Practice Test One, Math 292, Spring 2018

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1. Find the general solution of the differential equation

$$x'(t) = -\frac{2}{t}x(t) + t^3 x(t)^2 + t^{-5}$$

for $t > 0$.

2. **(a)** Let $v(x) = x/\ln(x)$, which is continuous on $1 < x < \infty$. For all x_0 in this interval, find all solutions of

$$x'(t) = v(x(t)) \ , \qquad x(0) = x_0 \ . \tag{$*$}$$

For which values of t is each solution defined?

(b) Find an explicit formula for the flow transformation $\varphi_t(x)$ on the interval $(1, \infty)$ such that for each $x_0 \in (1, \infty)$, $x(t) := \varphi_t(x_0)$ is the unique solution to $(*)$ for all t for which the solution is deined.

(c) Consider the equations

$$x'(t) = \sqrt{x(t)(1 - x(t))} \qquad \text{and} \qquad y'(t) = y^2(t)(1 - y(t))^2 \ ,$$

with the initial conditions $x(0) = y(0) = 1/2$.

One of these solutions stays in the interval $(0, 1)$ for all t, and the other reaches the right endpoint in a finite time T. Which one is it, and what is the value of T?

3. **(a)** Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -2 \end{bmatrix} .$$

Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}. (Note that this system is recursively coupled.)

(b) For the same matrix A and for $\mathbf{f}(t) = (0, e^t, 0)$, find the solution of

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t) \qquad \text{with} \qquad \mathbf{x}(0) = (1, 2, 1) \ .$$

(c) Consider the matrix $B = \begin{bmatrix} 7 & 9 \\ -1 & 1 \end{bmatrix}$. Compute the matrix exponential e^{tB}. Is there any choice of $\mathbf{x}_0$ so that if $\mathbf{x}(t)$ is the solution of $\mathbf{x}'(t) = B\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$, the both $\lim_{t\to\infty} \|\mathbf{x}(t)\| = \infty$ and $\lim_{t\to-\infty} \|\mathbf{x}(t)\| = \infty$?

4. Consider the vector field

$$\mathbf{v}(x,y) = ((x+y)(x-y-1), (x+y-2)(x-y+1)) \ .$$

(a) Find all equilibrium points of $\mathbf{v}$, and determine which, if any, are asymptotically stable, and which if any are unstable. Sketch the flow lines in the vicinity of each equilibrium point.

(b) Do the same for

$$\mathbf{v}(x,y) = ((x+y-2)(x-y+1) \ , (x+y)(x-y-1)) \ .$$

5. Let $M = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 8 & 2 \\ 2 & 2 \end{bmatrix}$.

(a) Consider the equation $\mathbf{x}'' = -A\mathbf{x}$. Find an equivalent equation of the form $\mathbf{y}'' = -K\mathbf{y}$, and write down the general solution of this equation. Then write down the solution of $\mathbf{x}'' = -A\mathbf{x}$ with $\mathbf{x}(0) = (3, 2)$ and $\mathbf{x}'(0) = \mathbf{0}$.

(b)

Let $\mathbf{g} = (4, 1)$. Consider the differential equation

$$M\mathbf{x}''(t) + A\mathbf{x}(t) = \mathbf{g}\cos(\omega t)$$

for some positive number ω. Find the solution for all $\omega > 0$. For which values of ω, if any, is there resonance?