Solutions for Homework Assignment 6, Math 292, Spring 2018

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1. **(a)** Find the general solution of the homogeneous differential equation

$$t^2x'' + 3tx' + x = 0 \ .$$

(b) Find the general solution to the equation

$$t^2x'' + 3tx' + x = \ln(t)$$

(c) Find the solution of

$$t^2x'' + 3tx' + x = \ln(t)$$

that satisfies $x(1) = 0$ and $x(3) = 0$, or explain why there is no such solution.

SOLUTION **(a)** Looking for a solution of the form $x(t) = t^\alpha$, we find $\alpha^2 + 2\alpha + 1 = 0$, so $\alpha = -1$, one solution is $x_1(t) = 1/t$. Since $P = 3/t$,

$$v = \int \frac{1}{x_1^2} e^{-\int P} = \ln(t) \ ,$$

so the second solutions is $x_2(t) = \ln(t)/t^2$. Hence the general solutions is

$$x(t) = c_1\frac{1}{t} + c_2\frac{\ln(t)}{t} \ .$$

(b) We must compute

$$M(s) = \begin{bmatrix} x_1(s) & x_2(s) \\ x_1'(s) & x_2'(s) \end{bmatrix} \quad \text{and} \quad N(t,s) = \begin{bmatrix} x_1(s) & x_2(s) \\ x_1(t) & x_2(t) \end{bmatrix} \ ,$$

and then a particular solutions is

$$x_p(t) = \int_{t_0}^t \frac{\det(N(t,s))}{\det(M(s))} r(s)\mathrm{d}s \ .$$

We find $\det(M(s)) = s^{-3}$ and $\det(N(t,s)) = \frac{1}{st}\ln(s/t)$, and $r(s) = \ln(s)/s^2$. Altogether, taking $t_0 = 1$, we find

$$x_p(t) = \frac{1}{t}\int_1^t (\ln(t)\ln(s) - (\ln(s))^2)\mathrm{d}s = \frac{2}{t} + \frac{\ln(t)}{t} + \ln(t) - 2 \ .$$

Because the first two terms are multiples of x_1 and x_2, we may ignore them and simplify our particular solutions to

$$x_p(t) = \ln(t) + 2 \ .$$

This is simple enough to check; it is indeed a solution. Hence the general solution is

$$x(t) := c_1 \frac{1}{t} + c_2 \frac{\ln(t)}{t} + \ln(t) - 2 \ .$$

(c) We find $x(1) = c_1 - 2$, and so $c_1 = 2$. Then

$$x(3) = -2\ln(3) + \left(\frac{c_2}{3} + 1\right)\ln(3) - 2 \ ,$$

from which one finds $c_2 = 3 + 6/\ln(3)$.
2. (a) The equation

$$xu'' - u' + (1-x)u = 0 \tag{0.1}$$

has one solution of the form $u_1(x) = e^{\alpha x}$. Find this solution and a second independent solution $u_2(x)$.
(b) Find functions p and q so that with $\mathcal{L}u = (pu')' + q$, $\mathcal{L}u = 0$ if and only if $xu'' - u' + (1-x)u = 0$.
(c) For the Sturm-Liouville operator $\mathcal{L}$ from part **(b)**, find the Green's function for solving $\mathcal{L}u = f$ subject to $u(1) = u(2) = 0$, or explain why it does not exist. Finally, find the solution of $\mathcal{L}u = x^2$ subject to $u(1) = u(2) = 0$.
(d) Solve $\mathcal{L}u = x^2$ subject to Neumann boundary conditions on $[1, 2]$.

SOLUTION (a) We find that $u_1 = e^x$, and then

$$v = \int \frac{1}{u_1^2} e^{-\int P} = -\frac{1}{4}(2x+1)e^{-2x} \ .$$

Therefore, dropping an irrelevant minus sign and constant, we get $u_2 = \left(x + \frac{1}{2}\right) e^{-x}$. One readily checks that this is a solution.
(b) Note that $(x^{-1}u')' = x^{-1}u'' - x^{-2}u'$. Therefore,

$$x^2 \left[(x^{-1}u')' + \frac{1-x}{x^2} u \right] = xu'' - u' + (1-x)u \ .$$

Hence $p = 1/x$ and $q = (1-x)/x^2$.
(c) Note that $u_2(1) = \frac{3}{2}e^{-1} = \frac{3}{2}e^{-2}u_1(1)$. Therefore

$$\widetilde{u}_1(x) := 2u_2(x) - 3e^{-2}u_1(x) = (2x+1)e^{-x} - 3e^{x-2}$$

is a solution of the homogeneous equation with $\widetilde{u}_1(1) = 0$. Likewise,

$$u_1(2) = e^2 = \frac{2}{5}e^4 u_2(2) \ .$$

Therefore

$$\widetilde{u}_2(x) := 2u_2(x) - 5e^{-4}u_1(x) = (2x+1)e^{-x} - 5e^{x-4}$$

is a solution of the homogeneous equation with $\widetilde{u}_2(2) = 0$.

From this point on *in this part*, we drop the tildes, and work with these solutions of the homogeneous equation: Then the Green's Function is given by

$$G(x, y) = \frac{1}{C} \begin{cases} u_1(x)u_2(y) & y \geq x \\ u_1(y)u_2(x) & x \geq y \ , \end{cases} \tag{0.2}$$

and C itself is simply given by $C = -u_2(1)u_1'(1)p(1) = 5e^{-4} - 3e^{-2}$. The solution then is

$$u(x) = \frac{u_2(x)}{C} \int_1^x ((2y+1)e^{-y} - 3e^{y-2})y^2 \mathrm{d}y + \frac{u_1(x)}{C} \int_x^2 ((2y+1)e^{-y} - 5e^{y-4})y^2 \mathrm{d}y \ .$$

(d) For this, we find the general solution of $\mathcal{L}u = x^2$, which by part **(b)** is the same as

$$xu'' - u' + (1-x)u = x^4 \ .$$

We return to the original u_1 and u_2.

