Solutions for Selected Exercises from Chapter 3

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February 25, 2018

3.2 Consider the vector field $\mathbf{v}(\mathbf{x})$ defined on $U = \{(x, y) \ : \ x > |y|\}$ defined by

$$\mathbf{v}(x, y) = \left(x - y\sqrt{x^2 - y^2} \ , \ y - x\sqrt{x^2 - y^2}\right) .$$

Notice that on the boundary of U, the vector field is tangent to the boundary, so that the vector field does not ever carry the solution out of U.

Define the change of variables

$$u(x, y) = \frac{1}{2}\left(\frac{x + y}{x - y}\right) \qquad \text{and} \qquad v(x, y) = \sqrt{x^2 - y^2} .$$

(a) Let $\mathbf{x}(t)$ be any continuously differentiable curve in $\mathbb{R}^2$ with values in U. Define a curve

$$\mathbf{u}(t) = (u(\mathbf{x}(t)), v(\mathbf{x}(t))) .$$

Find a vector field $\mathbf{w}$ defined on an open set $V \subset \mathbb{R}^2$ so that

$$\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t)) \qquad \text{if and only if} \qquad \mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t)) .$$

(b) Find the general solution of $\mathbf{u}'(t) = \mathbf{w}(\mathbf{u}(t))$ with $\mathbf{u}(0) \in V$.

(c) Find the general solution of $\mathbf{x}'(t) = \mathbf{w}(\mathbf{u}(t))$ with $\mathbf{x}(0) \in U$, and find the corresponding flow transformation.

SOLUTION: By the Chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}'(t) = [D_{u,v}(\mathbf{x}(t)]\mathbf{x}'(t) = [D_{\mathbf{u}}(\mathbf{x}(t)]\mathbf{v}(\mathbf{x}(t)) .$$

Computing the Jacobian, we find

$$[D_{u,v}(\mathbf{x}] = \begin{bmatrix} -y(x-y)^{-2} & x(x-y)^{-2} \\ -x(x^2-y^2)^{-1/2} & y(x^2-y^2)^{-1/2} \end{bmatrix}$$

and then

$$[D_{u,v}(\mathbf{x}]\mathbf{v}(\mathbf{x}) = \left(\sqrt{x^2 - y^2}(x + y)/(x - y) \ , \ \sqrt{x^2 - y^2}\right) = (2uv, v) .$$

Thus we define $\mathbf{w}(u,v) = (2uv, v)$ and then $\mathbf{u}' = \mathbf{w}(\mathbf{u})$ is the equivalent equation in the new variables.

As (x,y) ranges over U, (u,v) ranges over V, the positive quadrant. Since the determinant of $[D_{u,v}(\mathbf{x})]$ is not zero on U, the change of variables is invertible there – but you can also check this explicitly; see part **c** below.

(b) We only need to solve the decoupled system $u' = 2uv$ and $v' = v$. The solution of the latter is $v(t) = e^t v_0$, and then the first equation becomes

$$\frac{u'}{u} = 2v_0 e^t \ ,$$

and integrating we find

$$\ln(u(t)/u_0) = 2v_0(e^t - 1) \ .$$

That is,

$$u(t) = u_0 \exp(2v_0(e^t - 1)) \ .$$

Altogether,

$$\mathbf{u}(t) = \big(u_0 \exp(2v_0(e^t - 1)) \ , \ 2v_0 e^t\big) \ .$$

(c) We now need to invert the change of variables. From $x + y = 2u(x - y)$, multiplying both sides by $x + y$ first, and then $x - y$, we deduce $x + y = \sqrt{2uv}$ and $x - y = v/\sqrt{2u}$. Therefore

$$x = \frac{1}{2}\left(\sqrt{2u} + \frac{1}{\sqrt{2u}}\right) v \quad \text{and} \quad y = \frac{1}{2}\left(\sqrt{2u} - \frac{1}{\sqrt{2u}}\right) v \ . \tag{0.1}$$

Now, given (x_0, y_0) define (u_0, v_0) by

$$u_0 = \frac{1}{2}\left(\frac{x_0 + y_0}{x_0 - y_0}\right) \qquad \text{and} \qquad v_0 = \sqrt{x_0^2 - y_0^2} \ .$$

Then in terms of the u, v variables, the solutions we seek is

$$\mathbf{u}(t) = \left(\frac{1}{2}\left(\frac{x_0 + y_0}{x_0 - y_0}\right) \exp(2\sqrt{x_0^2 - y_0^2}(e^t - 1)) \ , \ 2\sqrt{x_0^2 - y_0^2} e^t\right) \ .$$

Finally, substituting $u(t)$ and $v(t)$ into the change of variables (0.1) gives us $\Psi_t(x_0, y_0)$. That is,

$$\Psi_t(x_0, y_0) = v(t)\frac{1}{2}\left(\sqrt{2u(t)} + \frac{1}{\sqrt{2u(t)}} \ , \ \sqrt{2u(t)} - \frac{1}{\sqrt{2u(t)}}\right)$$

for this $u(t)$ and $v(t)$.

4. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} -4 & 2 \\ 5 & -1 \end{bmatrix} \ .$$

(a) Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}.

(b) Find all $\mathbf{x}_0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

SOLUTION:

5. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix} .$$

(a) Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}.

