Practice Test Two, Math 292, 2018

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1. **(a)** Find the general solution of

$$x^2u''(x) - xu'(x) - 3u(x) = 0$$

for $x > 0$.

(b) Find the general solution of

$$x^2u''(x) - xu'(x) - 3u(x) = x - 3$$

(c) Find the solution of

$$x^2u''(x) - xu'(x) - 3u(x) = x - 3$$

with $u(1) = u(2) = 0$.

2. **(a)** Find functions p and q such that if we define $\mathcal{L}u = (pu')' + qu$, u satisfies $xu'' - (1+x)u' + u = 0$ if and only if $\mathcal{L}u = 0$ everywhere on $(0, \infty)$.

(b) Are there any solutions of $\mathcal{L}u = 0$ that have more than one zero on $(0, \infty)$? Justify your answer: First find an equation of the form

$$y'' + V(x)y(x)$$

so that every solution of this equation on $(0, \infty)$ is a nonzero multiple of a solution of $\mathcal{L}u = 0$, and apply the Strum oscillation theorems.

(c) Find the Green's function for $\mathcal{L}u = f$ subject to $u(1) = u(2) = 0$, and find the solution of

$$\mathcal{L}u = e^x \ .$$

3. Let $\mathcal{L}$ be the Sturm-Liouville operator defined by

$$\mathcal{L}u(x) = (1 + x^2)((1 + x^2)u'(x))' \ .$$

(a) Find all eigenvalues and eigenfunctions $\mathcal{L}u(x) = \lambda u(x)$ subject to $u(0) = 0$ and $u(1) = 0$. *Hint: recall that the derivative of* $\arctan(x)$ *is* $(1 + x^2)^{-1}$*, and consider the function* v *defined by* $u(x) = v(\arctan(x))$*, and compute* $\mathcal{L}u(x)$ *in terms of* v*.*

(b) Solve the wave equation

$$\frac{\partial}{\partial t} h(x,t) = \mathcal{L} h(x,t)$$

subject to $h(0,t) = h(1,t) = 0$ for all t, and subject to

$$h(x,0) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} h(x,0) = \frac{x - x^3}{(1+x^2)^2} \ .$$

Hint: If the function on the right looks unfamiliar after you have completed part **(a)**, *compute* $\mathcal{L}$ *applied to this function.*

(c) Consider the wave equation

$$\frac{\partial}{\partial t} h(x,t) = \mathcal{L} h(x,t)$$

subject to $h(0,t) = h(1,t) = 0$ for all t, and subject to

$$h(x,0) = x - x^3 \quad \text{and} \quad \frac{\partial}{\partial t} h(x,0) = 0 \ .$$

Find integrals giving numbers β_k so that the solution $h(x,t)$ is given by

$$h(x,t) = \sum_{k=1}^{\infty} \cos(\sqrt{|\lambda_k|} t) u_k(x)$$

where $\{\lambda_k\}$ and $\{u_k\}$ are the eigenvectors and eigenvalue sequence found in part **(a)**.

4. Consider the equation $\sqrt{1+x^2}u'' + xu' = \lambda u$.

(a) .Write this equation in Sturm-Liouville form as an eigenvalue equation. That is, find positive functions $\rho(x)$ and $p(x)$ so that $\mathcal{L}u(x) = \lambda u(x)$ with

$$\sqrt{1+x^2}u'' + xu' = \frac{1}{\rho(x)}(p(x)u')' = \mathcal{L}u(x) \ .$$

(b) Find a function $V_\lambda(x)$ such that if u is any non-trivial solution of $\sqrt{1+x^2}u'' + xu' = \lambda u$, there is a solution $y(x)$ of

$$y''(x) + V_\lambda(x) y(x) = 0$$

that has its zeros in the same places as $u(x)$.

(c) Find a number κ_0 so that for $\lambda > \kappa_0$ you know that all solutions of $\mathcal{L}u = \lambda u$ have at most one zero in $(0,1)$. Justify your answer.

(d) Find a number κ_1 so that for $\lambda < \kappa_1$ you know that all solutions of $\mathcal{L}u = \lambda u$ with $u(0) = 0$ have a zero in $(0,1)$. Justify your answer.

(e) Let λ_1 be the largest eigenvalue of $\mathcal{L}$ for $u(0) = u(1) = 0$. Find numbers a and b so that you know that $a \le \lambda_1 \le b$. Justify your answer.