We must compute

$$M(y) = \begin{bmatrix} u_1(y) & u_2(y) \\ u_1'(y) & u_2'(y) \end{bmatrix} \quad \text{and} \quad N(x, y) = \begin{bmatrix} u_1(y) & u_2(y) \\ u_1(x) & u_2(x) \end{bmatrix} \ ,$$

and then a particular solutions is

$$u_p(x) = \int_{x_0}^x \frac{\det(N(x, y))}{\det(M(y))} r(y) \mathrm{d}y \ .$$

We find $\det(M(y)) = -2y$ and $\det(N(x, y)) = xe^{y-x} - ye^{x-y} + \sinh(y - x)$, and $r(y) = y^3$. Altogether, taking $x_0 = 1$, we find

$$u_p(x) = -\frac{1}{2} \int_1^x (xe^{y-x} - ye^{x-y} + \sinh(y-x))y^2 \mathrm{d}y \ .$$

Doing the integral one finds

$$u_p = \frac{1}{2}(19 + x)\cosh(x - 1) + \frac{1}{2}(18 - x)\cosh(x - 1) - x^3 - x^2 - 4x - 4 \ .$$

Differentiating,

$$u_p' = \frac{1}{2}(19 - x)\cosh(x - 1) + \frac{1}{2}(18 + x)\cosh(x - 1) - 3x^2 - 2x - 4 \ ,$$

and therefore

$$u_p'(1) = 0 \quad \text{and} \quad u_p'(2) + \frac{17}{2}\cosh(1) + 10\sinh(1) - 20 \ .$$

The general solution is

$$c_1u_1 + c_2u_2 + u_p \ .$$

Evaluating the derivative at $x = 1$ and $x = 2$ give the system

$$\begin{aligned} c_1e - \frac{1}{2e}c_2 &= 0 \\ c_1e^2 + \frac{3}{2e^2}c_2 &= -\frac{17}{2}\cosh(1) - 10\sinh(1) + 20 \end{aligned}$$

The first equation tells us that $c_1 = \frac{1}{2e^2}c_2$, and then the second equation tells us

$$c_2 = \left(\frac{2e^2}{e^2-3}\right)\left(-\frac{17}{2}\cosh(1) - 10\sinh(1) + 20\right) ,$$

so that finally

$$c_1 = \left(\frac{1}{e^2-3}\right)\left(-\frac{17}{2}\cosh(1) - 10\sinh(1) + 20\right) ,$$

3: Solve the equation

$$\frac{\partial}{\partial t}h(x,t) = (1+x)^2\frac{\partial^2}{\partial^2 x}h(x,t)$$

for $x \in (0,1)$, $t > 0$, subject to

$$h(0,t) = h(1,t) = 0$$

and

$$h(x,0) = (1+x)^{1/2}\sin\left(\pi\frac{\ln(1+x))}{\ln(2)}\right) .$$

SOLUTION Let $\mathcal{L}u = (1+x)^2u''(x)$. The eigenvalue problem is

$$(1+x)^2u''(x) - \lambda u(x) = 0 .$$

It is natural to look for solutions of the form $u(x) = (1+x)^\alpha$. This yields the equation $\alpha(\alpha-1)+\lambda = 0$. Proceeding as in an example in the text, we find the eigenvalues

$$\lambda_k = -\frac{1}{4} - \frac{k^2\pi^2}{(\ln(2))^2} ,$$

and the eigenfunctions

$$u_k(x) = (1+x)^{1/2}\sin\left(k\pi\frac{\ln(1+x)}{\ln(2)}\right) .$$

Notice that $h(x,0) = u_1(x)$, and therefore, the unique solution of the equation is the special solutions

$$h(x,t) = \exp\left(t\left(-\frac{1}{4} - \frac{\pi^2}{(\ln(2))^2}\right)\right)(1+x)^{1/2}\sin\left(\pi\frac{\ln(1+x))}{\ln(2)}\right) .$$

4: **(a)** Consider the equation $\mathcal{L}u = 0$ where

$$\mathcal{L}u = \left(\frac{1}{x}u'\right)' + \frac{1-2x}{x^2}u .$$

(a) Are there any solutions of this equation that have more than one zero on $[1,\infty)$? Justify your answer: First find an equation of the form

$$y'' + V(x)y(x)$$

so that every solution of this equation on $(0,\infty)$ is a nonzero multiple of a solution of $\mathcal{L}u = 0$, and apply the Strum oscillation theorems.

(b) Does $\mathcal{L}u = f(x)$ have a solution satisfying $u(1) = u(2) = 0$ for all continuous $f(x)$ on $[1,2]$? Justify your answer.

SOLUTION (a) We write the equation is standard form

$$u'' - \frac{1}{x}u' + \left(\frac{1}{x} - 2\right) u \ .$$

Then $P = -1/x$ and $Q = 1/x - 2$, so

$$V = Q - \frac{1}{4}P^2 - \frac{1}{2}P' = \left(\frac{1}{x} - 2\right) - \frac{3}{4x^2} = \frac{4x - 8x^2 - 3}{4x^2} < 0 \ .$$

Since V is negative, solutions will have at most one zero, so there are no such solutions.

(b) Yes. The only way this can fail to happen is if there is a non-trivial solution of $\mathcal{L}u = 0$ that satisfies $u(1) = u(2) = 0$. But by part **(a)**, non-trivial solutions of $\mathcal{L}u = 0$ have a most one zero.