Homework Assignment 4, Math 292, Spring 2018

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1. Let A be the matrix $A = \begin{bmatrix} 0 & 1 \\ -\kappa & 0 \end{bmatrix}$.

(a) Compute A^2, A^3 and A^4 Observe the patterns, and deduce a formula for A^k for all positive integers k. (You will probably want to consider even and odd k separately.)

(b) Use the results of part **(a)** to compute e^{tA}.

2. In this problem we consider driven oscillations with friction taken into account. We will consider a fricative force of the form $-ax'(t)$ where $a > 0$. That is the force is a negative multiple of the velocity. Combining this with the spring force, again assumed to be given by Hooke's Law, we have the Newton equation

$$mx''(t) = -kx(t) - ax'(t) + f(t) \tag{0.1}$$

where m is the mass, k is the spring constant, and $f(t)$ is the driving force.

(a) Introduce $y(t) = x'(t)$, and $\mathbf{x}(t) = (x(t), y(y))$ and $\mathbf{g}(t) = (0, \frac{1}{m}f(t))$. Find a 2×2 matrix B so that (0.1) is equivalent to

$$\mathbf{x}'(t) = B\mathbf{x}(t) + \mathbf{g}(t)\ .$$

(b) Compute e^{tB}. There will be three cases, according to whether $(a/m)^2 > 4(k/m)$, $(a/m)^2 = 4(k/m)$ and $(a/m)^2 < 4(k/m)$.

3. **(a)** Continuing with Exercise 2, use Duhamel's formula to-find integral formulas for the solution of (0.1). You will need 3 formulas, depending on whether $(a/m)^2 > 4(k/m)$, $(a/m)^2 = 4(k/m)$ or $(a/m)^2 < 4(k/m)$.

(b) Solve (0.1) with $x(0) = 0$,x'(0) =0, $f(t) = \cos(t)$, $m = 1$, $a = 1$ and $k = 5/4$.

(c) Solve (0.1) with $x(0) = 0$,x'(0) =0, $f(t) = \cos(t)$, $m = 1$, $a = 1$ and $k = 1/4$.

4. Consider the matrices

$$M := \frac{1}{5}\begin{bmatrix} 8 & 6 \\ 6 & 17 \end{bmatrix} \quad \text{and} \quad A := \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix} .$$

(a) Compute the matrix square root $M^{1/2}$.

(b) Compute the matrix $L := M^{-1/2}AM^{-1/2}$.

(c) Let $\mathbf{u}_1 = \frac{1}{\sqrt{5}}(2,1)$ and let $\mathbf{u}_2 = \frac{1}{\sqrt{5}}(1,-2)$. Find solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ of

$$\mathbf{y}'' = -L\mathbf{y}$$

that satisfy $\mathbf{y}_j(0) = \mathbf{u}_j$ and $\mathbf{y}_j'(0) = \mathbf{0}$, $j = 1, 2$. Find solutions $\mathbf{y}_3$ and $\mathbf{y}_4$ of the same equation that satisfy $\mathbf{y}_{j+2}(0) = \mathbf{0}$ and $\mathbf{y}_{j+2}'(0) = \mathbf{u}_j$, $j = 1, 2$.

(d) Find the solution of $M\mathbf{x}'' = -A\mathbf{x}$ with $\mathbf{x}(0) = 0$ and $\mathbf{x}'(0) = (3, -1)$.