(b) Find all $\mathbf{x}_0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

SOLUTION: The characteristic polynomial is $t^2 - 6t + 9$ so $\mu = 3$ is the only eigenvalue. The vector $(1, 2)$ is a corresponding eigenvector. In this case $(A - 3I)^2 = 0$, and so

$$e^{tA} = e^{3t}(I + t(A - 3I) = e^{3t} \begin{bmatrix} 1 + 2t & -t \\ 4t & 1 - 2t \end{bmatrix} .$$

The solution with $\mathbf{x}(t) = (x_0, y_0)$ then is

$$e^{3t}((1 + 2t)x_0 - y_0 \ , \ 4x_0 + (1 - 2t)y_0) \ .$$

For large t, the leading term is $2te^{3t}(x_0 \ , \ -y_0)$, and this tends to $(0, 0)$ only for $(x_), y_0) = (0, 0)$.

6. Consider the differential equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} .$$

(a) Find the general solution $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ in closed form. That is, compute e^{tA}.

(b) Find all $\mathbf{x}_0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

SOLUTION: The characteristic polynomial is $t^2 - 3t + -10$ so the eigenvalues are $\mu_1 = 5$ and $mu_2 = -2$. A pair of corresponding eigenvectors is $\mathbf{v}_1 = (2, 1)$ and $\mathbf{v}_2 = (1, -3)$. Define $M(t) = [e^{5t}\mathbf{v}_1, e^{3t}\mathbf{v}_2]$. Then

$$e^{tA} = M(t)M(0)^{-1} = \frac{1}{7} \begin{bmatrix} 6e^{5t} + e^{-2t} & 2e^{5t} - 2e^{-2t} \\ 3e^{5t} - 3e^{-2t} & e^{5t} + 6e^{-2t} \end{bmatrix} .$$

The general solution is of the form $\alpha e^{5t}\mathbf{v}_1 + \beta e^{-2t}\mathbf{v}_2$. Evidently this goes to $\mathbf{0}$ as $t \to \infty$ if and only if $\alpha = 0$, and this is the case if an only if $\mathbf{x}_0$ is a multiple of $\mathbf{v}_2$.

9 Compute e^{tA} for

$$A = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} .$$

SOLUTION: The characteristic polynomial is $t^3 - 3t^2 + 3t - 1$ so $\mu = 1$ is the only eigenvalue. The vectors $\mathbf{v}_1(1, 0, 2)$ and $\mathbf{v}_2 = (3, 4, 0)$ are two linearly independent eigenvectors, but the rank of $A - I$ is one, and a third linearly independent vector cannot be found. However, in this case $(A - I)^2 = 0$, and so

$$e^{tA} = e^t(I + t(A - I) = e^t \begin{bmatrix} 1+4t & -3t & -2t \\ 8t & 1-6t & -4t \\ -4t & 3t & 1+2t \end{bmatrix} .$$

11. Compute e^{tA} for

$$A = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 2 & 2 \\ -2 & 2 & 1 \end{bmatrix} .$$

SOLUTION: The characteristic polynomial is $t^3 - 7t^2 + 15t - 9$ so $\mu_1 = 1$ and $\mu_2 = 3$ are the only eigenvalues. Since μ_1 is not repeated, we only seek an eigenvector for it: We easily find $\mathbf{v}_1 = (-1, -1, 1)$. Since μ_2 is repeated, we may need to work with generalized eigenvectors. (Indeed, this is the case.) We compute $A - 3I$ and find that it has two linearly independent rows. Hence the rank is 2, and we cannot find more than one linearly independent eigenvector – but we do find one: $\mathbf{v}_2 = (1, 1, 0)$. We now form

$$(A - 3I)^2 = 4 \begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} .$$

All of the rows are multiples of one another, so the rank is 1, and we can find two linearly independent solutions of $(A - 3I)\mathbf{x} = 0$. One is $\mathbf{v}_2$. To get another, we seek a vector that is orthogonal to both $\mathbf{v}_2$ and $(1, -1, 1)$. A simple choice is $\mathbf{v}_3 = (1, -1, -2)$.

Then

$$\mathbf{x}_3(t) := e^t A\mathbf{v}_3 = e^{3t}(I + t[A - 3I)\mathbf{v}_3 = e^{3t}(\mathbf{v}_3 + t(-2, -2, 0)) = e^{3t}(1 - 2t, -1 - 2t, -2) .$$

The other two solutions as $\mathbf{x}_1(t) = e^t\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{3t}\mathbf{v}_2$. We then form $M(t) = [\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)]$, and

$$e^{tA} = M(t)M(0)^{-1} = \begin{bmatrix} e^{3t}(2-t) - e^t & te^{3t} + e^t - e^{3t} & e^{3t} - e^t \\ e^{3t} - e^t - te^{3t} & te^{3t} + e^t & e^{3t} - e^t \\ e^t - e^{3t} & e^{3t} - e^t & e^t \end{bmatrix}$$

0.1 Remark. Notice that the solution to 3.9 is much simpler than the solution to 3.11 because in the case of 3.9, there was only one eigenvalue, and then we did not need to work explicitly with generalized eigenvectors. When there is more than one eigenvalue, and one is "missing" eigenvectors for at least one of these eigenvalues, then one must proceed as we did in 3.11, working directly with generalized eigenvectors.