lambda_1 < -0.42$.