Homework Assignment 7, Math 292, Spring 2019

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1. Find the minimum value and minimizing function $y(x)$ for

$$I[y] = \int_0^1 (1+x)(y'(x))^2 \mathrm{d}x$$

subject to $y(0) = 0$ and $y(1) = 1$.

2. Find the minimum value and minimizing function $y(x)$ for

$$I[y] = \int_0^1 \left[(y'(x))^2 + \frac{y^2(x)}{x^2}\right] \mathrm{d}x$$

subject to $y(0) = 0$ and $y(1) = 1$.

3. Let $L > 0$ and let

$$I[y] = \int_0^L [(y')^2 - y^2 - (\sin x)y]\mathrm{d}x \ .$$

Consider the problem of minimizing $I[y]$ subject to $y(0) = y(L) = 0$. Find the corresponding Euler-Lagrange equation. For which values of L does it have a solution subject to the boundary conditions, and for which values of L is it a minimum?

4: For all continuously differentiable functions $y(x)$ on $(1,4)$ such that $y(1) = 1$ and $y(4) = 2$, define

$$I[y] = \int_1^4 y^2(x)(y'(x))^2 \mathrm{d}x \ .$$

(a) Find and solve the Euler-Lagrange equation for these boundary conditions.

(b) Use the identity

$$y^2(4) - y^2(1) = \int_1^4 2y(x)y'(x)\mathrm{d}x$$

valid for any continuously differentiable functions, and the Cauchy-Schwarz inequality for integrals, to prove that the solution you found in part **(a)** is the minimizer of the functional.

5. Find the minimum value and minimizing function $y(x)$ for

$$I[y] = \int_0^1 y^2(x)\mathrm{d}x$$

subject to the constraints $\int_0^1 x^2 y(x)\mathrm{d}x = 1$ and $\int_0^1 y(x)\mathrm{d}x = 1$. There are many connections between things we have discussed this year. The easiest way to think about this problem is in terms of the Gram-Schmidt Algorithm. If we define $v_1(x) = x^2$, and $v_2(x) = 1$, and define the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\mathrm{d}x$, and $\|f\|^2 = \langle f, f\rangle$, the function to be minimized is $\|y\|^2$, and the constraints are $\langle y, v_1\rangle = 1$ and $\langle y, v_2\rangle = 1$. How would you find a vector $\mathbf{y} \in \mathbb{R}^3$ with minimal norm such that for given vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, $\mathbf{y}\cdot\mathbf{v}_1 = 1$ and $\mathbf{y}\cdot\mathbf{v}_2 = 1$? Compare this approach with the Lagrange multiplier approach.