Solutions for Selected Exercises from Chapter 4

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4.2 Let $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$. Find the solution of the system

$$\mathbf{x}'(t) = A\mathbf{x}(t) + e^{-8t}(t,t) \qquad \text{with} \qquad \mathbf{x}(0) = (0,1)\ .$$

SOLUTION: We apply Duhamel's formula. We first compute

$$e^{tA} = \frac{1}{2}\begin{bmatrix} e^{8t}+e^{2t} & e^{8t}-e^{2t} \\ e^{8t}-e^{2t} & e^{8t}+e^{2t} \end{bmatrix}\ .$$

Notice that $(1,1)$ is an eogenvector of A with the eigenvalue 8. Hence

$$e^{(t-s)A}e^{-8s}(s,s) = e^{-8s}e^{(t-s)A}(s,s) = se^{-8s}e^{(t-s)8}(1,1) = e^{8t}se^{-16s}(1,1)\ .$$

Therefore,

$$\int_0^t e^{(t-s)A}e^{-8s}(s,s)\mathrm{d}s = e^{8t}\frac{1}{256}(1-16te^{-16t}-e^{-16t})(1,1)\ .$$

Combining this with

$$e^{tA}\mathbf{x}_0 = \frac{1}{2}(e^{8t}-e^{2t}, e^{8t}+e^{2t})\ .$$

we get,

$$\mathbf{x}(t) = \frac{1}{2}(e^{8t}-e^{2t}, e^{8t}+e^{2t}) + e^{8t}\frac{1}{256}(1-16te^{-16t}-e^{-16t})(1,1)\ .$$

4.8 Consider the vector field

$$\mathbf{v}(x,y) = ((x+y)(x-y-1), (x+y-2)(x-y+1))\ .$$

(a) Find all equilibrium points of $\mathbf{v}$, and determine which, if any, are asymptotically stable, and which if any are unstable.

(b) Do the same for

$$\mathbf{v}(x,y) = ((x+y-2)(x-y+1)\ , (x+y)(x-y-1))\ .$$

SOLUTION: If $(x+y)(x-y-1)=0$, then either $y=-x$ or else $y=x-1$. In the first case, $(x+y-2)(x-y+1)=0$ reduces to $x=-1/2$, yielding the equilibrium point $(-1/2,1/2)$. In the second case, $(x+y-2)(x-y+1)=0$ reduces to $x=3/2$, yielding the equilibrium point $(3/2,1/2)$. We next compute the Jacobian

$$D_{\mathbf{v}}(x,y)] = \begin{bmatrix} 2x-1 & -2y-1 \\ 2x-1 & -2y+3 \end{bmatrix} .$$

Evaluating at $(-1/2,1/2)$, we find

$$A = \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}$$

The eigenvalues are $\pm 2^{3/2}$ and so this equilibrium point is unstable.

Evaluating at $(3/2,1/2)$, we find

$$A = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$$

The eigenvalues are $2 \pm 2i$ and so this equilibrium point is unstable.

(b) Since the components of $\mathbf{v}$ have been swapped, the equilibrium points are the same as the ones we found in **(b)**. Hence the equlibria are $(-1/2,1/2)$ and $(3/2,1/2)$.

We next compute the Jacobian

$$D_{\mathbf{v}}(x,y)] = \begin{bmatrix} 2x-1 & -2y+3 \\ 2x-1 & -2y-1 \end{bmatrix} .$$

Evaluating at $(-1/2,1/2)$, we find

$$A = -\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$$

which is -1 times a matrix we deat with above. The eigenvlaues are $-2 \pm 2i$, and this equilibrium is assymptotically stable – solutions spiral in toward it.

Evaluating at $(3/2,1/2)$, we find

$$A = -\begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}$$

which is -1 times a matrix we dealt with above. The eigenvlaues are The eigenvalues are $\pm 2^{3/2}$ and so this equilibrium point is unstable.

4.9 Consider the vector field

$$\mathbf{v}(x,y) = (x - xy, y + 2xy) .$$

(a) Find all equilibrium points of $\mathbf{v}$, and determine which, if any, are asymptotically stable, and which if any are unstable.

(b) Do the same for

$$\mathbf{v}(x,y) = (y + 2xy \ , x - xy) .$$

SOLUTION: If $x - yx = 0$, either $x = 0$ or $y = 1$. In the first case, $y + 2xy = 0$ reduces to $y = 0$, giving the equilibrium point $(0, 0)$. In the second case, $y + 2xy = 0$ reduce to $1 + 2x = 0$, so $x = -1/2$ giving the equilibrium point $(-1/2, 1)$.

We next compute the Jacobian

$$D_{\mathbf{v}}(x,y)] = \begin{bmatrix} 1-y & -x \\ 2y & 2x+1 \end{bmatrix} .$$

Evaluating at $(0, 0)$, we find

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvalue 1, and so this equilibrium point is unstable.

Evaluating at $(-1/2, 1)$, we find

$$A = \begin{bmatrix} 0 & 1/2 \\ 2 & 0 \end{bmatrix}$$

The eigenvalues are ± 1 and so this equilibrium point is unstable.

(b) Since the components of $\mathbf{v}$ have been swapped, the equilibrium points are the same as the ones we found in **(b)**. Hence the equlibria are $(0, 0)$ and $(-1/2, 1)$.

We next compute the Jacobian

$$D_{\mathbf{v}}(x,y)] = \begin{bmatrix} 2y & 2x+1 \\ 1-y & -x \end{bmatrix} .$$

Evaluating at $(0, 0)$, we find

$$A = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The eigenvlaues are ± 1, and this equilibrium is unstable.

Evaluating at $(-1/2, 1)$, we find

$$A = -\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

which is -1 times a matrix we delat with above. The eigenvlaues are The eigenvlaues are 2 and 1/2, and this equilibrium is unstable